STUDIA MATHEMATICA, T. XXXIV. (1970)

Continuity of seminorms and linear mappings on a space with Schauder basis *

by

OSCAR T. JONES (Baton Rouge)

1. Introduction. In this paper we study conditions which ensure continuity of seminorms or of linear mappings on a space with Schauder basis. Interest in Schauder basis has been largely concentrated on Banach and Fréchet spaces, although generalization of the concept to non-metrizable spaces is straightforward (see [2], for example) and examples exist in abundance. The relationships of the conditions considered here, however, are most clearly understood in the context of a general locally convex linear topological space.

Throughout the paper $(X, \mathscr{T})$, or simply X, will denote a locally convex linear topological space with topology $\mathscr{T}$, $X^* = (X, \mathscr{T})^*$ the conjugate space of X, $\{x_n\}$ a Schauder basis of X, $\{f_n\}$ the coefficient functionals biorthogonal to $\{x_n\}$, and S_n $(n = 1, 2, \ldots)$ the n^{th} partial sum operator defined by

$$S_n(x) = \sum_{K=1}^{n} f_K(x) x_K \qquad (x \in X).$$

Thus for each $x \in X$ we have

$$x = \sum_{K=1}^{\infty} f_K(x) x_K = \lim_{n\to\infty} S_n(x)$$

2. Seminorms and Schauder bases.

THEOREM 1. *Let X be a barrelled space with Schauder basis $\{x_n\}$. Each seminorm p on X satisfying*

(1) $$\lim_{n\to\infty} p\big(S_n(x)\big) = p(x) \qquad (x \in X)$$

is continuous.

---

* Portions of the article are from the author's doctoral dissertation at Florida State University (1966), research for which was supported by National Science Foundation Grand GP-2179

The above assertion follows from the fact that in any barralled space the point-wise limit of a sequence of continuous seminorms is continuous, provided that this limit exists for all vectors in the given space (see e.g. [6], Theorem 8.12).

Some implications of this theorem are discussed below. For the present we observe that a partial converse is valid.

THEOREM 2. *Let X be a space with Schauder basis $\{x_n\}$, and suppose each seminorm on X satisfying* (1) *is continuous. Then X is quasi-barrelled.*

Proof. Let U be a bound absorbing barrel in X, and let p be its gauge. For each $x \in X$, $\{S_n(x): n = 1, 2, \ldots\}$ is bounded and is therefore absorbed by U. Thus we may define a seminorm p' on X by

$$p'(x) = \sup_m p\big(S_m(x)\big) \qquad (x \in X).$$

Since U is closed, we have $p(x) \leqslant p'(x)$ on X. Now for each $x \in X$,

$$\begin{aligned}\lim_{n\to\infty} p'\big(S_n(x)\big) &= \lim_{n\to\infty} \sup_m p\big(S_m \circ S_n(x)\big) = \lim_{n\to\infty} \sup_{1\leqslant m\leqslant n} p\big(S_m(x)\big)\\ &= \sup_m p\big(S_m(x)\big) = p'(x).\end{aligned}$$

Thus p' satisfies (1). It follows that p' is continuous, whence p is continuous, and U is a neighborhood of 0 in X.

3. Some related conditions on Schauder bases. Consider the following assertions which may or may not hold for the space $(X, \mathcal{T})$ with Schauder basis $\{x_n\}$. They are related to the condition of Theorem 1 and to each other in the manner subsequently described.

(A) If T is a linear mapping of X into a locally convex space Y such that

$$\lim_{n\to\infty} T\big(S_n(x)\big) = T(x) \qquad (x \in X) \tag{2}$$

then T is continuous.

(A$'$) If a linear functional f on X satisfies

$$\lim_{n\to\infty} f\big(S_n(x)\big) = f(x) \qquad (x \in X),$$

then f is continuous.

(B) If p is a seminorm on X such that

$$\lim_{n\to\infty} p\big(x - S_n(x)\big) = 0 \qquad (x \in X), \tag{3}$$

then p is continuous.

(C) There exists no locally convex topology $\mathcal{T}'$ for X which is strictly stronger than $\mathcal{T}$ such that $\{x_n\}$ is a basis for $(X, \mathcal{T}')$.

THEOREM 3. (A), (B), *and* (C) *are equivalent assertions.*

Proof. We show (A) $\Rightarrow$ (C) $\Rightarrow$ (B) $\Rightarrow$ (A).

Assume (A), and let $\mathcal{T}'$ be a locally convex topology for X which is stronger than $\mathcal{T}$ and such that $\{x_n\}$ is a basis for $(X, \mathcal{T}')$. The identity map of $(X, \mathcal{T})$ onto $(X, \mathcal{T}')$ satisfies (2) and is therefore continuous. Thus $\mathcal{T} = \mathcal{T}'$.

Given (C), let p be a seminorm on X satisfying (3). Let Φ be a collection of seminorms on X such that $\mathcal{T} = \sigma(X, \Phi)$. Let Φ' be Φ augmented by p. Then $\{x_n\}$ is a basis for X relative to the topology $\sigma(X, \Phi')$. Thus $\sigma(X, \Phi) = \sigma(X, \Phi')$, whence p is $\mathcal{T}$-continuous.

Assume (B). Let T be a linear mapping of X into a locally convex space Y, and suppose T satisfies (2). Let U be a closed convex circled neighborhood of 0 in Y. If q is the gauge of U, then $p = q \circ T$ is a seminorm on X. We define a seminorm p' on X by

$$p'(x) = \sup_m p\big(S_m(x)\big) \qquad (x \in X).$$

Since $\lim\limits_{n\to\infty} p\big(S_n(x)\big) = p(x)$, we see that $p'(x) \geqslant p(x)$ for all $x \in X$. Let $x \in X$. Given $\varepsilon > 0$, there exists N such that for $n, m \geqslant N$,

$$p\big(S_m(x) - S_n(x)\big) = q\big[T\big(S_m(x)\big) - T\big(S_n(x)\big)\big] \leqslant \varepsilon.$$

Now if $n \geqslant N$,

$$p'\big(x - S_n(x)\big) = \sup_m p\big(S_m(x) - S_m \circ S_n(x)\big) = \sup_{m>n} p\big(S_m(x) - S_n(x)\big) \leqslant \varepsilon.$$

Thus $\lim\limits_{n\to\infty} p'\big(x - S_n(x)\big) = 0$ for $x \in X$, so p' is continuous, whence p is continuous. Then $V = \{x: p(x) \leqslant 1\}$ is a neighborhood of 0 in X. But $V = T^{-1}[U]$. It follows that T is continuous. This completes the proof of the theorem.

Remark. It is clear that (A) implies (A$'$). On the other hand, if the topology $\mathcal{T}$ of X is Mackey and (A$'$) is satisfied, then (A) holds. For if $\mathcal{T}'$ is a locally convex topology for X such that $\{x_n\}$ is a basis for $(X, \mathcal{T}')$, then $(X, \mathcal{T}')^* \subset (X, \mathcal{T})^*$, so $\mathcal{T}'$ cannot be strictly stronger than $\mathcal{T}$. This verifies (C), hence (A). It follows from the seminorm inequality

$$|p(x) - p(y)| \leqslant p(x - y)$$

that the condition of Theorem 1 implies (B).

The following example suggested by J. R. Retherford shows that they are not equivalent. It is an example of a basis of a non-barrelled space with basis satisfying condition (A). The space is not even quasi-barrelled, which in view of Theorem 2 shows that (A) and the condition of Theorem 1 are not equivalent.

Example 1. With the usual norms, l^1 and m are Banach spaces, and $(l^1)^* = m$. Let X be the space m with the Mackey topology derived from the natural pairing of l^1 and m. X is not quasi-barrelled, hence neither barrelled nor bound ([6], Problem 20A). In the terminology of Köthe [8], Section 30, l^1 is the α-dual of m. It follows ([8], p. 417 (10)) that the unit vectors of m form a basis for X. It is easy to see that the basis satisfies (A′), hence (A).

4. Equicontinuity of partial sum operators. Bases with the property that the sequence of partial sum operators $\{S_n\}$ is equicontinuous have been studied in [9]. We observe that this property is implied by condition (A), but the converse does not hold.

Let $\{x_n\}$ be a Schauder basis for X, and let $\mathscr{U}$ be a base for the neighborhood system of 0 in X consisting of closed convex circled sets. For $U \in \mathscr{U}$ define

$$U' = \{x \in X\colon S_n(x) \in U \text{ for all } n\}.$$

$\mathscr{U}' = \{U'\colon U \in \mathscr{U}\}$ is a local base for a locally convex topology $\mathscr{T}'$ for X; $\mathscr{T}'$ is stronger than the original topology $\mathscr{T}$ of X, and $\{x_n\}$ is a Schauder basis for $(X, \mathscr{T}')$. These assertions are verified in a somewhat more general setting in [10].

Since

$$\bigcap_{n=1}^{\infty} S_n^{-1}[U'] = U' \qquad (U \in \mathscr{U}),$$

we see that $\{S_n\}$ is equicontinuous relative to $\mathscr{T}'$. In view of Theorem 3 (A $\Rightarrow$ C) we have the following result:

THEOREM 4. *If a basis satisfies condition* (A), *then the partial sum operators are equicontinuous.*

Example 2. Let X be l^1, and let $\mathscr{T}$ be the topology induced on X by the norm of c_0. $(X, \mathscr{T})$ is a normed space with closed unit sphere $U = \{a = (a_i) \in l^1\colon \sup_i |a_i| \leqslant 1\}$, and the unit vectors form a Schauder basis. The sequence of partial sum operators is equicontinuous, since

$$\bigcap_{n=1}^{\infty} S_n^{-1}[U] = \bigcap_{n=1}^{\infty} \{a \in l^1\colon S_n(a) \in U\} = \bigcap_{n=1}^{\infty} \{a \in l^1\colon \sup_{1 \leqslant i \leqslant n} |a_i| \leqslant 1\} = \{a \in l^1\colon \sup_i |a_i| \leqslant 1\} = U.$$

This basis does not satisfy (A), since it is a basis for l^1 with its natural topology, which is strictly stronger than $\mathscr{T}$.

5. Applications. Let $B(X, Y)$ be the space of continuous linear mappings of X into a linear topological space Y. If X is bound, this is precisely the space of bounded linear mappings of X into Y. In this case, if Y is complete, then $B(X, Y)$ is complete relative to the topology of uniform convergence on bounded subsets of X ([6], Theorem 8.15). The following completeness theorem is somewhat more general in the case of spaces X satisfying a basis requirement. The proof uses a version of the classical Moore theorem on interchange of limits. Usually stated for nets in a complete metric space ([3], I. 7.6, for example), it is also valid for nets in a complete locally convex space. Only technical changes in the standard proof are required.

THEOREM 5. *Let X have a basis $\{x_n\}$ satisfying condition* (A), *and let Y be a complete locally convex space. Then $B(X, Y)$ is complete relative to the topology of uniform convergence on bounded subsets of X.*

Proof. Let $\{T_\alpha\}$ be a Cauchy net in $B(X, Y)$ relative to the topology of uniform convergence on bounded subsets of X. For each $x \in X$, $\{T_\alpha(x)\}$ is a Cauchy net in Y, and therefore converges to a limit in Y which we denote by $T(x)$. T is linear on X to Y, and a standard argument shows that convergence of $\{T_\alpha\}$ to T is uniform on bounded subsets of X. For each $x \in X$ we have

$$\begin{aligned} \lim_{n\to\infty} T\big(S_n(x)\big) &= \lim_{n\to\infty}\lim_{\alpha} T_\alpha\big(S_n(x)\big) \\ &= \lim_{\alpha}\lim_{n\to\infty} T_\alpha\big(S_n(x)\big) \\ &= \lim_{\alpha} T_\alpha(x) = T(x). \end{aligned}$$

The interchange of limits is justified by the convergence of $\{T_\alpha\big(S_n(x)\big)\}$ to $T\big(S_n(x)\big)$ uniformly with respect to n, since $\{S_n(x)\}$ is bounded in X. It follows that T is continuous, and the proof is complete.

Theorem 5 implies that if a space X has a basis satisfying (A), then X^* with the strong topology is complete. A slightly stronger result is given in Theorem 6 below. In particular, the strong dual of a barrelled space with Schauder basis is complete. This is not true of barrelled spaces in general [7].

THEOREM 6. *Let X have a basis satisfying* (A′); *i.e., every linear functional f on X such that*

$$\lim_{n\to\infty} f\big(S_n(x)\big) = f(x) \qquad (x \in X)$$

is continuous. Then X^* is complete relative to the strong topology.

Proof. The proof is similar to that of Theorem 5, with obvious modifications.

References

[1] M. G. Arsove, *Similar bases and isomorphisms in Fréchet spaces*, Math. Ann. 135 (1958), p. 283–293.

[2] — and R. E. Edwards, *Generalized bases in topological linear spaces*, Studia Math. 19 (1960), p. 95–113.

[3] N. Dunford and J. T. Schwartz, *Linear operators*, Part I, New York 1958.

[4] I. A. Ezrohi, *The general form of linear operators in spaces with a countable basis*, Doklady Akad. Nauk SSSR (N. S.) 59 (1948), p. 1537–1540 (Russian) [Abstract, Math. Reviews 9 (1948), p. 448.]

[5] O. T. Jones and J. R. Retherford, *On similar bases in barrelled spaces*, Proc. Amer. Math. Soc. 18 (1967), p. 677–680.

[6] J. L. Kelley, I. Namioka et al., *Linear topological spaces*, Princeton 1963.

[7] Y. Komura, *Some examples on linear topological spaces*, Math. Ann. 153 (1964), p. 150–162.

[8] G. Köthe, *Topologische lineare Räume I*, Die Grund. der Math. Wiss., Band 107, Berlin 1960.

[9] C. W. McArthur and J. R. Retherford, *Uniform and equicontinuous Schauder bases of subspaces*, Canad. J. Math. 17 (1965), p. 207–212.

[10] — *Some remarks on bases in linear topological spaces*, Math. Ann. 164 (1966), p. 38–41.

STETSON UNIVERSITY and LOUISIANA STATE UNIVERSITY

Reçu par la Rédaction le 1. 11. 1968

STUDIA MATHEMATICA, T. XXXIV. (1970)

Les nombres de Pisot et l'analyse harmonique

par

Y. MEYER (Paris)

Le but de ce travail était de démontrer qu'un ensemble de nombres réels, du type Cantor, construit à l'aide d'un rapport de dissection (constant) ξ est un ensemble de synthèse dès que ξ^{-1} est un nombre de Pisot(1). Le résultat que nous avons en fait prouvé est plus général et moins précis (th. X) mais, chemin faisant, nous avons eu l'occasion de relier les nombres de Pisot et aussi les nombres de Salem à divers points de l'analyse harmonique.

Dans la partie I sont étudiés les *ensembles harmonieux* de nombres réels: nous appelons ainsi toute partie Λ de $\boldsymbol{R}$ telle que les restrictions à Λ des caractères du groupe $\boldsymbol{R}$, muni de la topologie discrète, soient uniformément approchables, sur Λ, par des caractères du groupe $\boldsymbol{R}$ muni de la topologie usuelle. Ces ensembles Λ sont, s'ils ne sont pas trop dispersés, caractérisés par le théorème IV.

Dans la partie II, une catégorie plus vaste d'ensembles Λ de nombres réels est étudiée: on dit que Λ a un *compact associé* s'il existe un nombre réel ε positif, non nul, et un ensemble compact K de la droite réelle tels que, pour tout polynôme trigonométrique à spectre dans Λ, $P(x)$, on ait

$$\sup_{x\in K}|P(x)| \geqslant \varepsilon \sup_{x\in R}|P(x)|.$$

Ces derniers ensembles jouent, pour le problème de la synthèse, un rôle analogue à celui des progressions arithmétiques intervenant dans un théorème de C. Herz ([3], th. VII, p. 124) et c'est, grâce à ce fait, que nous obtenons le théorème X dans le partie III.

I. LES ENSEMBLES HARMONIEUX

1. Propriétés générales des ensembles harmonieux

1.1. Le groupe additif des nombres réels est noté $\boldsymbol{R}$ et le groupe multiplicatif des nombres complexes de module 1 est noté $\boldsymbol{T}$. Sauf avis contraire, les topologies considérées sur $\boldsymbol{R}$ et $\boldsymbol{T}$ sont les topologies usuelles.

(1) L'auteur vient de prouver ce résultat quand, en outre, $0 < \xi < 1/3$.