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Orthonormal basis in the space $C_1[0,1]$

by

J. RADECKI (Poznań)

Introduction. Let $\{h_n(t)\}$, $n = 1, 2, \ldots$, denote the orthonormal Haar system, i.e.

$$h_1(t) \equiv 1 \quad \text{in } [0,1],$$

$$h_{2^n+1}(1) = -\sqrt{2^n},$$

$$h_{2^n+k}(t) = \begin{cases} \sqrt{2^n} & \text{in } \left[\frac{k-1}{2^n}, \frac{2k-1}{2^{n+1}}\right), \\ -\sqrt{2^n} & \text{in } \left[\frac{2k-1}{2^{n+1}}, \frac{k}{2^n}\right), \\ 0 & \text{elsewhere in } [0,1], \end{cases} \tag{1}$$

where $n = 0, 1, 2, \ldots$ and $k = 1, 2, \ldots, 2^n$. Ciesielski [1] proved that the functions

$$1, \quad \int_0^t h_n(s)\,ds \quad (0 \leqslant t \leqslant 1, n = 1, 2, \ldots) \tag{2}$$

form a Schauder basis in the space $C[0,1]$ of continuous functions and that for every function $f \epsilon C[0,1]$ we have

$$f(t) = f(0) + \sum_{n=1}^{\infty} \int_0^1 h_n(s)\,df(s) \int_0^t h_n(\tau)\,d\tau, \tag{3}$$

where the series is uniformly convergent on $[0,1]$. By the way, let us remark that this result may be at once generalized as follows. Functions

$$1, t, \ldots, t^p, \quad \int_0^t (t-s)^p h_n(s)\,ds \quad (n = 1, 2, \ldots)$$

form a basis in the space $C_p[0,1]$ of functions with continuous derivatives of order p, and for every $f \epsilon C_p[0,1]$ we have

$$f^{(r)}(t) = \frac{d^r}{dt^r}\left[f(0) + \frac{f'(0)}{1!}t + \ldots + \frac{f^{(p)}(0)}{p!}t^p\right] + \\ + \frac{1}{(p-r)!}\sum_{n=1}^{\infty} \int_0^1 h_n(\tau)\,df^{(p)}(\tau) \int_0^t (t-s)^{p-r} h_n(s)\,ds \quad (r = 0, 1, \ldots, p),$$

where the series converges uniformly on $[0,1]$.

By the orthogonalization process of (2) we obtain the Franklin orthonormal system.

In this paper we form — by means of Haar functions — an orthonormal basis in the space $C_1[0,1]$. The main results are contained in Section 1. The construction and some properties of this basis are given in Section 2. In Section 3 we prove inequalities of the Bernstein-Zygmund type.

The results of this paper may be developed in various directions. The author himself intends to devote his next papers to this task.

1. Fundamental lemma and theorem. Let l denote an integer such that $1 \leqslant l \leqslant n$ and let the segment $[0,1]$ be dissected by points ξ_i, where $\xi_i = i/2n$ for $i = 0, 1, \ldots, 2l$ and $\xi_i = (2i-2l)/2n$ for $i = 2l+1, \ldots, l+n$. We say that the function φ is an *element of the class* $A_{n,l}$ if and only if $\varphi \in C_1[0,1]$ and if $\varphi(t) = w_k(t)$ for $t \in (\xi_k, \xi_{k+1})$ $(k = 0, 1, \ldots, l+n-1)$, where $w_k(t)$ denotes a polynomial of degree not greater than 2.

LEMMA. *The necessary and sufficient condition for $\varphi \in A_{n,l}$ is*

$$(4)\quad \begin{cases} w_k(t) = \dfrac{\eta_{k+1}+\eta_k}{2} + \dfrac{\eta_{k+1}-\eta_k}{h}(t-\xi_k) + \\ \qquad + \dfrac{1}{2}(\eta_{k+2}-2\eta_{k+1}+\eta_k)\left(\dfrac{t-\xi_k}{h}\right)^2 \quad \text{for } k = 0, 1, \ldots, 2l-1, \\ w_{2l}(t) = \dfrac{\eta_{2l+1}+\eta_{2l}}{2} + \dfrac{\eta_{2l+1}-\eta_{2l}}{h}(t-\xi_{2l}) + \\ \qquad + \dfrac{1}{4}(2\eta_{2l+2}-7\eta_{2l+1}+5\eta_{2l})\left(\dfrac{t-\xi_{2l}}{2h}\right)^2, \\ w_{2l+1}(t) = \dfrac{2\eta_{2l+2}+3\eta_{2l+1}-\eta_{2l}}{4} + \dfrac{1}{2}(2\eta_{2l+2}-3\eta_{2l+1}+\eta_{2l})\dfrac{t-\xi_{2l+1}}{2h} + \\ \qquad + \dfrac{1}{4}(2\eta_{2l+3}-4\eta_{2l+2}+3\eta_{2l+1}-\eta_{2l})\left(\dfrac{t-\xi_{2l+1}}{2h}\right)^2, \\ w_k(t) = \dfrac{\eta_{k+1}+\eta_k}{2} + \dfrac{\eta_{k+1}-\eta_k}{2h}(t-\xi_k) + \\ \qquad + \dfrac{1}{2}(\eta_{k+2}-2\eta_{k+1}+\eta_k)\left(\dfrac{t-\xi_k}{2h}\right)^2 \quad \text{for } k = 2l+2, \ldots, l+n-1, \end{cases}$$

where $h = 1/2n$ and the parameters $\eta_0, \ldots, \eta_{l+n+1}$ are mutually independent.

Proof. We verify directly that $w_k(\xi_{k+1}) = w_{k+1}(\xi_{k+1})$ and $w_k'(\xi_{k+1}) = w_{k+1}'(\xi_{k+1})$ for $k = 0, 1, \ldots, l+n-2$; this proves the sufficiency. Let

$\varphi \in A_{n,l}$. Hence we have $\varphi(t) = w_k(t) = a_k + b_k(t-\xi_k) + c_k(t-\xi_k)^2$ for $t \in (\xi_k, \xi_{k+1})$ and $w_k(\xi_{k+1}) = w_{k+1}(\xi_{k+1})$ for $k = 0, 1, \ldots, l+n-2$; so

$$a_{k+1} = a_k + b_k(\xi_{k+1}-\xi_k) + c_k(\xi_{k+1}-\xi_k)^2$$

and

$$a_k = a_0 + \sum_{i=0}^{k-1} b_i(\xi_{i+1}-\xi_i) + \sum_{i=0}^{k-1} c_i(\xi_{i+1}-\xi_i)^2$$

for $k = 1, 2, \ldots, l+n-1$.

Let us write

$$B_0 = a_0, \quad B_k = a_0 + \sum_{i=0}^{k-1} b_i(\xi_{i+1}-\xi_i),$$

$$C_0 = 0, \quad C_k = \sum_{i=0}^{k-1} c_i(\xi_{i+1}-\xi_i)^2$$

for $k = 1, 2, \ldots, l+n$; then

$$b_k = \frac{B_{k+1}-B_k}{\xi_{k+1}-\xi_k}, \quad c_k = \frac{C_{k+1}-C_k}{(\xi_{k+1}-\xi_k)^2}$$

and

$$(5)\quad w_k(t) = B_k + C_k + \frac{B_{k+1}-B_k}{\xi_{k+1}-\xi_k}(t-\xi_k) + \frac{C_{k+1}-C_k}{(\xi_{k+1}-\xi_k)^2}(t-\xi_k)^2 \quad (k = 0, 1, \ldots, l+n-1).$$

Thus, from the conditions $w_k'(\xi_{k+1}) = w_{k+1}'(\xi_{k+1})$ we obtain

$$\frac{B_{k+1}-B_k}{\xi_{k+1}-\xi_k} + 2\frac{C_{k+1}-C_k}{\xi_{k+1}-\xi_k} = \frac{B_{k+2}-B_{k+1}}{\xi_{k+2}-\xi_{k+1}}$$

and, in particular,

$$C_{k+1}-C_k = \frac{B_{k+2}-B_{k+1}-(B_{k+1}-B_k)}{2} \quad \text{for } k = 0, 1, \ldots, 2l-2.$$

Hence

$$C_k = \frac{B_{k+1}-B_k}{2} - \frac{B_1-B_0}{2} \quad (k = 0, 1, \ldots, 2l-1).$$

Finally, let $B_k - \frac{1}{2}(B_1-B_0) = \eta_k$. Equality (5) assumes the form

$$w_k(t) = \frac{\eta_{k+1}+\eta_k}{2} + \frac{\eta_{k+1}-\eta_k}{h}(t-\xi_k) + \frac{1}{2}(\eta_{k+2}-2\eta_{k+1}+\eta_k)\left(\frac{t-\xi_k}{h}\right)^2 \quad \text{for } k = 0, 1, \ldots, 2l-1.$$

But for $i = 2l, \ldots, l+n-1$ we have $\xi_{i+1}-\xi_i = 1/n = 2h$.

Repeating the above argument, we infer that for $k = 2l, \ldots, l+n-1$ the polynomials have the form

$$w_k(t) = \frac{a_{k+1}+a_k}{2} + \frac{a_{k+1}-a_k}{2h}(t-\xi_k) + \frac{1}{2}(a_{k+2}-2a_{k+1}+a_k)\left(\frac{t-\xi_k}{2h}\right)^2.$$

By means of the conditions $w_{2l-1}(\xi_{2l}) = w_{2l}(\xi_{2l})$, $w'_{2l-1}(\xi_{2l}) = w'_{2l}(\xi_{2l})$, we have $a_{2l+1}+a_{2l} = \eta_{2l+1}+\eta_{2l}$, $a_{2l+1}-a_{2l} = 2(\eta_{2l+1}-\eta_{2l})$; hence $a_{2l} = \frac{1}{2}(-\eta_{2l+1}+3\eta_{2l})$, $a_{2l+1} = \frac{1}{2}(3\eta_{2l+1}-\eta_{2l})$.

Setting $a_k = \eta_k$ for $k = 2l+2, \ldots, l+n+1$, we obtain (4).

THEOREM. *If $f \in C[0,1]$, $n \geqslant 7$, $1 \leqslant l \leqslant n$, and*

$$\int_0^1 [\varphi(t)-f(t)]^2 dt = \inf_{\psi \in A_{n,l}} \int_0^1 [\psi(t)-f(t)]^2 dt,$$

then $\|\varphi\| \leqslant 2763\|f\|$. Moreover, if $f \in C_1[0,1]$, then also

$$\|\varphi'\| \leqslant 1020\|f'\|, \quad \textit{where } \|f\| = \max_{t\in[0,1]}|f(t)|.$$

Proof. Let $\varphi(t) = w_k(t)$ for $t \in (\xi_k, \xi_{k+1})$ $(k = 0, 1, \ldots, l+n-1)$, where the polynomials w_k are defined by (4), and let

$$F(\eta_0, \ldots, \eta_{l+n+1}) = \int_0^1 [\varphi(t)-f(t)]^2 dt.$$

The functional $\int_0^1 [\psi(t)-f(t)]^2 dt$ attains its minimum for $\psi(t) = \varphi(t)$; hence

$$\frac{\partial F}{\partial \eta_i} = 0 \quad \text{for } i = 0, 1, \ldots, l+n+1. \tag{6}$$

Let us write

$$a_{n,l} = \max_{0\leqslant i\leqslant l+n+1} |\eta_i|, \quad b_{n,l} = \max_{1\leqslant i\leqslant l+n+1} \frac{|\eta_i-\eta_{i-1}|}{h}.$$

Now, let us consider the following cases.

1° Let $2 \leqslant l \leqslant n-5$. System (6) assumes the form

$$\frac{1}{2}\frac{\partial F}{\partial \eta_0} = \int_{\xi_0}^{\xi_1} [w_0(t)-f(t)]\left[\frac{1}{2} - \frac{t-\xi_0}{h} + \frac{1}{2}\left(\frac{t-\xi_0}{h}\right)^2\right] dt = 0,$$

$$\frac{1}{2}\frac{\partial F}{\partial \eta_1} = \int_{\xi_0}^{\xi_1} [w_0(t)-f(t)]\left[\frac{1}{2} + \frac{t-\xi_0}{h} - \left(\frac{t-\xi_0}{h}\right)^2\right] dt + \int_{\xi_1}^{\xi_2} [w_1(t)-f(t)]\left[\frac{1}{2} - \frac{t-\xi_1}{h} + \frac{1}{2}\left(\frac{t-\xi_1}{h}\right)^2\right] dt = 0,$$

$$\frac{1}{2}\frac{\partial F}{\partial \eta_r} = \int_{\xi_{r-2}}^{\xi_{r-1}} [w_{r-2}(t)-f(t)]\frac{1}{2}\left(\frac{t-\xi_{r-2}}{h}\right)^2 + \int_{\xi_{r-1}}^{\xi_r} [w_{r-1}(t)-f(t)]\left[\frac{1}{2} + \frac{t-\xi_{r-1}}{h} - \left(\frac{t-\xi_{r-1}}{h}\right)^2\right] dt + \int_{\xi_r}^{\xi_{r+1}} [w_r(t)-f(t)]\left[\frac{1}{2} - \frac{t-\xi_r}{h} + \frac{1}{2}\left(\frac{t-\xi_r}{h}\right)^2\right] dt = 0$$

$$\text{for } r = 2, 3, \ldots, 2l-1,$$

$$\frac{1}{2}\frac{\partial F}{\partial \eta_{2l}} = \int_{\xi_{2l-2}}^{\xi_{2l-1}} [w_{2l-2}(t)-f(t)]\frac{1}{2}\left(\frac{t-\xi_{2l-2}}{h}\right)^2 dt + \int_{\xi_{2l-1}}^{\xi_{2l}} [w_{2l-1}(t)-f(t)]\left[\frac{1}{2} + \frac{t-\xi_{2l-1}}{h} - \left(\frac{t-\xi_{2l-1}}{h}\right)^2\right] dt + \int_{\xi_{2l}}^{\xi_{2l+1}} [w_{2l}(t)-f(t)]\left[\frac{1}{2} - 2\frac{t-\xi_{2l}}{2h} + \frac{5}{4}\left(\frac{t-\xi_{2l}}{2h}\right)^2\right] dt + \int_{\xi_{2l+1}}^{\xi_{2l+2}} [w_{2l+1}(t)-f(t)]\left[-\frac{1}{4} + \frac{1}{2}\frac{t-\xi_{2l+1}}{2h} - \frac{1}{4}\left(\frac{t-\xi_{2l+1}}{2h}\right)^2\right] dt = 0,$$

$$\frac{1}{2}\frac{\partial F}{\partial \eta_{2l+1}} = \int_{\xi_{2l-1}}^{\xi_{2l}} [w_{2l-1}(t)-f(t)]\frac{1}{2}\left(\frac{t-\xi_{2l-1}}{h}\right)^2 dt + \int_{\xi_{2l}}^{\xi_{2l+1}} [w_{2l}(t)-f(t)]\left[\frac{1}{2} + 2\frac{t-\xi_{2l}}{2h} - \frac{7}{4}\left(\frac{t-\xi_{2l}}{2h}\right)^2\right] dt + \int_{\xi_{2l+1}}^{\xi_{2l+2}} [w_{2l+1}(t)-f(t)]\left[\frac{3}{4} - \frac{3}{2}\frac{t-\xi_{2l+1}}{2h} + \frac{3}{4}\left(\frac{t-\xi_{2l+1}}{2h}\right)^2\right] dt = 0,$$

$$\frac{1}{2}\frac{\partial F}{\partial \eta_{2l+2}} = \int_{\xi_{2l}}^{\xi_{2l+1}} [w_{2l}(t)-f(t)]\frac{1}{2}\left(\frac{t-\xi_{2l}}{2h}\right)^2 dt + \int_{\xi_{2l+1}}^{\xi_{2l+2}} [w_{2l+1}(t)-f(t)]\left[\frac{1}{2} + \frac{t-\xi_{2l+1}}{2h} - \left(\frac{t-\xi_{2l+1}}{2h}\right)^2\right] dt + \int_{\xi_{2l+2}}^{\xi_{2l+3}} [w_{2l+2}(t)-f(t)]\left[\frac{1}{2} - \frac{t-\xi_{2l+2}}{2h} + \frac{1}{2}\left(\frac{t-\xi_{2l+2}}{2h}\right)^2\right] dt = 0,$$

$$\frac{1}{2}\,\frac{\partial F}{\partial \eta_{2l+3}} = \int\limits_{\xi_{2l+1}}^{\xi_{2l+2}} [w_{2l+1}(t)-f(t)]\frac{1}{2}\left(\frac{t-\xi_{2l+1}}{2h}\right)^2 dt + \int\limits_{\xi_{2l+2}}^{\xi_{2l+3}} [w_{2l+2}(t)-f(t)]\left[\frac{1}{2}+\frac{t-\xi_{2l+2}}{2h}-\left(\frac{t-\xi_{2l+2}}{2h}\right)^2\right]dt + \int\limits_{\xi_{2l+3}}^{\xi_{2l+4}} [w_{2l+3}(t)-f(t)]\left[\frac{1}{2}-\frac{t-\xi_{2l+3}}{2h}+\frac{1}{2}\left(\frac{t-\xi_{2l+3}}{2h}\right)^2\right]dt = 0,$$

$$\frac{1}{2}\,\frac{\partial F}{\partial \eta_{r}} = \int\limits_{\xi_{r-2}}^{\xi_{r-1}} [w_{r-2}(t)-f(t)]\frac{1}{2}\left(\frac{t-\xi_{r-2}}{2h}\right)^2 dt + \int\limits_{\xi_{r-1}}^{\xi_{r}} [w_{r-1}(t)-f(t)]\left[\frac{1}{2}+\frac{t-\xi_{r-1}}{2h}-\left(\frac{t-\xi_{r-1}}{2h}\right)^2\right]dt + \int\limits_{\xi_{r}}^{\xi_{r+1}} [w_{r}(t)-f(t)]\left[\frac{1}{2}-\frac{t-\xi_{r}}{2h}+\frac{1}{2}\left(\frac{t-\xi_{r}}{2h}\right)^2\right]dt = 0$$

for $r = 2l+4, \ldots, l+n-1$,

$$\frac{1}{2}\,\frac{\partial F}{\partial \eta_{l+n}} = \int\limits_{\xi_{l+n-2}}^{\xi_{l+n-1}} [w_{l+n-2}(t)-f(t)]\frac{1}{2}\left(\frac{t-\xi_{l+n-2}}{2h}\right)^2 dt + \int\limits_{\xi_{l+n-1}}^{\xi_{l+n}} [w_{l+n-1}(t)-f(t)]\left[\frac{1}{2}+\frac{t-\xi_{l+n-1}}{2h}-\left(\frac{t-\xi_{l+n-1}}{2h}\right)^2\right]dt = 0,$$

$$\frac{1}{2}\,\frac{\partial F}{\partial \eta_{l+n+1}} = \int\limits_{\xi_{l+n-1}}^{\xi_{l+n}} [w_{l+n-1}(t)-f(t)]\frac{1}{2}\left(\frac{t-\xi_{l+n-1}}{2h}\right)^2 dt = 0;$$

hence

$$(7)\qquad 6\eta_0+13\eta_1+\eta_2 = \frac{120}{h}\int\limits_{\xi_0}^{\xi_1} f(t)\left[\frac{1}{2}-\frac{t-\xi_0}{h}+\frac{1}{2}\left(\frac{t-\xi_0}{h}\right)^2\right]dt \equiv P_0,$$

$$13\eta_0+60\eta_1+26\eta_2+\eta_3 = \frac{120}{h}\left\{\int\limits_{\xi_0}^{\xi_1} f(t)\left[\frac{1}{2}+\frac{t-\xi_0}{h}-\left(\frac{t-\xi_0}{h}\right)^2\right]dt + \int\limits_{\xi_1}^{\xi_2} f(t)\left[\frac{1}{2}-\frac{t-\xi_1}{h}+\frac{1}{2}\left(\frac{t-\xi_1}{h}\right)^2\right]dt\right\} \equiv P_1,$$

$$\eta_{r-2}+26\eta_{r-1}+66\eta_r+26\eta_{r+1}+\eta_{r+2} = \frac{120}{h}\left\{\int\limits_{\xi_{r-2}}^{\xi_{r-1}} f(t)\frac{1}{2}\left(\frac{t-\xi_{r-2}}{h}\right)^2 dt + \int\limits_{\xi_{r-1}}^{\xi_{r}} f(t)\left[\frac{1}{2}+\frac{t-\xi_{r-1}}{h}-\left(\frac{t-\xi_{r-1}}{h}\right)^2\right]dt + \int\limits_{\xi_{r}}^{\xi_{r+1}} f(t)\left[\frac{1}{2}-\frac{t-\xi_{r}}{h}+\frac{1}{2}\left(\frac{t-\xi_{r}}{h}\right)^2\right]dt\right\} \equiv P_r$$

for $r = 2, 3, \ldots, 2l-1$,

$$\eta_{2l-2}+26\eta_{2l-1}+78\eta_{2l}-21\eta_{2l+1}-23\eta_{2l+2}-\eta_{2l+3} = \frac{120}{h}\left\{\int\limits_{\xi_{2l-2}}^{\xi_{2l-1}} f(t)\frac{1}{2}\left(\frac{t-\xi_{2l-2}}{h}\right)^2 dt + \int\limits_{\xi_{2l-1}}^{\xi_{2l}} f(t)\left[\frac{1}{2}+\frac{t-\xi_{2l-1}}{h}-\left(\frac{t-\xi_{2l-1}}{h}\right)^2\right]dt + \int\limits_{\xi_{2l}}^{\xi_{2l+1}} f(t)\left[\frac{1}{2}-2\,\frac{t-\xi_{2l}}{2h}+\frac{5}{4}\left(\frac{t-\xi_{2l}}{2h}\right)^2\right]dt + \int\limits_{\xi_{2l+1}}^{\xi_{2l+2}} f(t)\left[-\frac{1}{4}+\frac{1}{2}\,\frac{t-\xi_{2l+1}}{2h}-\frac{1}{4}\left(\frac{t-\xi_{2l+1}}{2h}\right)^2\right]dt\right\} \equiv P_{2l},$$

$$\eta_{2l-1}-21\eta_{2l}+240\eta_{2l+1}+77\eta_{2l+2}+3\eta_{2l+3} = \frac{120}{h}\left\{\int\limits_{\xi_{2l-1}}^{\xi_{2l}} f(t)\frac{1}{2}\left(\frac{t-\xi_{2l-1}}{h}\right)^2 dt + \int\limits_{\xi_{2l}}^{\xi_{2l+1}} f(t)\left[\frac{1}{2}+2\,\frac{t-\xi_{2l}}{2h}-\frac{7}{4}\left(\frac{t-\xi_{2l}}{2h}\right)^2\right]dt + \int\limits_{\xi_{2l+1}}^{\xi_{2l+2}} f(t)\left[\frac{3}{4}-\frac{3}{2}\,\frac{t-\xi_{2l+1}}{2h}+\frac{3}{4}\left(\frac{t-\xi_{2l+1}}{2h}\right)^2\right]dt\right\} \equiv P_{2l+1},$$

$$-23\eta_{2l}+77\eta_{2l+1}+132\eta_{2l+2}+52\eta_{2l+3}+2\eta_{2l+4} = \frac{120}{h}\left\{\int\limits_{\xi_{2l}}^{\xi_{2l+1}} f(t)\frac{1}{2}\left(\frac{t-\xi_{2l}}{2h}\right)^2 dt + \int\limits_{\xi_{2l+1}}^{\xi_{2l+2}} f(t)\left[\frac{1}{2}+\frac{t-\xi_{2l+1}}{2h}-\left(\frac{t-\xi_{2l+1}}{2h}\right)^2\right]dt + \int\limits_{\xi_{2l+2}}^{\xi_{2l+3}} f(t)\left[\frac{1}{2}-\frac{t-\xi_{2l+2}}{2h}+\frac{1}{2}\left(\frac{t-\xi_{2l+2}}{2h}\right)^2\right]dt\right\} \equiv P_{2l+2},$$

$$-\eta_{2l}+3\eta_{2l+1}+52\eta_{2l+2}+132\eta_{2l+3}+52\eta_{2l+4}+2\eta_{2l+5}$$

$$\begin{aligned}=\frac{120}{h}\Big\{\int_{\xi_{2l+1}}^{\xi_{2l+2}} f(t)\,\frac{1}{2}\left(\frac{t-\xi_{2l+1}}{2h}\right)^2 dt &+\int_{\xi_{2l+2}}^{\xi_{2l+3}} f(t)\left[\frac{1}{2}+\frac{t-\xi_{2l+2}}{2h}-\left(\frac{t-\xi_{2l+2}}{2h}\right)^2\right]dt+\\ &+\int_{\xi_{2l+3}}^{\xi_{2l+4}} f(t)\left[\frac{1}{2}-\frac{t-\xi_{2l+3}}{2h}+\frac{1}{2}\left(\frac{t-\xi_{2l+3}}{2h}\right)^2\right]dt\Big\}\equiv P_{2l+3},\end{aligned}$$

$$\eta_{r-2}+26\eta_{r-1}+66\eta_r+26\eta_{r+1}+\eta_{r+2}$$

$$\begin{aligned}=\frac{120}{h}\Big\{\int_{\xi_{r-2}}^{\xi_{r-1}} f(t)\,\frac{1}{2}\left(\frac{t-\xi_{r-2}}{2h}\right)^2 dt &+\int_{\xi_{r-1}}^{\xi_{r}} f(t)\left[\frac{1}{2}+\frac{t-\xi_{r-1}}{2h}-\left(\frac{t-\xi_{r-1}}{2h}\right)^2\right]dt+\\ &+\int_{\xi_{r}}^{\xi_{r+1}} f(t)\left[\frac{1}{2}-\frac{t-\xi_{r}}{2h}+\frac{1}{2}\left(\frac{t-\xi_{r}}{2h}\right)^2\right]dt\Big\}\equiv P_r\end{aligned}$$

$$\text{for } r=2l+4,\ldots,l+n-1,$$

$$\eta_{l+n-2}+26\eta_{l+n-1}+60\eta_{l+n}+13\eta_{l+n+1}$$

$$=\frac{120}{2h}\Big\{\int_{\xi_{l+n-2}}^{\xi_{l+n-1}} f(t)\,\frac{1}{2}\left(\frac{t-\xi_{l+n-2}}{2h}\right)^2 dt+\int_{\xi_{l+n-1}}^{\xi_{l+n}} f(t)\left[\frac{1}{2}+\frac{t-\xi_{l+n-1}}{2h}-\left(\frac{t-\xi_{l+n-1}}{2h}\right)^2\right]dt\Big\}\equiv P_{l+n},$$

$$\eta_{l+n-1}+13\eta_{l+n}+6\eta_{l+n+1}$$

$$=\frac{120}{2h}\int_{\xi_{l+n-1}}^{\xi_{l+n}} f(t)\,\frac{1}{2}\left(\frac{t-\xi_{l+n-1}}{2h}\right)^2 dt\equiv P_{l+n+1}.$$

Applying the Mean-Value Theorem to the right-hand sides of (7), we obtain

(8)
$$\begin{aligned}
P_0 &= 20f(\lambda_0^{(0)}),\\
P_1 &= 20[4f(\lambda_0^{(1)})+f(\lambda_1^{(1)})],\\
P_r &= 20[f(\lambda_{r-2}^{(r)})+4f(\lambda_{r-1}^{(r)})+f(\lambda_r^{(r)})],\quad r=2,3,\ldots,2l-1,\\
P_{2l} &= \frac{120}{h}\Big\{\frac{h}{6}f(\lambda_{2l-2}^{(2l)})+\frac{2h}{3}f(\lambda_{2l-1}^{(2l)})+f(\lambda_{2l}^{(2l)})\int_{\xi_{2l}}^{\xi_{2l+1}}\frac{1}{2}\,dt+\\
&\quad+f(\mu_{2l})\int_{\xi_{2l}}^{\xi_{2l+1}}\left[-2\frac{t-\xi_{2l}}{2h}+\frac{5}{4}\left(\frac{t-\xi_{2l}}{2h}\right)^2\right]dt+\\
&\quad+f(\lambda_{2l+1}^{(2l)})\int_{\xi_{2l+1}}^{\xi_{2l+2}}\left[-\frac{1}{4}+\frac{1}{2}\,\frac{t-\xi_{2l+1}}{2h}-\frac{1}{4}\left(\frac{t-\xi_{2l+1}}{2h}\right)^2\right]dt\Big\}\\
&= 20[f(\lambda_{2l-2}^{(2l)})+4f(\lambda_{2l-1}^{(2l)})+6f(\lambda_{2l}^{(2l)})-7f(\mu_{2l})-f(\lambda_{2l+1}^{(2l)})],\\
P_{2l+1} &= 20[f(\lambda_{2l-1}^{(2l+1)})+11f(\lambda_{2l}^{(2l+1)})+3f(\lambda_{2l+1}^{(2l+1)})],\\
P_{2l+2} &= 20[2f(\lambda_{2l}^{(2l+2)})+8f(\lambda_{2l+1}^{(2l+2)})+2f(\lambda_{2l+2}^{(2l+2)})],\\
P_{2l+3} &= 20[2f(\lambda_{2l+1}^{(2l+3)})+8f(\lambda_{2l+2}^{(2l+3)})+2f(\lambda_{2l+3}^{(2l+3)})],\\
P_r &= 20[f(\lambda_{r-2}^{(r)})+4f(\lambda_{r-1}^{(r)})+f(\lambda_r^{(r)})],\quad r=2l+4,\ldots,l+n-1,\\
P_{l+n} &= 20[f(\lambda_{l+n-2}^{(l+n)})+4f(\lambda_{l+n-1}^{(l+n)})],\\
P_{l+n+1} &= 20f(\lambda_{l+n-1}^{(l+n+1)}),\quad\text{where } \lambda_j^{(i)}\in[\xi_j,\xi_{j+1}],\ \mu_{2l}\in[\xi_{2l},\xi_{2l+1}].
\end{aligned}$$

In particular, we obtain

(9)
$$\begin{aligned}
|P_0| &\leqslant 20\,\|f\|,\\
|P_1| &\leqslant 20\cdot 5\,\|f\|,\\
|P_r| &\leqslant 20\cdot 6\,\|f\|,\quad r=2,3,\ldots,2l-1,\\
|P_{2l}| &\leqslant 20\cdot 19\,\|f\|,\\
|P_{2l+1}| &\leqslant 20\cdot 15\,\|f\|,\\
|P_{2l+2}| &\leqslant 20\cdot 12\,\|f\|,\\
|P_{2l+3}| &\leqslant 20\cdot 12\,\|f\|,\\
|P_r| &\leqslant 20\cdot 6\,\|f\|,\quad r=2l+4,\ldots,l+n-1,\\
|P_{l+n}| &\leqslant 20\cdot 5\,\|f\|,\\
|P_{l+n+1}| &\leqslant 20\,\|f\|.
\end{aligned}$$

By means of the first three equations of (7) we obtain the following linear combination: $P_2-3P_1+12P_0=34\eta_0+2\eta_1+23\eta_3+\eta_4$; hence $34|\eta_0|\leqslant 26a_{n,l}+|P_2|+3|P_1|+12|P_0|$, and by (9),

(10)
$$|\eta_0|\leqslant\frac{13}{17}a_{n,l}+60\cdot\frac{11}{34}\,\|f\|.$$

From the first of (7) we get $13|\eta_1|\leqslant 7a_{n,l}+|P_0|$; thus with respect to (9) we have

(11)
$$|\eta_1|\leqslant\frac{7}{13}a_{n,l}+60\cdot\frac{1}{39}\,\|f\|.$$

If $r=2,3,\ldots,2l-1$, then $66|\eta_r|\leqslant 54a_{n,l}+|P_r|$, and so

(12)
$$|\eta_r|\leqslant\frac{9}{11}a_{n,l}+60\cdot\frac{1}{33}\,\|f\|.$$

From the equation with index $2l$, in system (7), we obtain $78|\eta_{2l}| \leqslant 72a_{n,l}+|P_{2l}|$, and by (9),

$$(13) \qquad |\eta_{2l}| \leqslant \frac{12}{13} a_{n,l}+60\cdot\frac{19}{3\cdot 78} \|f\|.$$

From the next equation we have $240|\eta_{2l+1}| \leqslant 102 a_{n,l}+|P_{2l+1}|$, and so

$$(14) \qquad |\eta_{2l+1}| \leqslant \frac{51}{120} a_{n,l}+60\cdot\frac{1}{48} \|f\|.$$

From the equations with indices $2l+1$, $2l+2$ we obtain the identity

$$3P_{2l+2}-P_{2l+1} = -\eta_{2l-1}-48\eta_{2l}-9\eta_{2l+1}+319\eta_{2l+2}+153\eta_{2l+3}+6\eta_{2l+4},$$

which implies $319|\eta_{2l+2}| \leqslant 217 a_{n,l}+3|P_{2l+2}|+|P_{2l+1}|$, and by (9) we have

$$(15) \qquad |\eta_{2l+2}| \leqslant \frac{217}{319} a_{n,l}+60\cdot\frac{17}{319} \|f\|.$$

From the equation with the index $2l+3$ we obtain $132|\eta_{2l+3}| \leqslant 110 a_{n,l}+ +|P_{2l+3}|$, and so

$$(16) \qquad |\eta_{2l+3}| \leqslant \frac{5}{6} a_{n,l}+60\cdot\frac{1}{33} \|f\|,$$

and from the next equations we have $66|\eta_r| \leqslant 54 a_{n,l}+|P_r|$, and so

$$(17) \qquad |\eta_r| \leqslant \frac{9}{11} a_{n,l}+60\cdot\frac{1}{33} \|f\| \quad \text{for } r = 2l+4, \ldots, l+n-1.$$

From the last equation of (7) we obtain $13|\eta_{l+n}| \leqslant 7 a_{n,l}+|P_{l+n+1}|$, and so

$$(18) \qquad |\eta_{l+n}| \leqslant \frac{7}{13} a_{n,l}+60\cdot\frac{1}{39} \|f\|.$$

From the identity $P_{l+n-1}-3P_{l+n}+12P_{l+n+1} = \eta_{l+n-3}+23\eta_{l+n-2}+ +2\eta_{l+n}+34\eta_{l+n+1}$, which can be obtained from the last three equations of (7), we have

$$34|\eta_{l+n+1}| \leqslant 26 a_{n,l}+|P_{l+n-1}|+3|P_{l+n}|+12|P_{l+n+1}|;$$

hence

$$(19) \qquad |\eta_{l+n+1}| \leqslant \frac{13}{17} a_{n,l}+60\cdot\frac{11}{34} \|f\|.$$

From inequalities (10)-(19) we infer that

$$|\eta_r| \leqslant \frac{12}{13} a_{n,l}+60\cdot\frac{11}{34} \|f\| \quad \text{for } r = 0, 1, \ldots, l+n+1;$$

hence $a_{n,l} \leqslant \frac{12}{13} a_{n,l}+60\cdot\frac{11}{34} \|f\|$, and so

$$(20) \qquad a_{n,l} \leqslant 60\cdot\frac{11\cdot 13}{34} \|f\|.$$

From the first three equations of (7) we obtain $14P_0-4P_1+P_2 = 33(\eta_0-\eta_1)+(\eta_1-\eta_2)-23(\eta_2-\eta_3)-(\eta_3-\eta_4)$; hence

$$(21) \qquad 33\frac{|\eta_1-\eta_0|}{h} \leqslant 25 b_{n,l}+\frac{1}{h}|14P_0-4P_1+P_2|.$$

Moreover, $P_0+P_1-P_2 = 18(\eta_0-\eta_1)+65(\eta_1-\eta_2)+26(\eta_2-\eta_3)+ +(\eta_3-\eta_4)$, and so

$$(22) \qquad 65\frac{|\eta_2-\eta_1|}{h} \leqslant 45 b_{n,l}+\frac{1}{h}|P_0+P_1-P_2|.$$

For $r = 3, 4, \ldots, 2l-1$ we have $P_r-P_{r-1} = (\eta_{r-2}-\eta_{r-3})+26(\eta_{r-1}- -\eta_{r-2})+66(\eta_r-\eta_{r-1})+26(\eta_{r+1}-\eta_r)+(\eta_{r+2}-\eta_{r+1})$ and so

$$(23) \qquad 66\frac{|\eta_r-\eta_{r-1}|}{h} \leqslant 54 b_{n,l}+\frac{1}{h}|P_r-P_{r-1}|.$$

Let $\eta_r = \eta_{r-1}+x_r$ for $r = 2l-2, \ldots, 2l+6$. The equations with indices $2l-1, \ldots, 2l+4$ assume the following form:

$$120\eta_{2l-3}+119x_{2l-2}+93x_{2l-1}+27x_{2l}+x_{2l+1} = P_{2l-1},$$

$$60\eta_{2l-3}+60x_{2l-2}+59x_{2l-1}+33x_{2l}-45x_{2l+1}-24x_{2l+2}-x_{2l+3} = P_{2l},$$

$$300(\eta_{2l-3}+x_{2l-2}+x_{2l-1})+299x_{2l}+320x_{2l+1}+80x_{2l+2}+3x_{2l+3} = P_{2l+1},$$

$$240(\eta_{2l-3}+x_{2l-2}+x_{2l-1}+x_{2l})+263x_{2l+1}+ \\ +186x_{2l+2}+54x_{2l+3}+2x_{2l+4} = P_{2l+2},$$

$$240(\eta_{2l-3}+x_{2l-2}+x_{2l-1}+x_{2l})+241x_{2l+1}+ \\ +238x_{2l+2}+186x_{2l+3}+54x_{2l+4}+2x_{2l+5} = P_{2l+3},$$

$$120(\eta_{2l-3}+x_{2l-2}+x_{2l-1}+x_{2l}+x_{2l+1}+x_{2l+2}) \\ +119x_{2l+3}+93x_{2l+4}+27x_{2l+5}+x_{2l+6} = P_{2l+4};$$

hence

$$-x_{2l-2}-25x_{2l-1}-39x_{2l}+91x_{2l+1}+48x_{2l+2}+2x_{2l+3}=P_{2l-1}-2P_{2l},$$
$$-5x_{2l-1}-134x_{2l}-545x_{2l+1}-200x_{2l+2}-8x_{2l+3}=5P_{2l}-P_{2l+1},$$
$$(24)\qquad -2x_{2l}-\frac{35}{2}x_{2l+1}-305x_{2l+2}-129x_{2l+3}-5x_{2l+4}=2P_{2l+1}-\frac{5}{2}P_{2l+2},$$
$$22x_{2l+1}-52x_{2l+2}-132x_{2l+3}-52x_{2l+4}-2x_{2l+5}=P_{2l+2}-P_{2l+3},$$
$$x_{2l+1}-2x_{2l+2}-52x_{2l+3}-132x_{2l+4}-52x_{2l+5}-2x_{2l+6}=P_{2l+3}-2P_{2l+4}.$$

Adding the second equation of (24) to the first one multiplied by 6, we obtain

$$6(P_{2l-1}-2P_{2l})+5P_{2l}-P_{2l+1}$$
$$=-6x_{2l-2}-155x_{2l-1}-368x_{2l}+x_{2l+1}+88x_{2l+2}+4x_{2l+3};$$

hence

$$(25)\qquad 368\frac{|x_{2l}|}{h}\leqslant 254b_{n,l}+\frac{1}{h}\left(6|P_{2l-1}-2P_{2l}|+|5P_{2l}-P_{2l+1}|\right).$$

From the second equation of (24) we obtain

$$(26)\qquad 545\frac{|x_{2l+1}|}{h}\leqslant 347b_{n,l}+\frac{1}{h}|5P_{2l}-P_{2l+1}|$$

and from the third one,

$$(27)\qquad 305\frac{|x_{2l+2}|}{h}\leqslant 153{,}5b_{n,l}+\frac{1}{h}\left|2P_{2l+1}-\frac{5}{2}P_{2l+2}\right|.$$

Now, we multiply the fourth equation of (24) by -6 and add to the third one

$$-2x_{2l}-149{,}5x_{2l+1}+7x_{2l+2}+663x_{2l+3}+307x_{2l+4}+12x_{2l+5}$$
$$=2P_{2l+1}-\frac{5}{2}P_{2l+2}-6(P_{2l+2}-P_{2l+3});$$

hence

$$(28)\qquad 663\frac{|x_{2l+3}|}{h}\leqslant 477{,}5b_{n,l}+\frac{1}{h}\left(\left|2P_{2l+1}-\frac{5}{2}P_{2l+2}\right|+6|P_{2l+2}-P_{2l+3}|\right).$$

By means of the last two equations of (24) we obtain the identity

$$39x_{2l+1}-94x_{2l+2}-4x_{2l+3}+556x_{2l+4}+256x_{2l+5}+10x_{2l+6}$$
$$=2(P_{2l+2}-P_{2l+3})-5(P_{2l+3}-2P_{2l+4}),$$

from which we infer that

$$(29)\qquad 556\frac{|x_{2l+4}|}{h}\leqslant 403b_{n,l}+\frac{1}{h}\left(2|P_{2l+2}-P_{2l+3}|+5|P_{2l+3}-2P_{2l+4}|\right).$$

For $r=2l+5,\ldots,l+n-1$ we have from (7)

$$(30)\qquad 66\frac{|\eta_r-\eta_{r-1}|}{h}\leqslant 54b_{n,l}+\frac{1}{h}|P_r-P_{r-1}|.$$

From the last three equations of (7) we obtain

$$P_{l+n+1}+P_{l+n}-P_{l+n-1}=(\eta_{l+n-2}-\eta_{l+n-3})+$$
$$+26(\eta_{l+n-1}-\eta_{l+n-2})+65(\eta_{l+n}-\eta_{l+n-1})+18(\eta_{l+n+1}-\eta_{l+n});$$

hence

$$(31)\qquad 65\frac{|\eta_{l+n}-\eta_{l+n-1}|}{h}\leqslant 45b_{n,l}+\frac{1}{h}|P_{l+n+1}+P_{l+n}-P_{l+n-1}|.$$

Analogously, we have

$$14P_{l+n+1}-4P_{l+n}+P_{l+n-1}$$
$$=33(\eta_{l+n+1}-\eta_{l+n})+(\eta_{l+n}-\eta_{l+n-1})-23(\eta_{l+n-1}-\eta_{l+n-2})-$$
$$-(\eta_{l+n-2}-\eta_{l+n-3});$$

hence

$$(32)\qquad 33\frac{|\eta_{l+n+1}-\eta_{l+n}|}{h}\leqslant 25b_{n,l}+\frac{1}{h}|14P_{l+n+1}-4P_{l+n}+P_{l+n-1}|.$$

Applying formulas (8) and the Mean-Value Theorem, we obtain

$$14P_0-4P_1+P_2=20\{14[f(\lambda_0^{(0)})-f(\lambda_0^{(1)})]-[f(\lambda_0^{(1)})-f(\lambda_0^{(2)})]-$$
$$-[f(\lambda_0^{(1)})-f(\lambda_2^{(2)})]-4[f(\lambda_1^{(1)})-f(\lambda_1^{(2)})]\}$$
$$=20[14f'(\theta_1)(\lambda_0^{(0)}-\lambda_0^{(1)})-f'(\theta_2)(\lambda_0^{(1)}-\lambda_0^{(2)})-$$
$$-f'(\theta_3)(\lambda_0^{(1)}-\lambda_2^{(2)})-4f'(\theta_4)(\lambda_1^{(1)}-\lambda_1^{(2)})],$$

where θ_i denote the intermediate points. Since $\lambda_j^{(i)}\in[\xi_j,\xi_{j+1}]$, we have

$$(33)\qquad \frac{1}{h}|14P_0-4P_1+P_2|\leqslant\frac{20}{h}\|f'\|(14h+h+3h+4h)=20\cdot 22\|f'\|.$$

Analogously, we have

$$P_0+P_1-P_2=20\{[f(\lambda_0^{(0)})-f(\lambda_0^{(2)})]+[f(\lambda_1^{(1)})-f(\lambda_2^{(2)})]+4[f(\lambda_0^{(1)})-f(\lambda_1^{(2)})]\};$$

hence

$$(34)\qquad \frac{1}{h}|P_0+P_1-P_2| \leqslant \frac{20}{h}\|f'\|(h+2h+4\cdot 2h) = 20\cdot 11\|f'\|.$$

For $r = 3, 4, \ldots, 2l-1$ we have

$$P_r-P_{r-1} = 20\{[f(\lambda_{r-2}^{(r)})-f(\lambda_{r-3}^{(r-1)})]+ \\ +[f(\lambda_r^{(r)})-f(\lambda_{r-1}^{(r-1)})]+4[f(\lambda_{r-1}^{(r)})-f(\lambda_{r-2}^{(r-1)})]\};$$

hence

$$(35)\qquad \frac{1}{h}|P_r-P_{r-1}| \leqslant \frac{20}{h}\|f'\|(2h+2h+4\cdot 2h) = 20\cdot 12\|f'\|.$$

Analogously, we have

$$P_{2l-1}-2P_{2l} = 20\{[f(\lambda_{2l-3}^{(2l-1)})-f(\lambda_{2l-2}^{(2l)})]+[f(\lambda_{2l-1}^{(2l-1)})-f(\lambda_{2l-2}^{(2l)})]+ \\ +4[f(\lambda_{2l-2}^{(2l-1)})-f(\lambda_{2l-1}^{(2l)})]+4[f(\mu_{2l})-f(\lambda_{2l-1}^{(2l)})]+ \\ +10[f(\mu_{2l})-f(\lambda_{2l}^{(2l)})]+2[f(\lambda_{2l+1}^{(2l)})-f(\lambda_{2l}^{(2l)})]\};$$

hence

$$(36)\qquad \frac{1}{h}|P_{\ l-1}-2P_{2l}| \\ \leqslant \frac{20}{h}\|f'\|(2h+2h+4\cdot 2h+4\cdot 3h+10\cdot 2h+2\cdot 4h) = 20\cdot 52\|f'\|,$$

$$5P_{2l}-P_{2l+1} = 20\{5[f(\lambda_{2l-2}^{(2l)})-f(\mu_{2l})]+20[f(\lambda_{2l-1}^{(2l)})-f(\mu_{2l})]+ \\ +10[f(\lambda_{2l}^{(2l)})-f(\mu_{2l})]+5[f(\lambda_{2l}^{(2l)})-f(\lambda_{2l+1}^{(2l)})]+[f(\lambda_{2l}^{(2l)})-f(\lambda_{2l-1}^{(2l+1)})]+ \\ +11[f(\lambda_{2l}^{(2l)})-f(\lambda_{2l}^{(2l+1)})]+3[f(\lambda_{2l}^{(2l)})-f(\lambda_{2l+1}^{(2l+1)})]\};$$

hence

$$(37)\qquad \frac{1}{h}|5P_{2l}-P_{2l+1}| \\ \leqslant \frac{20}{h}\|f'\|(5\cdot 4h+20\cdot 3h+10\cdot 2h+5\cdot 4h+ \\ +3h+11\cdot 2h+3\cdot 4h) = 20\cdot 157\|f'\|,$$

$$2P_{2l+1}-\frac{5}{2}P_{2l+2} = 20\{2[f(\lambda_{2l-1}^{(2l+1)})-f(\lambda_{2l}^{(2l+2)})]+ \\ +3[f(\lambda_{2l}^{(2l+1)})-f(\lambda_{2l}^{(2l+2)})]+19[f(\lambda_{2l}^{(2l+1)})-f(\lambda_{2l+1}^{(2l+2)})]+ \\ +[f(\lambda_{2l+1}^{(2l+1)})-f(\lambda_{2l+1}^{(2l+2)})]+5[f(\lambda_{2l+1}^{(2l+1)})-f(\lambda_{2l+2}^{(2l+2)})]\};$$

hence

$$(38)\qquad \frac{1}{h}\left|2P_{2l+1}-\frac{5}{2}P_{2l+2}\right| \leqslant \frac{20}{h}\|f'\|(2\cdot 3h+3\cdot 2h+19\cdot 4h+2h+5\cdot 4h) \\ = 20\cdot 110\|f'\|,$$

$$P_{2l+2}-P_{2l+3} = 40\{[f(\lambda_{2l}^{(2l+2)})-f(\lambda_{2l+1}^{(2l+3)})]+ \\ +4[f(\lambda_{2l+1}^{(2l+2)})-f(\lambda_{2l+2}^{(2l+3)})]+[f(\lambda_{2l+2}^{(2l+2)})-f(\lambda_{2l+3}^{(2l+3)})]\};$$

hence

$$(39)\qquad \frac{1}{h}|P_{2l+2}-P_{2l+3}| \leqslant \frac{40}{h}\|f'\|(4h+4\cdot 4h+4h) = 20\cdot 48\|f'\|,$$

$$P_{2l+3}-2P_{2l+4} = 40\{[f(\lambda_{2l+1}^{(2l+3)})-f(\lambda_{2l+2}^{(2l+4)})]+ \\ +4[f(\lambda_{2l+2}^{(2l+3)})-f(\lambda_{2l+3}^{(2l+4)})]+[f(\lambda_{2l+3}^{(2l+3)})-f(\lambda_{2l+4}^{(2l+4)})]\};$$

hence

$$(40)\qquad \frac{1}{h}|P_{2l+3}-2P_{2l+4}| \leqslant \frac{40}{h}\|f'\|(4h+4\cdot 4h+4h) = 20\cdot 48\|f'\|.$$

For $r = 2l+5, \ldots, l+n-1$ we have

$$P_r-P_{r-1} = 20\{[f(\lambda_{r-2}^{(r)})-f(\lambda_{r-3}^{(r-1)})]+ \\ +4[f(\lambda_{r-1}^{(r)})-f(\lambda_{r-2}^{(r-1)})]+[f(\lambda_r^{(r)})-f(\lambda_{r-1}^{(r-1)})]\};$$

hence

$$(41)\qquad \frac{1}{h}|P_r-P_{r-1}| \leqslant \frac{20}{h}\|f'\|(4h+4\cdot 4h+4h) = 20\cdot 24\|f'\|$$

and

$$P_{l+n+1}+P_{l+n}-P_{l+n-1} = 20\{[f(\lambda_{l+n-1}^{(l+n+1)})-f(\lambda_{l+n-1}^{(l+n-1)})]+ \\ +4[f(\lambda_{l+n-1}^{(l+n)})-f(\lambda_{l+n-2}^{(l+n-1)})]+ \\ +[f(\lambda_{l+n-2}^{(l+n)})-f(\lambda_{l+n-3}^{(l+n-1)})]\},$$

hence

$$(42)\qquad \frac{1}{h}|P_{l+n+1}+P_{l+n}-P_{l+n-1}| \leqslant \frac{20}{h}\|f'\|(2h+4\cdot 4h+4h) = 20\cdot 22\|f'\|,$$

$$14P_{l+n+1}-4P_{l+n}+P_{l+n-1} \\ = 20\{14[f(\lambda_{l+n-1}^{(l+n+1)})-f(\lambda_{l+n-1}^{(l+n)})]-4[f(\lambda_{l+n-2}^{(l+n)})-f(\lambda_{l+n-2}^{(l+n-1)})]- \\ -[f(\lambda_{l+n-1}^{(l+n)})-f(\lambda_{l+n-3}^{(l+n-1)})]-[f(\lambda_{l+n-1}^{(l+n)})-f(\lambda_{l+n-1}^{(l+n-1)})]\};$$

hence

$$(43)\qquad \frac{1}{h}|14P_{l+n+1}-4P_{l+n}+P_{l+n-1}| \leqslant \frac{20}{h}\|f'\|(14\cdot 2h+4\cdot 2h+6h+2h) = 20\cdot 44\|f'\|.$$

With respect to the inequalities (33)-(43) the estimations (21), (22), (23) and (25)-(32) assume the following form:

$$\frac{|\eta_1-\eta_0|}{h} \leqslant \frac{25}{33}b_{n,l}+20\cdot\frac{2}{3}\|f'\|,$$

$$\frac{|\eta_2-\eta_1|}{h} \leqslant \frac{9}{13}b_{n,l}+20\cdot\frac{11}{65}\|f'\|,$$

$$\frac{|\eta_r-\eta_{r-1}|}{h} \leqslant \frac{9}{11}b_{n,l}+20\cdot\frac{2}{11}\|f'\| \quad \text{for } r = 3, 4, \ldots, 2l-1,$$

$$\frac{|\eta_{2l}-\eta_{2l-1}|}{h} \leqslant \frac{127}{184}b_{n,l}+20\cdot\frac{469}{368}\|f'\|,$$

$$\frac{|\eta_{2l+1}-\eta_{2l}|}{h} \leqslant \frac{347}{545}b_{n,l}+20\cdot\frac{157}{545}\|f'\|,$$

$$\frac{|\eta_{2l+2}-\eta_{2l+1}|}{h} \leqslant \frac{153{,}5}{305}b_{n,l}+20\cdot\frac{110}{305}\|f'\|,$$

$$\frac{|\eta_{2l+3}-\eta_{2l+2}|}{h} \leqslant \frac{477{,}5}{663}b_{n,l}+20\cdot\frac{398}{663}\|f'\|,$$

$$\frac{|\eta_{2l+4}-\eta_{2l+3}|}{h} \leqslant \frac{403}{556}b_{n,l}+20\cdot\frac{84}{139}\|f'\|,$$

$$\frac{|\eta_r-\eta_{r-1}|}{h} \leqslant \frac{9}{11}b_{n,l}+20\cdot\frac{4}{11}\|f'\| \quad \text{for } r = 2l+5, \ldots, l+n-1,$$

$$\frac{|\eta_{l+n}-\eta_{l+n-1}|}{h} \leqslant \frac{9}{13}b_{n,l}+20\cdot\frac{22}{65}\|f'\|,$$

$$\frac{|\eta_{l+n+1}-\eta_{l+n}|}{h} \leqslant \frac{25}{33}b_{n,l}+20\cdot\frac{4}{3}\|f'\|.$$

From these inequalities it follows that

$$\frac{|\eta_r-\eta_{r-1}|}{h} \leqslant \frac{9}{11}b_{n,l}+20\cdot\frac{4}{3}\|f'\| \quad \text{for } r = 1, \ldots, l+n+1,$$

or

$$b_{n,l} \leqslant \frac{9}{11}b_{n,l}+20\cdot\frac{4}{3}\|f'\|;$$

hence

$$(44)\qquad b_{n,l} \leqslant \frac{440}{3}\|f'\|.$$

2° Let $l = n-4$; then equations (6) assume the following form:

$$(45)\qquad 6\eta_0+13\eta_1+\eta_2 = \frac{120}{h}\int_{\xi_0}^{\xi_1} f(t)\left[\frac{1}{2}-\frac{t-\xi_0}{h}+\frac{1}{2}\left(\frac{t-\xi_0}{h}\right)^2\right]dt \equiv P_0,$$

$$13\eta_0+60\eta_1+26\eta_2+\eta_3$$
$$= \frac{120}{h}\left\{\int_{\xi_0}^{\xi_1} f(t)\left[\frac{1}{2}+\frac{t-\xi_0}{h}-\left(\frac{t-\xi_0}{h}\right)^2\right]dt + \right.$$
$$\left. + \int_{\xi_1}^{\xi_2} f(t)\left[\frac{1}{2}-\frac{t-\xi_1}{h}+\frac{1}{2}\left(\frac{t-\xi_1}{h}\right)^2\right]dt\right\} \equiv P_1,$$

$$\eta_{r-2}+26\eta_{r-1}+66\eta_r+26\eta_{r+1}+\eta_{r+2}$$
$$= \frac{120}{h}\left\{\int_{\xi_{r-2}}^{\xi_{r-1}} f(t)\frac{1}{2}\left(\frac{t-\xi_{r-2}}{h}\right)^2 dt + \right.$$
$$+ \int_{\xi_{r-1}}^{\xi_r} f(t)\left[\frac{1}{2}+\frac{t-\xi_{r-1}}{h}-\left(\frac{t-\xi_{r-1}}{h}\right)^2\right]dt +$$
$$\left. + \int_{\xi_r}^{\xi_{r+1}} f(t)\left[\frac{1}{2}-\frac{t-\xi_r}{h}+\frac{1}{2}\left(\frac{t-\xi_r}{h}\right)^2\right]dt\right\} \equiv P_r,$$
$$r = 2, 3, \ldots, 2n-9,$$

$$\eta_{2n-10}+26\eta_{2n-9}+78\eta_{2n-8}-21\eta_{2n-7}-23\eta_{2n-6}-\eta_{2n-5}$$
$$= \frac{120}{h}\left\{\int_{\xi_{2n-10}}^{\xi_{2n-9}} \frac{1}{2}f(t)\left(\frac{t-\xi_{2n-10}}{h}\right)^2 dt + \right.$$
$$+ \int_{\xi_{2n-9}}^{\xi_{2n-8}} f(t)\left[\frac{1}{2}+\frac{t-\xi_{2n-9}}{h}-\left(\frac{t-\xi_{2n-9}}{h}\right)^2\right]dt +$$
$$+ \int_{\xi_{2n-8}}^{\xi_{2n-7}} f(t)\left[\frac{1}{2}-2\frac{t-\xi_{2n-8}}{2h}+\frac{5}{4}\left(\frac{t-\xi_{2n-8}}{2h}\right)^2\right]dt +$$
$$\left. + \int_{\xi_{2n-7}}^{\xi_{2n-6}} f(t)\left[-\frac{1}{4}+\frac{1}{2}\frac{t-\xi_{2n-7}}{2h}-\frac{1}{4}\left(\frac{t-\xi_{2n-7}}{2h}\right)^2\right]dt\right\} \equiv P_{2n-8},$$

$$\eta_{2n-9}-21\eta_{2n-8}+240\eta_{2n-7}+77\eta_{2n-6}+3\eta_{2n-5}$$

$$=\frac{120}{h}\Bigg\{\int_{\xi_{2n-9}}^{\xi_{2n-8}} f(t)\,\frac{1}{2}\left(\frac{t-\xi_{2n-9}}{h}\right)^2 dt+\int_{\xi_{2n-8}}^{\xi_{2n-7}} f(t)\left[\frac{1}{2}+2\,\frac{t-\xi_{2n-8}}{2h}-\frac{7}{4}\left(\frac{t-\xi_{2n-8}}{2h}\right)^2\right]dt+\int_{\xi_{2n-7}}^{\xi_{2n-6}} f(t)\left[\frac{3}{4}-\frac{3}{2}\,\frac{t-\xi_{2n-7}}{2h}+\frac{3}{4}\left(\frac{t-\xi_{2n-7}}{2h}\right)^2\right]dt\Bigg\}\equiv P_{2n-7},$$

$$-23\eta_{2n-8}+77\eta_{2n-7}+132\eta_{2n-6}+52\eta_{2n-5}+2\eta_{2n-4}$$

$$=\frac{120}{h}\Bigg\{\int_{\xi_{2n-8}}^{\xi_{2n-7}} f(t)\,\frac{1}{2}\left(\frac{t-\xi_{2n-8}}{2h}\right)^2 dt+\int_{\xi_{2n-7}}^{\xi_{2n-6}} f(t)\left[\frac{1}{2}+\frac{t-\xi_{2n-7}}{2h}-\left(\frac{t-\xi_{2n-7}}{2h}\right)^2\right]dt+\int_{\xi_{2n-6}}^{\xi_{2n-5}} f(t)\left[\frac{1}{2}-\frac{t-\xi_{2n-6}}{2h}+\frac{1}{2}\left(\frac{t-\xi_{2n-6}}{2h}\right)^2\right]dt\Bigg\}\equiv P_{2n-6},$$

$$-\eta_{2n-8}+3\eta_{2n-7}+52\eta_{2n-6}+132\eta_{2n-5}+52\eta_{2n-4}+2\eta_{2n-3}$$

$$=\frac{120}{h}\Bigg\{\int_{\xi_{2n-7}}^{\xi_{2n-6}} f(t)\,\frac{1}{2}\left(\frac{t-\xi_{2n-7}}{2h}\right)^2 dt+\int_{\xi_{2n-6}}^{\xi_{2n-5}} f(t)\left[\frac{1}{2}+\frac{t-\xi_{2n-6}}{2h}-\left(\frac{t-\xi_{2n-6}}{2h}\right)^2\right]dt+\int_{\xi_{2n-5}}^{\xi_{2n-4}} f(t)\left[\frac{1}{2}-\frac{t-\xi_{2n-5}}{2h}+\frac{1}{2}\left(\frac{t-\xi_{2n-5}}{2h}\right)^2\right]dt\Bigg\}\equiv P_{2n-5},$$

$$\eta_{2n-6}+26\eta_{2n-5}+60\eta_{2n-4}+13\eta_{2n-3}$$

$$=\frac{120}{2h}\Bigg\{\int_{\xi_{2n-6}}^{\xi_{2n-5}} f(t)\,\frac{1}{2}\left(\frac{t-\xi_{2n-6}}{2h}\right)^2 dt+\int_{\xi_{2n-5}}^{\xi_{2n-4}} f(t)\left[\frac{1}{2}+\frac{t-\xi_{2n-5}}{2h}-\left(\frac{t-\xi_{2n-5}}{2h}\right)^2\right]dt\Bigg\}\equiv P_{2n-4},$$

$$\eta_{2n-5}+13\eta_{2n-4}+6\eta_{2n-3}$$

$$=\frac{120}{2h}\int_{\xi_{2n-5}}^{\xi_{2n-4}} f(t)\,\frac{1}{2}\left(\frac{t-\xi_{2n-5}}{2h}\right)^2 dt\equiv P_{2n-3}.$$

Proceeding as in case 1°, we obtain

$$|\eta_r|\leqslant\frac{12}{13}\,a_{n,n-4}+60\cdot\frac{11}{34}\,\|f\|\quad\text{for } r=0,1,\ldots,2n-4. \tag{46}$$

In order to estimate the parameter η_{2n-3} we form the following linear combination, by means of the last three equations of system (45),

$$24P_{2n-3}-6P_{2n-4}+P_{2n-5}=-\eta_{2n-8}+3\eta_{2n-7}+46\eta_{2n-6}+4\eta_{2n-4}+68\eta_{2n-3},$$

from which it follows

$$68\,|\eta_{2n-3}|\leqslant 54\,a_{n,n-4}+24\,|P_{2n-3}|+6\,|P_{2n-4}|+|P_{2n-5}|$$
$$\leqslant 54a_{n,n-4}+60\,\|f\|\left(24\cdot\frac{1}{3}+6\cdot\frac{5}{3}+4\right).$$

Hence

$$|\eta_{2n-3}|\leqslant\frac{27}{34}\,a_{n,n-4}+60\cdot\frac{11}{34}\,\|f\|. \tag{47}$$

From (46) and (47) we get

$$a_{n,n-4}\leqslant\frac{12}{13}\,a_{n,n-4}+60\cdot\frac{11}{34}\,\|f\|,$$

and so

$$a_{n,n-4}\leqslant 60\cdot\frac{11\cdot 13}{34}\,\|f\|. \tag{48}$$

In the same manner as in 1° we obtain

$$\frac{|\eta_r-\eta_{r-1}|}{h}\leqslant\frac{9}{11}\,b_{n,n-4}+20\cdot\frac{4}{3}\,\|f'\|\quad\text{for } r=1,2,\ldots,2n-5. \tag{49}$$

Let $\eta_r-\eta_{r-1}=x_r$ $(r=2n-7,\ldots,2n-3)$. The last four equations of system (45) assume the following form:

$$240\eta_{2n-8}+263x_{2n-7}+186x_{2n-6}+54x_{2n-5}+2x_{2n-4}=P_{2n-6},$$
$$240\eta_{2n-8}+241x_{2n-7}+238x_{2n-6}+186x_{2n-5}+54x_{2n-4}+2x_{2n-3}=P_{2n-5},$$
$$100\,(\eta_{2n-8}+x_{2n-7}+x_{2n-6})+99x_{2n-5}+73x_{2n-4}+13x_{2n-3}=P_{2n-4},$$
$$20\,(\eta_{2n-8}+x_{2n-7}+x_{2n-6}+x_{2n-5})+19x_{2n-4}+6x_{2n-3}=P_{2n-3};$$

hence

$$\begin{aligned}
22x_{2n-7}-52x_{2n-6}-132x_{2n-5}-52x_{2n-4}-2x_{2n-3} &= P_{2n-6}-P_{2n-5},\\
\tag{50} \frac{5}{2}x_{2n-7}-5x_{2n-6}-129x_{2n-5}-303x_{2n-4}-73x_{2n-3} &= \frac{5}{2}P_{2n-5}-6P_{2n-4},\\
x_{2n-5}+22x_{2n-4}+17x_{2n-3} &= 5P_{2n-3}-P_{2n-4}.
\end{aligned}$$

From the last equation of system (50), we get

$$\tag{51} \frac{|x_{2n-4}|}{h}\leqslant\frac{9}{11}b_{n,n-4}+\frac{1}{22h}|5P_{2n-3}-P_{2n-4}|,$$

and from the linear combination

$$-(P_{2n-6}-P_{2n-5})+\left(\frac{5}{2}P_{2n-5}-6P_{2n-4}\right)+11(5P_{2n-3}-P_{2n-4})$$
$$= -19{,}5x_{2n-7}+47x_{2n-6}+14x_{2n-5}-9x_{2n-4}+116x_{2n-3}$$

we have

$$\tag{52} \frac{|x_{2n-3}|}{h}\leqslant\frac{89{,}5}{116}b_{n,n-4}+$$
$$+\frac{1}{116h}\left(\left|P_{2n-6}-P_{2n-5}\right|+\left|\frac{5}{2}P_{2n-5}-6P_{2n-4}\right|+11\left|5P_{2n-3}-P_{2n-4}\right|\right).$$

Because of

$$\begin{aligned}
P_{2n-6} &= 20\cdot 2[f(\lambda_{2n-8}^{(2n-6)})+4f(\lambda_{2n-7}^{(2n-6)})+f(\lambda_{2n-6}^{(2n-6)})],\\
P_{2n-5} &= 20\cdot 2[f(\lambda_{2n-7}^{(2n-5)})+4f(\lambda_{2n-6}^{(2n-5)})+f(\lambda_{2n-5}^{(2n-5)})],\\
P_{2n-4} &= 20[f(\lambda_{2n-6}^{(2n-4)})+4f(\lambda_{2n-5}^{(2n-4)})],\\
P_{2n-3} &= 20f(\lambda_{2n-5}^{(2n-3)}),\quad \text{where } \lambda_i^{(j)}\in[\xi_i,\xi_{i+1}],
\end{aligned}$$

we have

$$\begin{aligned}
\frac{1}{h}|P_{2n-6}-P_{2n-5}| &= \frac{40}{h}|f(\lambda_{2n-8}^{(2n-6)})-f(\lambda_{2n-7}^{(2n-5)})+\\
&\quad +4[f(\lambda_{2n-7}^{(2n-6)})-f(\lambda_{2n-6}^{(2n-5)})]+f(\lambda_{2n-6}^{(2n-6)})-f(\lambda_{2n-5}^{(2n-5)})|\\
&\leqslant\frac{40}{h}\|f'\|(4h+4\cdot 4h+4h)=20\cdot 48\|f'\|,
\end{aligned}$$

$$\begin{aligned}
\frac{1}{h}\left|\frac{5}{2}P_{2n-6}-6P_{2n-4}\right| &= \frac{20}{h}|5[f(\lambda_{2n-7}^{(2n-5)})-f(\lambda_{2n-6}^{(2n-4)})]+\\
&\quad +f(\lambda_{2n-6}^{(2n-5)})-f(\lambda_{2n-6}^{(2n-4)})+19[f(\lambda_{2n-6}^{(2n-5)})-\\
&\quad -f(\lambda_{2n-5}^{(2n-4)})]+5[f(\lambda_{2n-5}^{(2n-5)})-f(\lambda_{2n-5}^{(2n-4)})]|\\
&\leqslant\frac{20}{h}\|f'\|(5\cdot 4h+2h+19\cdot 4h+5\cdot 2h)=20\cdot 108\|f'\|,
\end{aligned}$$

$$\begin{aligned}
\frac{1}{h}|5P_{2n-3}-P_{2n-4}| &= \frac{20}{h}|f(\lambda_{2n-5}^{(2n-3)})-f(\lambda_{2n-6}^{(2n-4)})+4[f(\lambda_{2n-5}^{(2n-3)})-f(\lambda_{2n-5}^{(2n-4)})]|\\
&\leqslant\frac{20}{h}\|f'\|(4h+4\cdot 2h)=20\cdot 12\|f'\|.
\end{aligned}$$

Hence, from (51) and (52) we obtain

$$\tag{53} \frac{|\eta_{2n-4}-\eta_{2n-5}|}{h}\leqslant\frac{9}{11}b_{n,n-4}+20\cdot\frac{6}{11}\|f'\|,$$

$$\tag{54} \frac{|\eta_{2n-3}-\eta_{2n-4}|}{h}\leqslant\frac{89{,}5}{116}b_{n,n-4}+20\cdot\frac{72}{29}\|f'\|.$$

From inequalities (49), (53) and (54) it follows that

$$\frac{|\eta_r-\eta_{r-1}|}{h}\leqslant\frac{9}{11}b_{n,n-4}+20\cdot\frac{72}{29}\|f'\|\quad\text{for } r=1,\ldots,2n-3;$$

hence

$$\tag{55} b_{n,n-4}\leqslant\frac{10\cdot 11\cdot 72}{29}\|f'\|.$$

3° If $l=n-3$, then system (6) assumes the form

$$\tag{56} 6\eta_0+13\eta_1+\eta_2=\frac{120}{h}\int\limits_{\xi_0}^{\xi_1}f(t)\left[\frac{1}{2}-\frac{t-\xi_0}{h}+\frac{1}{2}\left(\frac{t-\xi_0}{h}\right)^2\right]dt\equiv P_0,$$

$$\begin{aligned}
13\eta_0+60\eta_1+26\eta_2+\eta_3 &\\
= \frac{120}{h}\Bigg\{\int\limits_{\xi_0}^{\xi_1}f(t)\left[\frac{1}{2}+\frac{t-\xi_0}{h}-\left(\frac{t-\xi_0}{h}\right)^2\right]dt &+\\
+\int\limits_{\xi_1}^{\xi_2}f(t)\left[\frac{1}{2}-\frac{t-\xi_1}{h}+\frac{1}{2}\left(\frac{t-\xi_1}{h}\right)^2\right]dt\Bigg\} &\equiv P_1,
\end{aligned}$$

$$\eta_{r-2}+26\eta_{r-1}+66\eta_r+26\eta_{r+1}+\eta_{r+2}$$
$$=\frac{120}{h}\left\{\int_{\xi_{r-2}}^{\xi_{r-1}} f(t)\frac{1}{2}\left(\frac{t-\xi_{r-2}}{h}\right)^2 dt+\int_{\xi_{r-1}}^{\xi_r} f(t)\left[\frac{1}{2}+\frac{t-\xi_{r-1}}{h}-\left(\frac{t-\xi_{r-1}}{h}\right)^2\right]dt+\right.$$
$$\left.+\int_{\xi_r}^{\xi_{r+1}} f(t)\left[\frac{1}{2}-\frac{t-\xi_r}{h}+\frac{1}{2}\left(\frac{t-\xi_r}{h}\right)^2\right]dt\right\}\equiv P_r,\quad r=2,3,\ldots,2n-5,$$

$$\eta_{2n-8}+26\eta_{2n-7}+78\eta_{2n-6}-21\eta_{2n-5}-23\eta_{2n-4}-\eta_{2n-3}$$
$$=\frac{120}{h}\left\{\int_{\xi_{2n-8}}^{\xi_{2n-7}} f(t)\frac{1}{2}\left(\frac{t-\xi_{2n-8}}{h}\right)^2 dt+\int_{\xi_{2n-7}}^{\xi_{2n-6}} f(t)\left[\frac{1}{2}+\frac{t-\xi_{2n-7}}{h}-\left(\frac{t-\xi_{2n-7}}{h}\right)^2\right]dt+\right.$$
$$+\int_{\xi_{2n-6}}^{\xi_{2n-5}} f(t)\left[\frac{1}{2}-2\,\frac{t-\xi_{2n-6}}{2h}+\frac{5}{4}\left(\frac{t-\xi_{2n-6}}{2h}\right)^2\right]dt+$$
$$\left.+\int_{\xi_{2n-5}}^{\xi_{2n-4}} f(t)\left[-\frac{1}{4}+\frac{1}{2}\,\frac{t-\xi_{2n-5}}{2h}-\frac{1}{4}\left(\frac{t-\xi_{2n-5}}{2h}\right)^2\right]dt\right\}\equiv P_{2n-6},$$

$$\eta_{2n-7}-21\eta_{2n-6}+240\eta_{2n-5}+77\eta_{2n-4}+3\eta_{2n-3}$$
$$=\frac{120}{h}\left\{\int_{\xi_{2n-7}}^{\xi_{2n-6}} f(t)\frac{1}{2}\left(\frac{t-\xi_{2n-7}}{h}\right)^2 dt+\right.$$
$$+\int_{\xi_{2n-6}}^{\xi_{2n-5}} f(t)\left[\frac{1}{2}+2\,\frac{t-\xi_{2n-6}}{2h}-\frac{7}{4}\left(\frac{t-\xi_{2n-6}}{2h}\right)^2\right]dt+$$
$$\left.+\int_{\xi_{2n-5}}^{\xi_{2n-4}} f(t)\left[\frac{3}{4}-\frac{3}{2}\,\frac{t-\xi_{2n-5}}{2h}+\frac{3}{4}\left(\frac{t-\xi_{2n-5}}{2h}\right)^2\right]dt\right\}\equiv P_{2n-5},$$

$$-23\eta_{2n-6}+77\eta_{2n-5}+132\eta_{2n-4}+52\eta_{2n-3}+2\eta_{2n-2}$$
$$=\frac{120}{h}\left\{\int_{\xi_{2n-6}}^{\xi_{2n-5}} f(t)\frac{1}{2}\left(\frac{t-\xi_{2n-6}}{2h}\right)^2 dt+\int_{\xi_{2n-5}}^{\xi_{2n-4}} f(t)\left[\frac{1}{2}+\frac{t-\xi_{2n-5}}{2h}-\left(\frac{t-\xi_{2n-5}}{2h}\right)^2\right]dt\right.$$
$$\left.+\int_{\xi_{2n-4}}^{\xi_{2n-3}} f(t)\left[\frac{1}{2}-\frac{t-\xi_{2n-4}}{2h}+\frac{1}{2}\left(\frac{t-\xi_{2n-4}}{2h}\right)^2\right]dt\right\}\equiv P_{2n-4},$$

$$-\eta_{2n-6}+3\eta_{2n-5}+52\eta_{2n-4}+120\eta_{2n-3}+26\eta_{2n-2}$$
$$=\frac{120}{h}\left\{\int_{\xi_{2n-5}}^{\xi_{2n-4}} f(t)\frac{1}{2}\left(\frac{t-\xi_{2n-5}}{2h}\right)^2 dt+\right.$$
$$\left.+\int_{\xi_{2n-4}}^{\xi_{2n-3}} f(t)\left[\frac{1}{2}+\frac{t-\xi_{2n-4}}{2h}-\left(\frac{t-\xi_{2n-4}}{2h}\right)^2\right]dt\right\}\equiv P_{2n-3},$$

$$\eta_{2n-4}+13\eta_{2n-3}+6\eta_{2n-2}=\frac{120}{2h}\int_{\xi_{2n-4}}^{\xi_{2n-3}} f(t)\frac{1}{2}\left(\frac{t-\xi_{2n-4}}{2h}\right)^2 dt\equiv P_{2n-2}.$$

As in 1°, we find

$$(57)\qquad |\eta_r|\leqslant\frac{12}{13}a_{n,n-3}+60\cdot\frac{11}{34}\|f\|\quad\text{for } r=0,1,\ldots,2n-3.$$

By means of the last four equations of system (56) we obtain the combination

$$P_{2n-5}-3P_{2n-4}+9P_{2n-3}-71P_{2n-2}$$
$$=\eta_{2n-7}+39\eta_{2n-6}+36\eta_{2n-5}+78\eta_{2n-4}+4\eta_{2n-3}-198\eta_{2n-2},$$

which implies

$$198\,|\eta_{2n-2}|\leqslant 158a_{n,n-3}+|P_{2n-5}|+3\,|P_{2n-4}|+9\,|P_{2n-3}|+71\,|P_{2n-2}|.$$

But

$$|P_{2n-5}|\leqslant 60\cdot 5\,\|f\|,\qquad |P_{2n-4}|\leqslant 60\cdot 4\,\|f\|,$$
$$|P_{2n-3}|\leqslant 60\cdot\frac{10}{3}\,\|f\|,\qquad |P_{2n-2}|\leqslant 60\cdot\frac{1}{3}\,\|f\|;$$

hence

$$(58)\qquad |\eta_{2n-2}|\leqslant\frac{79}{99}a_{n,n-3}+60\cdot\frac{106}{3\cdot 99}\|f\|.$$

From (57) and (58) it follows that

$$|\eta_r|\leqslant\frac{12}{13}a_{n,n-3}+60\cdot\frac{106}{3\cdot 99}\|f\|\quad\text{for } r=0,1,\ldots,2n-2;$$

hence

$$(59)\qquad a_{n,n-3}\leqslant 60\cdot\frac{13\cdot 106}{3\cdot 99}\|f\|.$$

Analogously to 1°, we obtain

$$(60)\qquad \frac{|\eta_r-\eta_{r-1}|}{h}\leqslant\frac{9}{11}b_{n,n-3}+20\cdot\frac{4}{3}\|f'\|\quad\text{for } r=1,\ldots,2n-4.$$

Let $\eta_r-\eta_{r-1}=x_r$ $(r=2n-6,\ldots,2n-2)$; then the last four equations of system (56) assume the following form:

$$300\eta_{2n-7}+299x_{2n-6}+320x_{2n-5}+80x_{2n-4}+3x_{2n-3}=P_{2n-5},$$
$$240\eta_{2n-7}+240x_{2n-6}+263x_{2n-5}+186x_{2n-4}+54x_{2n-3}+2x_{2n-2}=P_{2n-4},$$
$$200\eta_{2n-7}+200x_{2n-6}+201x_{2n-5}+198x_{2n-4}+146x_{2n-3}+26x_{2n-2}=P_{2n-3},$$
$$20\eta_{2n-7}+20x_{2n-6}+20x_{2n-5}+20x_{2n-4}+19x_{2n-3}+6x_{2n-2}=P_{2n-2};$$

hence

$$2x_{2n-6}+17{,}5x_{2n-5}+305x_{2n-4}+129x_{2n-3}+5x_{2n-2} = \frac{5}{2}P_{2n-4}-2P_{2n-5}, \tag{61}$$

$$54{,}5x_{2n-5}-129x_{2n-4}-303x_{2n-3}-73x_{2n-2} = \frac{5}{2}P_{2n-4}-3P_{2n-3},$$

$$-x_{2n-5}+2x_{2n-4}+44x_{2n-3}+34x_{2n-2} = 10P_{2n-2}-P_{2n-3}.$$

Now, from the linear combination

$$\frac{5}{2}P_{2n-4}-2P_{2n-5}+2\left(\frac{5}{2}P_{2n-4}-3P_{2n-3}\right)$$
$$= 2x_{2n-6}+126{,}5x_{2n-5}+47x_{2n-4}-477x_{2n-3}-141x_{2n-2}$$

of the first two equations of system (61) we obtain

$$447\frac{|x_{2n-3}|}{h} \leqslant 316{,}5b_{n,n-3}+\frac{1}{h}\left(\left|\frac{5}{2}P_{2n-4}-2P_{2n-5}\right|+2\left|\frac{5}{2}P_{2n-4}-3P_{2n-3}\right|\right) \tag{62}$$

and from

$$\frac{5}{2}P_{2n-4}-2P_{2n-5}+3\left(\frac{5}{2}P_{2n-4}-3P_{2n-3}\right)+18(10P_{2n-2}-P_{2n-3})$$
$$= 2x_{2n-6}+163x_{2n-5}-46x_{2n-4}+12x_{2n-3}+398x_{2n-2}$$

we get

$$398\frac{|x_{2n-2}|}{h} \tag{63}$$
$$\leqslant 223b_{n,n-3}+\frac{1}{h}\left(\left|\frac{5}{2}P_{2n-4}-2P_{2n-5}\right|+3\left|\frac{5}{2}P_{2n-4}-3P_{2n-3}\right|+\right.$$
$$\left.+18\,|10P_{2n-2}-P_{2n-3}|\right).$$

Because of

$$P_{2n-5} = 20[f(\lambda_{2n-7}^{(2n-5)})+11f(\lambda_{2n-6}^{(2n-5)})+3f(\lambda_{2n-5}^{(2n-5)})],$$
$$P_{2n-4} = 20[2f(\lambda_{2n-6}^{(2n-4)})+8f(\lambda_{2n-5}^{(2n-4)})+2f(\lambda_{2n-4}^{(2n-4)})],$$
$$P_{2n-3} = 20[2f(\lambda_{2n-5}^{(2n-3)})+8f(\lambda_{2n-4}^{(2n-3)})],$$
$$P_{2n-2} = 20f(\lambda_{2n-4}^{(2n-2)}), \quad \text{where } \lambda_i^{(j)}\in[\xi_i,\,\xi_{i+1}],$$

we have

$$\frac{1}{h}\left|\frac{5}{2}P_{2n-4}-2P_{2n-5}\right|$$
$$= \frac{20}{h}\,|2[f(\lambda_{2n-6}^{(2n-4)})-f(\lambda_{2n-7}^{(2n-5)})]+$$
$$+2[f(\lambda_{2n-6}^{(2n-4)})-f(\lambda_{2n-6}^{(2n-5)})]+f(\lambda_{2n-6}^{(2n-4)})-f(\lambda_{2n-5}^{(2n-5)})+$$
$$+20[f(\lambda_{2n-5}^{(2n-4)})-f(\lambda_{2n-6}^{(2n-5)})]+5[f(\lambda_{2n-4}^{(2n-4)})-f(\lambda_{2n-5}^{(2n-5)})]|$$
$$\leqslant \frac{20}{h}\,\|f'\|(2\cdot 3h+2\cdot 2h+4h+20\cdot 4h+5\cdot 4h) = 20\cdot 144\|f'\|,$$

$$\frac{1}{h}\left|\frac{5}{2}P_{2n-4}-3P_{2n-3}\right|$$
$$= \frac{20}{h}\,|5[f(\lambda_{2n-6}^{(2n-4)})-f(\lambda_{2n-5}^{(2n-3)})]+$$
$$+f(\lambda_{2n-5}^{(2n-4)})-f(\lambda_{2n-5}^{(2n-3)})+19[f(\lambda_{2n-5}^{(2n-4)})-f(\lambda_{2n-4}^{(2n-3)})]+$$
$$+5[f(\lambda_{2n-4}^{(2n-4)})-f(\lambda_{2n-4}^{(2n-3)})]|$$
$$\leqslant \frac{20}{h}\,\|f'\|(5\cdot 4h+2h+19\cdot 4h+5\cdot 2h) = 20\cdot 108\|f'\|,$$

$$\frac{1}{h}\,|10P_{2n-2}-P_{2n-3}|$$
$$= \frac{20}{h}\,|2[f(\lambda_{2n-4}^{(2n-2)})-f(\lambda_{2n-5}^{(2n-3)})]+8[f(\lambda_{2n-4}^{(2n-2)})-f(\lambda_{2n-4}^{(2n-3)})]|$$
$$\leqslant \frac{20}{h}\,\|f'\|(2\cdot 4h+8\cdot 2h) = 20\cdot 24\|f'\|.$$

From (62) and (63) we have

$$\frac{|\eta_{2n-3}-\eta_{2n-4}|}{h} \leqslant \frac{316{,}5}{477}b_{n,n-3}+20\cdot\frac{330}{477}\|f'\|, \tag{64}$$

$$\frac{|\eta_{2n-2}-\eta_{2n-3}|}{h} \leqslant \frac{223}{398}b_{n,n-3}+20\cdot\frac{870}{398}\|f'\|. \tag{65}$$

Finally, from estimations (60), (64) and (65) we infer that

$$\frac{|\eta_r-\eta_{r-1}|}{h} \leqslant \frac{9}{11}b_{n,n-3}+20\cdot 3\|f'\| \quad \text{for } r = 1, 2, \ldots, 2n-2;$$

hence

$$b_{n,n-3} \leqslant 330\|f'\|. \tag{66}$$

4° In the case $l = n-2$ we obtain the following equations:

$$6\eta_0+13\eta_1+\eta_2 = \frac{120}{h}\int_{\xi_0}^{\xi_1} f(t)\left[\frac{1}{2}-\frac{t-\xi_0}{h}+\frac{1}{2}\left(\frac{t-\xi_0}{h}\right)^2\right]dt \equiv P_0, \tag{67}$$

$$13\eta_0+60\eta_1+26\eta_2+\eta_3$$
$$= \frac{120}{h}\left\{\int_{\xi_0}^{\xi_1} f(t)\left[\frac{1}{2}+\frac{t-\xi_0}{h}-\left(\frac{t-\xi_0}{h}\right)^2\right]dt+\right.$$
$$\left.+\int_{\xi_1}^{\xi_2} f(t)\left[\frac{1}{2}-\frac{t-\xi_1}{h}+\frac{1}{2}\left(\frac{t-\xi_1}{h}\right)^2\right]dt\right\} \equiv P_1,$$

$$\eta_{r-2}+26\eta_{r-1}+66\eta_r+26\eta_{r+1}+\eta_{r+2}$$
$$= \frac{120}{h}\left\{\int_{\xi_{r-2}}^{\xi_{r-1}} f(t)\,\frac{1}{2}\left(\frac{t-\xi_{r-2}}{h}\right)^2 dt+\right.$$
$$+\int_{\xi_{r-1}}^{\xi_r} f(t)\left[\frac{1}{2}+\frac{t-\xi_{r-1}}{h}-\left(\frac{t-\xi_{r-1}}{h}\right)^2\right]dt+$$
$$\left.+\int_{\xi_r}^{\xi_{r+1}} f(t)\left[\frac{1}{2}-\frac{t-\xi_r}{h}+\frac{1}{2}\left(\frac{t-\xi_r}{h}\right)^2\right]dt\right\} \equiv P_r,$$
$$r = 2, 3, \ldots, 2n-5,$$

$$\eta_{2n-6}+26\eta_{2n-5}+78\eta_{2n-4}-21\eta_{2n-3}-23\eta_{2n-2}-\eta_{2n-1}$$
$$= \frac{120}{h}\left\{\int_{\xi_{2n-6}}^{\xi_{2n-5}} f(t)\,\frac{1}{2}\left(\frac{t-\xi_{2n-6}}{h}\right)^2 dt+\right.$$
$$+\int_{\xi_{2n-5}}^{\xi_{2n-4}} f(t)\left[\frac{1}{2}+\frac{t-\xi_{2n-5}}{h}-\left(\frac{t-\xi_{2n-5}}{h}\right)^2\right]dt+$$
$$+\int_{\xi_{2n-4}}^{\xi_{2n-3}} f(t)\left[\frac{1}{2}-2\,\frac{t-\xi_{2n-4}}{2h}+\frac{5}{4}\left(\frac{t-\xi_{2n-4}}{2h}\right)^2\right]dt+$$
$$\left.+\int_{\xi_{2n-3}}^{\xi_{2n-2}} f(t)\left[-\frac{1}{4}+\frac{1}{2}\,\frac{t-\xi_{2n-3}}{2h}-\frac{1}{4}\left(\frac{t-\xi_{2n-3}}{2h}\right)^2\right]dt\right\}$$
$$\equiv P_{2n-4},$$

$$\eta_{2n-5}-21\eta_{2n-4}+240\eta_{2n-3}+77\eta_{2n-2}+3\eta_{2n-1}$$
$$= \frac{120}{h}\left\{\int_{\xi_{2n-5}}^{\xi_{2n-4}} f(t)\,\frac{1}{2}\left(\frac{t-\xi_{2n-5}}{h}\right)^2 dt+\right.$$
$$+\int_{\xi_{2n-4}}^{\xi_{2n-3}} f(t)\left[\frac{1}{2}+2\,\frac{t-\xi_{2n-4}}{2h}-\frac{7}{4}\left(\frac{t-\xi_{2n-4}}{2h}\right)^2\right]dt+$$
$$\left.+\int_{\xi_{2n-3}}^{\xi_{2n-2}} f(t)\left[\frac{3}{4}-\frac{3}{2}\,\frac{t-\xi_{2n-3}}{2h}+\frac{3}{4}\left(\frac{t-\xi_{2n-3}}{2h}\right)^2\right]dt\right\} \equiv P_{2n-3},$$

$$-23\eta_{2n-4}+77\eta_{2n-3}+120\eta_{2n-2}+26\eta_{2n-1}$$
$$= \frac{120}{h}\left\{\int_{\xi_{2n-4}}^{\xi_{2n-3}} f(t)\,\frac{1}{2}\left(\frac{t-\xi_{2n-4}}{2h}\right)^2 dt+\right.$$
$$\left.+\int_{\xi_{2n-3}}^{\xi_{2n-2}} f(t)\left[\frac{1}{2}+\frac{t-\xi_{2n-3}}{2h}-\left(\frac{t-\xi_{2n-3}}{2h}\right)^2\right]dt\right\} \equiv P_{2n-2},$$

$$-\eta_{2n-4}+3\eta_{2n-3}+26\eta_{2n-2}+12\eta_{2n-1}$$
$$= \frac{120}{h}\int_{\xi_{2n-3}}^{\xi_{2n-2}} f(t)\,\frac{1}{2}\left(\frac{t-\xi_{2n-3}}{2h}\right)^2 dt \equiv P_{2n-1}.$$

As in case 1°, we obtain

$$|\eta_r| \leqslant \frac{12}{13}\,a_{n,n-2}+60\cdot\frac{11}{34}\,\|f\| \quad \text{for } r = 0, 1, \ldots, 2n-3. \tag{68}$$

From the last equation of system (67) we have

$$26\,|\eta_{2n-2}| \leqslant 16a_{n,n-2}+|P_{2n-1}|;$$

hence

$$|\eta_{2n-2}| \leqslant \frac{8}{13}\,a_{n,n-2}+60\cdot\frac{1}{39}\,\|f\|. \tag{69}$$

From the last four equations of system (67) we form the linear combination

$$P_{2n-4}-2P_{2n-3}+7P_{2n-2}-26P_{2n-1}$$
$$= \eta_{2n-6}+24\eta_{2n-5}-15\eta_{2n-4}-40\eta_{2n-3}-13\eta_{2n-2}-137\eta_{2n-1},$$

which implies

$$137\,|\eta_{2n-1}| \leqslant 93a_{n,n-2}+|P_{2n-4}|+2\,|P_{2n-3}|+7\,|P_{2n-2}|+26\,|P_{2n-1}|.$$

Hence

$$|\eta_{2n-1}| \leqslant \frac{93}{137}\,a_{n,n-2}+60\cdot\frac{171}{3\cdot 137}\,\|f\|. \tag{70}$$

From (68), (69) and (70) we deduce that

$$|\eta_r| \leqslant \frac{12}{13}\,a_{n,n-2}+60\cdot\frac{171}{3\cdot 137}\,\|f\| \quad \text{for } r = 0, 1, \ldots, 2n-1;$$

hence

$$a_{n,n-2} \leqslant \frac{20\cdot 171\cdot 13}{137}\,\|f\|. \tag{71}$$

As in 1°, we obtain

$$\frac{|\eta_r-\eta_{r-1}|}{h} \leqslant \frac{9}{11}\, b_{n,n-2}+20\cdot\frac{4}{3}\,\|f'\| \quad \text{for } r=1,2,\ldots,2n-3. \tag{72}$$

Let $\eta_r-\eta_{r-1}=x_r$ for $r=2n-5,\ldots,2n-1$; then the last four equations of system (67) assume the following form:

$$\begin{aligned}
60\eta_{2n-6}+59x_{2n-5}+33x_{2n-4}-45x_{2n-3}-24x_{2n-2}-x_{2n-1} &= P_{2n-4},\\
300(\eta_{2n-6}+x_{2n-5})+299x_{2n-4}+320x_{2n-3}+80x_{2n-2}+3x_{2n-1} &= P_{2n-3},\\
200(\eta_{2n-6}+x_{2n-5}+x_{2n-4})+223x_{2n-3}+146x_{2n-2}+26x_{2n-1} &= P_{2n-2},\\
40(\eta_{2n-6}+x_{2n-5}+x_{2n-4})+41x_{2n-3}+38x_{2n-2}+12x_{2n-1} &= P_{2n-1};
\end{aligned}$$

hence

$$\begin{aligned}
-5x_{2n-5}-134x_{2n-4}-545x_{2n-3}-200x_{2n-2}-8x_{2n-1} &= 5P_{2n-4}-P_{2n-3},\\
-2x_{2n-4}-29x_{2n-3}-278x_{2n-2}-72x_{2n-1} &= 2P_{2n-3}-3P_{2n-2},\\
18x_{2n-3}-44x_{2n-2}-34x_{2n-1} &= P_{2n-2}-5P_{2n-1}.
\end{aligned} \tag{73}$$

Analogously to 1°, we obtain

$$\frac{1}{h}\,|5P_{2n-4}-P_{2n-3}| \leqslant 20\cdot 157\,\|f'\|,$$

$$\begin{aligned}
\frac{1}{h}\,|2P_{2n-3}-3P_{2n-2}| &= \frac{40}{h}\,|f(\lambda_{2n-5}^{(2n-3)})-f(\lambda_{2n-3}^{(2n-2)})+11[f(\lambda_{2n-4}^{(2n-3)})-f(\lambda_{2n-3}^{(2n-2)})]+\\
&\quad +3[f(\lambda_{2n-3}^{(2n-2)})-f(\lambda_{2n-4}^{(2n-2)})]|\\
&\leqslant \frac{40}{h}\,\|f'\|(5h+11\cdot 4h+3\cdot 4h) = 40\cdot 61\,\|f'\|,
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{h}\,|P_{2n-2}-5P_{2n-1}|\\
&\quad = \frac{40}{h}\,|f(\lambda_{2n-4}^{(2n-2)})-f(\lambda_{2n-3}^{(2n-1)})+4[f(\lambda_{2n-3}^{(2n-2)})-f(\lambda_{2n-3}^{(2n-1)})]|\\
&\quad \leqslant \frac{40}{h}\,\|f'\|(4h+4\cdot 2h) = 40\cdot 12\,\|f'\|,
\end{aligned}$$

because of $\lambda_i^{(j)}\in[\xi_i,\xi_{i+1}]$. From the second equation of (73) we have

$$278\,\frac{|x_{2n-2}|}{h} \leqslant 103 b_{n,n-2}+\frac{1}{h}\,|2P_{2n-3}-3P_{2n-2}|;$$

hence

$$\frac{|x_{2n-2}|}{h} \leqslant \frac{103}{278}\, b_{n,n-2}+20\cdot\frac{61}{139}\,\|f'\|. \tag{74}$$

By means of equations (73) we form the linear combination

$$\begin{aligned}
5P_{2n-4}-P_{2n-3}-4(2P_{2n-3}-3P_{2n-2})+19(P_{2n-2}-5P_{2n-1})&\\
= -5x_{2n-5}-126x_{2n-4}-87x_{2n-3}+76x_{2n-2}-366x_{2n-1}&,
\end{aligned}$$

which implies

$$\begin{aligned}
366\,\frac{|x_{2n-1}|}{h} \leqslant 294 b_{n,n-2}+\frac{1}{h}\,(|5P_{2n-4}-P_{2n-3}|&+\\
+4\,|2P_{2n-3}-3P_{2n-2}|+19\,|P_{2n-2}-5P_{2n-1}|&).
\end{aligned}$$

Hence

$$\frac{|x_{2n-1}|}{h} \leqslant \frac{147}{183}\, b_{n,n-2}+20\cdot\frac{1101}{366}\,\|f'\|. \tag{75}$$

From inequalities (72), (74) and (75) we conclude that

$$\frac{|\eta_r-\eta_{r-1}|}{h} \leqslant \frac{9}{11}\, b_{n,n-2}+20\cdot\frac{34}{11}\,\|f'\| \quad \text{for } r=1,2,\ldots,2n-1;$$

hence

$$b_{n,n-2} \leqslant 340\,\|f'\|. \tag{76}$$

5° If $l=n-1$, then we obtain the following system of equations:

$$6\eta_0+13\eta_1+\eta_2 = \frac{120}{h}\int_{\xi_0}^{\xi_1} f(t)\left[\frac{1}{2}-\frac{t-\xi_0}{h}+\frac{1}{2}\left(\frac{t-\xi_0}{h}\right)^2\right]dt \equiv P_0, \tag{77}$$

$$\begin{aligned}
&13\eta_0+60\eta_1+26\eta_2+\eta_3\\
&\quad = \frac{120}{h}\left\{\int_{\xi_0}^{\xi_1} f(t)\left[\frac{1}{2}+\frac{t-\xi_0}{h}-\left(\frac{t-\xi_0}{h}\right)^2\right]dt+\right.\\
&\qquad \left.+\int_{\xi_1}^{\xi_2} f(t)\left[\frac{1}{2}-\frac{t-\xi_1}{h}+\frac{1}{2}\left(\frac{t-\xi_1}{h}\right)^2\right]dt\right\} \equiv P_1,
\end{aligned}$$

$$\begin{aligned}
&\eta_{r-2}+26\eta_{r-1}+66\eta_r+26\eta_{r+1}+\eta_{r+2}\\
&= \frac{120}{h}\left\{\int_{\xi_{r-2}}^{\xi_{r-1}} f(t)\,\frac{1}{2}\left(\frac{t-\xi_{r-2}}{h}\right)^2 dt+\int_{\xi_{r-1}}^{\xi_r} f(t)\left[\frac{1}{2}+\frac{t-\xi_{r-1}}{h}-\left(\frac{t-\xi_{r-1}}{h}\right)^2\right]dt+\right.\\
&\qquad \left.+\int_{\xi_r}^{\xi_{r+1}} f(t)\left[\frac{1}{2}-\frac{t-\xi_r}{h}+\frac{1}{2}\left(\frac{t-\xi_r}{h}\right)^2\right]dt\right\} \equiv P_r, \quad r=2,3,\ldots,2n-3,
\end{aligned}$$

$$\eta_{2n-4}+26\eta_{2n-3}+75\eta_{2n-2}-12\eta_{2n-1}-10\eta_{2n}$$

$$=\frac{120}{h}\Bigg\{\int\limits_{\xi_{2n-4}}^{\xi_{2n-3}} f(t)\,\frac{1}{2}\left(\frac{t-\xi_{2n-4}}{h}\right)^2 dt+\int\limits_{\xi_{2n-3}}^{\xi_{2n-2}} f(t)\left[\frac{1}{2}+\frac{t-\xi_{2n-3}}{h}-\left(\frac{t-\xi_{2n-3}}{h}\right)^2\right]dt+$$

$$+\int\limits_{\xi_{2n-2}}^{\xi_{2n-1}} f(t)\left[\frac{1}{2}-2\,\frac{t-\xi_{2n-2}}{2h}+\frac{5}{4}\left(\frac{t-\xi_{2n-2}}{2h}\right)^2\right]dt\Bigg\}\equiv P_{2n-2},$$

$$\eta_{2n-3}-12\eta_{2n-2}+213\eta_{2n-1}+38\eta_{2n}$$

$$=\frac{120}{h}\Bigg\{\int\limits_{\xi_{2n-3}}^{\xi_{2n-2}} f(t)\,\frac{1}{2}\left(\frac{t-\xi_{2n-3}}{h}\right)^2 dt+$$

$$+\int\limits_{\xi_{2n-2}}^{\xi_{2n-1}} f(t)\left[\frac{1}{2}+2\,\frac{t-\xi_{2n-2}}{2h}-\frac{7}{4}\left(\frac{t-\xi_{2n-2}}{2h}\right)^2\right]dt\Bigg\}$$

$$\equiv P_{2n-1},$$

$$-5\eta_{2n-2}+19\eta_{2n-1}+6\eta_{2n}=\frac{120}{2h}\int\limits_{\xi_{2n-2}}^{\xi_{2n-1}} f(t)\,\frac{1}{2}\left(\frac{t-\xi_{2n-2}}{2h}\right)^2 dt\equiv P_{2n}.$$

In the same manner as in 1°, we obtain

$$|\eta_r|\leqslant\frac{12}{13}\,a_{n,n-1}+60\cdot\frac{11}{34}\,\|f\|\quad\text{ for } r=0,1,\ldots,2n-3. \tag{78}$$

From the equation with the index $2n-2$ of system (77) we have

$$75\,|\eta_{2n-2}|\leqslant 49a_{n,n-1}+|P_{2n-2}|, \tag{79}$$

and from the last equation of this system we obtain

$$19\,|\eta_{2n-1}|\leqslant 11a_{n,n-1}+|P_{2n}|. \tag{80}$$

Since $P_{2n-2}-2P_{2n-1}+23P_{2n}=\eta_{2n-4}+24\eta_{2n-3}-16\eta_{2n-2}-\eta_{2n-1}+52\eta_{2n}$, we get

$$51\,|\eta_{2n}|\leqslant 42a_{n,n-1}+|P_{2n-2}|+2\,|P_{2n-2}|+23\,|P_{2n}|. \tag{81}$$

Because of $|P_{2n-2}|\leqslant 60\cdot 6\,\|f\|$, $|P_{2n-1}|\leqslant 60\cdot 4\,\|f\|$, $|P_{2n}|\leqslant 60\cdot\frac{1}{3}\,\|f\|$, inequalities (79), (80) and (81) assume the following form:

$$|\eta_{2n-2}|\leqslant\frac{49}{75}\,a_{n,n-1}+60\cdot\frac{2}{25}\,\|f\|, \tag{82}$$

$$|\eta_{2n-1}|\leqslant\frac{11}{19}\,a_{n,n-1}+60\cdot\frac{1}{3\cdot 19}\,\|f\|,\qquad |\eta_{2n}|\leqslant\frac{21}{26}\,a_{n,n-1}+60\cdot\frac{65}{3\cdot 52}\,\|f\|.$$

Now, from inequalities (78) and (82) we conclude that

$$|\eta_r|\leqslant\frac{12}{13}\,a_{n,n-1}+60\cdot\frac{65}{3\cdot 52}\,\|f\|\quad\text{ for } r=0,1,\ldots,2n,$$

and hence

$$a_{n,n-1}\leqslant 325\,\|f\|. \tag{83}$$

As in 1°, we obtain

$$\frac{|\eta_r-\eta_{r-1}|}{h}\leqslant\frac{9}{11}\,b_{n,n-1}+20\cdot\frac{4}{3}\,\|f'\|\quad\text{ for } r=1,2,\ldots,2n-3. \tag{84}$$

Setting $\eta_r-\eta_{r-1}=x_r$ $(r=2n-4,\ldots,2n)$ in the last four equations of system (77), we obtain

$$\begin{aligned}120\eta_{2n-5}+119x_{2n-4}+93x_{2n-3}+27x_{2n-2}+x_{2n-1}&=P_{2n-3},\\ 80(\eta_{2n-5}+x_{2n-4})+79x_{2n-3}+53x_{2n-2}-22x_{2n-1}-10x_{2n}&=P_{2n-2},\\ 240(\eta_{2n-5}+x_{2n-4}+x_{2n-3})+239x_{2n-2}+251x_{2n-1}+38x_{2n}&=P_{2n-1},\\ 20(\eta_{2n-5}+x_{2n-4}+x_{2n-3}+x_{2n-2})+25x_{2n-1}+6x_{2n}&=P_{2n},\end{aligned}$$

hence

$$\begin{aligned}-2x_{2n-4}-51x_{2n-3}-105x_{2n-2}-68x_{2n-1}+30x_{2n}&=2P_{2n-3}-3P_{2n-2},\\ -3x_{2n-3}-80x_{2n-2}-317x_{2n-1}-68x_{2n}&=3P_{2n-2}-P_{2n-1},\\ -x_{2n-2}-49x_{2n-1}-34x_{2n}&=P_{2n-1}-12P_{2n}.\end{aligned} \tag{85}$$

Adding the equations 1 and 3 of system (85), we have

$$-2x_{2n-4}-51x_{2n-3}-106x_{2n-2}+19x_{2n-1}-4x_{2n}$$
$$=2P_{2n-3}-3P_{2n-2}+P_{2n-1}-12P_{2n};$$

hence

$$106\,\frac{|x_{2n-2}|}{h}\leqslant 76b_{n,n-1}+\frac{1}{h}\left(|2P_{2n-3}-3P_{2n-2}|+|P_{2n-1}-12P_{2n}|\right). \tag{86}$$

From the last equation of (85) we obtain

$$49\,\frac{|x_{2n-1}|}{h}\leqslant 35b_{n,n-1}+\frac{1}{h}\,|P_{2n-1}-12P_{2n}| \tag{87}$$

and from the linear combination $3P_{2n-2}-P_{2n-1}-6(P_{2n-1}-12P_{2n})$ $=-3x_{2n-3}-74x_{2n-2}-23x_{2n-1}+136x_{2n}$ we have

$$136\,\frac{|x_{2n}|}{h}\leqslant 100b_{n,n-1}+\frac{1}{h}\left(|3P_{2n-2}-P_{2n-1}|+6\,|P_{2n-1}-12P_{2n}|\right). \tag{88}$$

In the same manner as in the previously considered cases, we obtain

$$\frac{1}{h}\,|2P_{2n-3}-3P_{2n-2}|\leqslant 20\cdot 64\,\|f'\|,$$

$$\frac{1}{h}\,|3P_{2n-2}-P_{2n-1}|\leqslant 20\cdot 82\,\|f'\|,$$

$$\frac{1}{h}\,|P_{2n-1}-12P_{2n}|\leqslant 20\cdot 25\,\|f'\|,$$

and inequalities (86), (87) and (88) assume the form

(89)
$$\frac{|x_{2n-2}|}{h}\leqslant\frac{38}{53}\,b_{n,n-1}+20\cdot\frac{89}{106}\,\|f'\|,$$

$$\frac{|x_{2n-1}|}{h}\leqslant\frac{35}{49}\,b_{n,n-1}+20\cdot\frac{25}{49}\,\|f'\|,$$

$$\frac{|x_{2n}|}{h}\leqslant\frac{25}{34}\,b_{n,n-1}+20\cdot\frac{29}{17}\,\|f'\|.$$

From (84) and (89) we infer that

$$\frac{|\eta_r-\eta_{r-1}|}{h}\leqslant\frac{9}{11}\,b_{n,n-1}+20\cdot 3\,\|f'\|\qquad\text{for } r=1,\ldots,2n;$$

hence

(90)
$$b_{n,n-1}\leqslant 330\,\|f'\|.$$

6° For $l=n$, we obtain the following equations:

$$6\eta_0+13\eta_1+\eta_2=\frac{120}{h}\int\limits_{\xi_0}^{\xi_1}f(t)\left[\frac{1}{2}-\frac{t-\xi_0}{h}+\frac{1}{2}\left(\frac{t-\xi_0}{h}\right)^2\right]dt,$$

$$13\eta_0+60\eta_1+26\eta_2+\eta_3$$

$$=\frac{120}{h}\left\{\int\limits_{\xi_0}^{\xi_1}f(t)\left[\frac{1}{2}+\frac{t-\xi_0}{h}-\left(\frac{t-\xi_0}{h}\right)^2\right]dt+\int\limits_{\xi_1}^{\xi_2}f(t)\left[\frac{1}{2}-\frac{t-\xi_1}{h}+\frac{1}{2}\left(\frac{t-\xi_1}{h}\right)^2\right]dt\right\},$$

$$\eta_{r-2}+26\eta_{r-1}+66\eta_r+26\eta_{r+1}+\eta_{r+2}$$

$$=\frac{120}{h}\left\{\int\limits_{\xi_{r-2}}^{\xi_{r-1}}f(t)\,\frac{1}{2}\left(\frac{t-\xi_{r-2}}{h}\right)^2dt+\int\limits_{\xi_{r-1}}^{\xi_r}f(t)\left[\frac{1}{2}+\frac{t-\xi_{r-1}}{h}-\left(\frac{t-\xi_{r-1}}{h}\right)^2\right]dt+\right.$$

$$\left.+\int\limits_{\xi_r}^{\xi_{r+1}}f(t)\left[\frac{1}{2}-\frac{t-\xi_r}{h}+\frac{1}{2}\left(\frac{t-\xi_r}{h}\right)^2\right]dt\right\},\qquad r=2,3,\ldots,2n-1,$$

$$\eta_{2n-2}+26\eta_{2n-1}+60\eta_{2n}+13\eta_{2n+1}$$

$$=\frac{120}{h}\left\{\int\limits_{\xi_{2n-2}}^{\xi_{2n-1}}f(t)\,\frac{1}{2}\left(\frac{t-\xi_{2n-2}}{h}\right)^2dt+\int\limits_{\xi_{2n-1}}^{\xi_{2n}}f(t)\left[\frac{1}{2}+\frac{t-\xi_{2n-1}}{h}-\left(\frac{t-\xi_{2n-1}}{h}\right)^2\right]dt\right\},$$

$$\eta_{2n-1}+13\eta_{2n}+6\eta_{2n+1}=\frac{120}{h}\int\limits_{\xi_{2n-1}}^{\xi_{2n}}f(t)\,\frac{1}{2}\left(\frac{t-\xi_{2n-1}}{h}\right)^2dt.$$

Repeating some parts of the previous considerations, we obtain the following estimations:

(91)
$$a_{n,n}\leqslant 60\cdot\frac{11\cdot 13}{34}\,\|f\|,\qquad b_{n,n}\leqslant\frac{440}{3}\,\|f'\|.$$

7° The case $l=1$ leads to the same argumentation as the case $l=n-4$. From results (20), (48), (59), (71), (83) and (91) we deduce that

(92)
$$a_{n,l}\leqslant 325\,\|f\|\qquad\text{for } l=1,2,\ldots,n,$$

and from (44), (55), (66), (76), (90) and (91) we obtain

(93)
$$b_{n,l}\leqslant 340\,\|f'\|\qquad\text{for } l=1,2,\ldots,n.$$

For $t\in[\xi_k,\xi_{k+1}]$ we have

$$|w_k(t)|\leqslant\begin{cases}5a_{n,l} & \text{if}\quad 0\leqslant k\leqslant 2l-1,\\ \dfrac{17}{2}\,a_{n,l} & \text{if}\quad k=2l,\\ 7a_{n,l} & \text{if}\quad k=2l+1,\\ 5a_{n,l} & \text{if}\quad 2l+2\leqslant k\leqslant l+n-1;\end{cases}$$

hence

$$\|\varphi\|\leqslant\frac{17}{2}\,a_{n,l}\leqslant\frac{17}{2}\cdot 325\,\|f\|<2763\,\|f\|.$$

Similarly,

$$|w_k'(t)|\leqslant\begin{cases}3b_{n,l} & \text{if}\quad 0\leqslant k\leqslant 2l-1\\ \dfrac{11}{4}\,b_{n,l} & \text{if}\quad k=2l\\ 2b_{n,l} & \text{if}\quad k=2l+1\\ \dfrac{3}{2}\,b_{n,l} & \text{if}\quad 2l+2\leqslant k\leqslant l+n-1;\end{cases}$$

hence $\|\varphi'\|\leqslant 3b_{n,l}\leqslant 3\cdot 340\,\|f'\|=1020\,\|f'\|$. So, the Theorem is proved.

2. Orthonormal basis in $C[0,1]$ and $C_1[0,1]$. Let $L_0(t)=1$, $L_1(t)=\sqrt{3}(2t-1)$, $H_n(t)$ $(n=1,2,\ldots)$ form the orthonormal system obtained by the orthogonalization of $1, t, \int_0^t (t-s)h_n(s)\,ds$ and let

$$S_n(f;t)=\int_0^1 f(s)L_0(s)\,ds\,L_0(t)+\int_0^1 f(s)L_1(s)\,ds\,L_1(t)+\sum_{i=1}^{n}\int_0^1 f(s)H_i(s)\,ds\,H_i(t).$$

THEOREM. *There exists a positive constant M such that for every function $f\in C[0,1]$ and every integer n we have*

$$\|S_n(f;t)\|\leqslant M\|f\|.$$

Moreover, if $f\in C_1[0,1]$, then there exists a positive constant M_1 such that

$$\|S_n'(f;t)\|\leqslant M_1\|f'\|.$$

Proof. From the definition of Haar functions it follows that $S_n(f;t)\in A_{2^m,k}$, where $n=2^m+k$ and $1\leqslant k\leqslant 2^m$. Hence, the well-known properties of partial sums of Fourier series and the previous theorem imply our assertion.

Let

$$\omega(\delta)=\sup\{a\colon a=|f(t_1)-f(t_2)|,\ |t_1-t_2|\leqslant\delta,\ t_1,t_2\in[0,1]\}$$

denote the modulus of continuity of the function f and $\omega_1(\delta)$ the same modulus of f'.

THEOREM. *If $f\in C_1[0,1]$, then there exist positive constants M and M_1 such that*

$$|S_n(f;t)-f(t)|\leqslant 2(M+1)\,\omega_1\left(\frac{1}{n}\right)$$

and

$$|S_n'(f;t)-f'(t)|\leqslant 2(M_1+1)\,\omega_1\left(\frac{1}{n}\right).$$

Proof. Let $n=2^m+k$ $(1\leqslant k\leqslant 2^m)$ and let $t_i=i/2^{m+1}$ for $i=0,1,\ldots,2k$, $t_i=(i-k)/2^m$ for $i=2k+1,\ldots,n$,

$$\sigma_n(f;t)=f(0)+f'(0)t+\sum_{i=1}^{n}\int_0^1 h_i(s)\,df'(s)\cdot\int_0^t (t-s)h_i(s)\,ds.$$

Since $\sigma_n'(f;t_i)=f'(t_i)$, we have for $t\in[t_i,t_{i+1}]$

$$|\sigma_n'(f;t)-f'(t)|\leqslant\omega_1(t_{i+1}-t_i)\leqslant\omega_1\left(\frac{1}{2^m}\right)\leqslant 2\omega_1\left(\frac{1}{n}\right)$$

and

$$|\sigma_n(f;t)-f(t)|=\left|\int_0^t[\sigma_n'(f;\tau)-f'(\tau)]\,d\tau\right|$$
$$\leqslant\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}|\sigma_n'(f;t)-f'(t)|\,dt+\int_{t_i}^{t}|\sigma_n'(f;\tau)-f'(\tau)|\,d\tau\leqslant 2\omega_1\left(\frac{1}{n}\right)t.$$

In virtue of the identity $S_n(\sigma_n;t)=\sigma_n(f;t)$ and by the previous Theorem, we have

$$|S_n'(f;t)-f'(t)|\leqslant|S_n'(\sigma_n-f;t)|+|\sigma_n'(f;t)-f'(t)|$$
$$\leqslant(M_1+1)\|\sigma_n'(f;t)-f'(t)\|\leqslant 2(M_1+1)\,\omega_1\left(\frac{1}{n}\right)$$

and analogously,

$$|S_n(f;t)-f(t)|\leqslant 2(M+1)\,\omega_1\left(\frac{1}{n}\right).$$

If we set in the above Theorem

$$f(t)=\int_0^t g(s)\,ds\qquad(g\in C[0,1])$$

we obtain the following

COROLLARY. *Functions $L_1'(t), H_n'(t)$ $(n=1,2,\ldots)$ form a basis in $C[0,1]$.*

THEOREM. *Functions $L_0(t), L_1(t), H_n(t)$ $(n=1,2,\ldots)$ form a basis in $C[0,1]$, and if $f\in C[0,1]$, then there exists a positive constant L such that*

$$|S_N(f;t)-f(t)|\leqslant L\omega\left(\frac{1}{N}\right).$$

Proof. Let $\varphi(t)=w_k(t)$ for $t\in[\xi_k,\xi_{k+1}]$, where the polynomials w_k are defined by formulas (4), and let us set $\eta_k=f(\xi_k)$ for $k=0,1,\ldots,n+l$, $\eta_{n+l+1}=f(\xi_{n+l})$. If $0\leqslant k\leqslant 2l-1$ and $t\in[\xi_k,\xi_{k+1}]$, then

$$|\varphi(t)-f(t)|\leqslant\frac{1}{2}|\eta_k-f(t)|+\frac{1}{2}|\eta_{k+1}-f(t)|+|\eta_{k+1}-\eta_k|+$$
$$+\frac{1}{2}(|\eta_{k+2}-\eta_{k+1}|+|\eta_{k+1}-\eta_k|)\leqslant 3\omega\left(\frac{1}{n}\right).$$

If $k=2l$, then

$$|\varphi(t)-f(t)|\leqslant\frac{1}{2}|\eta_{2l}-f(t)|+\frac{1}{2}|\eta_{2l+1}-f(t)|+2|\eta_{2l+1}-\eta_{2l}|+$$
$$+\frac{1}{4}(2|\eta_{2l+2}-\eta_{2l+1}|+5|\eta_{2l+1}-\eta_{2l}|)\leqslant\frac{19}{4}\,\omega\left(\frac{1}{n}\right).$$

If $k = 2l+1$, then

$$\begin{aligned}|\varphi(t)-f(t)| &\leqslant \frac{1}{4}\left(2|\eta_{2l+2}-f(t)|+2|\eta_{2l+1}-f(t)|+|\eta_{2l+1}-\eta_{2l}|\right)+\\ &\quad+\frac{1}{2}\left(2|\eta_{2l+2}-\eta_{2l+1}|+|\eta_{2l+1}-\eta_{2l}|\right)+\\ &\quad+\frac{1}{4}\left(2|\eta_{2l+3}-\eta_{2l+2}|+2|\eta_{2l+2}-\eta_{2l+1}|+|\eta_{2l+1}-\eta_{2l}|\right)\\ &\leqslant 4\omega\left(\frac{1}{n}\right).\end{aligned}$$

If $2l+2 \leqslant k \leqslant n+l-1$, then

$$\begin{aligned}|\varphi(t)-f(t)| &\leqslant \frac{1}{2}|\eta_k-f(t)|+\frac{1}{2}|\eta_{k+1}-f(t)|+|\eta_{k+1}-\eta_k|+\\ &\quad+\frac{1}{2}\left(|\eta_{k+2}-\eta_{k+1}|+|\eta_{k+1}-\eta_k|\right)\\ &\leqslant 3\omega\left(\frac{1}{n}\right).\end{aligned}$$

Hence we have

$$|\varphi(t)-f(t)| \leqslant \frac{19}{4}\,\omega\left(\frac{1}{n}\right)$$

for arbitrary $t \in [0,1]$.

Let $N = 2^m+k$, where m, k are positive integers and $1 \leqslant k \leqslant 2^m$. In particular, for $n = 2^m$ and $l = k$ we have $S_N(\varphi; t) = \varphi(t)$, and

$$\|\varphi-f\| \leqslant \frac{19}{4}\,\omega\left(\frac{1}{2^m}\right) \leqslant \frac{19}{4}\,\omega\left(\frac{2}{N}\right) \leqslant \frac{19}{2}\,\omega\left(\frac{1}{N}\right).$$

There exists a positive constant M such that

$$\|S_N(\varphi-f; t)\| \leqslant M\|\varphi-f\|,$$

or

$$\begin{aligned}|S_N(f;t)-f(t)| &\leqslant |S_N(f-\varphi;t)|+|\varphi-f| \leqslant (M+1)\|\varphi-f\| \leqslant (M+1)\,\frac{19}{2}\,\omega\left(\frac{1}{N}\right).\end{aligned}$$

3. Inequalities of the Bernstein-Zygmund type.

Lemma. *If $w(t)$ denotes a polynomial of degree not greater than 2, then*

$$(94)\qquad \int_a^b |w'(t)|^p\,dt \leqslant \frac{1}{(b-a)^{2p-1}}\left(20\int_a^b |w(t)|\,dt\right)^p \qquad (p \geqslant 1).$$

Proof. It can be seen that we may assume $a = 0$ and $b = 1$; the general case may immediately be obtained by the substitution $t = \dfrac{\tau-a}{b-a}$. Any polynomial of degree not greater than 2 may be written in the form $w(t) = \alpha+\beta t+\frac{1}{2}(\gamma-\beta)t^2$. For polynomials of degree 0, inequality (94) is obvious. For polynomials of degree 1 we have

$$\int_0^1 |w(t)|\,dt \geqslant \frac{\alpha^2+\beta^2-|\alpha\beta|}{2|\beta|} \geqslant \frac{|\beta|}{4} \qquad (\gamma = \beta \neq 0),$$

and so

$$\int_0^1 |w'(t)|^p\,dt = |\beta|^p \leqslant \left(4\int_0^1 |w(t)|\,dt\right)^p$$

and inequality (94) is also true. Without loss of generality we may assume that $\gamma-\beta > 0$. The polynomial w attains its minimum for $t_0 = -\beta/(\gamma-\beta)$. Let

$$\Phi(\alpha) = \int_0^1 |w(t)|\,dt.$$

Now, we are going to consider the following cases:

1° $t_0 \leqslant 0$, or $0 \leqslant \beta < \gamma$. If $\alpha \leqslant -(\gamma+\beta)/2$, then $w \leqslant 0$ in $[0,1]$, and

$$\Phi(\alpha) = -\left(\alpha+\frac{1}{2}\beta+\frac{\gamma-\beta}{6}\right).$$

If $-(\gamma+\beta)/2 < \alpha < 0$, then $0 < \tau = (-\beta+\sqrt{\Delta})/(\gamma-\beta) < 1$, where $\Delta = \beta^2-2\alpha(\gamma-\beta)$, and $w \leqslant 0$ in $[0,\tau]$, $w \geqslant 0$ in $[\tau, 1]$, and so

$$\begin{aligned}\Phi(\alpha) &= -\int_0^\tau w(t)\,dt+\int_\tau^1 w(t)\,dt\\ &= \frac{1}{2}\beta+\frac{\gamma-\beta}{6}-\beta\tau-\frac{1}{2}(\gamma-\beta)\tau^2+\beta\tau^2+\frac{2}{3}(\gamma-\beta)\tau^3\end{aligned}$$

and

$$\frac{d\Phi}{d\tau} = [\beta+(\gamma-\beta)\tau](2\tau-1).$$

The function Φ attains its minimum at $\tau = \frac{1}{2}$ or $a = -(\gamma+3\beta)/8$. For $a \geqslant 0$, we have

$$w(t) \geqslant \beta t + \frac{1}{2}(\gamma-\beta)t^2 \geqslant 0 \quad \text{and} \quad \Phi(a) = a + \frac{1}{2}\beta + \frac{\gamma-\beta}{6}.$$

The function $\Phi(a)$ is decreasing in the interval $(-\infty, -(\gamma+3\beta)/8)$ and increasing in the interval $(-(\gamma+3\beta)/8, \infty)$, and so

$$\inf_{-\infty<a<\infty} \Phi(a) = \Phi\left(-\frac{\gamma+3\beta}{8}\right) = \frac{\gamma+\beta}{8}. \tag{95}$$

Because of $w'(t) = \beta+(\gamma-\beta)t \geqslant 0$ in $[0,1]$,

$$\int_0^1 |w'(t)|^p\,dt = \frac{1}{p+1}\,\frac{\gamma^{p+1}-\beta^{p+1}}{\gamma-\beta}. \tag{96}$$

Applying the Mean-Value Theorem to the function x^{p+1}, we obtain

$$\frac{\gamma^{p+1}-\beta^{p+1}}{\gamma-\beta} = (p+1)\,\xi^p, \quad \text{where } \beta < \xi < \gamma.$$

Hence

$$\frac{1}{p+1}\,\frac{\gamma^{p+1}-\beta^{p+1}}{\gamma-\beta} < (\gamma+\beta)^p,$$

and by (95) and (96), we have

$$\int_0^1 |w'(t)|^p\,dt \leqslant (\gamma+\beta)^p \leqslant \Big(8\int_0^1 |w(t)|\,dt\Big)^p.$$

Inequality (94) holds in this case.

2° Let $0 < t_0 \leqslant \frac{1}{4}$ or $\gamma \geqslant -3\beta > 0$. If $a \leqslant -(\gamma+\beta)/2$, then $w(t) \leqslant 0$ in $[0,1]$ and

$$\Phi(a) = -\left(a+\frac{1}{2}\beta+\frac{\gamma-\beta}{6}\right).$$

If $-(\gamma+\beta)/2 < a \leqslant 0$, then

$$\Phi = \frac{1}{2}\beta+\frac{\gamma-\beta}{6}-\beta\tau+\frac{1}{2}(3\beta-\gamma)\tau^2+\frac{2}{3}(\gamma-\beta)\tau^3,$$

where

$$2t_0 \leqslant \tau = \frac{-\beta+\sqrt{\Delta}}{\gamma-\beta} < 1,$$

and

$$\frac{d\Phi}{d\tau} = [\beta+(\gamma-\beta)\tau](2\tau-1).$$

The function Φ attains its minimum at $\tau = \frac{1}{2}$ or $a = -(\gamma+3\beta)/8$. If $0 < a < \beta^2/2(\gamma-\beta)$, then the polynomial w has two different roots in the segment $[0,1]$, and

$$\Phi = \frac{1}{2}\beta+\frac{\gamma-\beta}{6}+\frac{1}{2}[\beta t_0+(\gamma-\beta)x^2](4x-1)-4\beta t_0 x-\frac{2}{3}(\gamma-\beta)x(3t_0^2+x^2),$$

where

$$0 < x = \frac{\sqrt{\Delta}}{\gamma-\beta} < t_0 \leqslant \frac{1}{4},$$

and

$$\frac{d\Phi}{dx} = (\gamma-\beta)x(4x-1) < 0 \quad \text{for } x \in (0, t_0).$$

Because of $a = -\frac{1}{2}[\beta t_0+(\gamma-\beta)x^2]$, the function $\Phi(a)$ is increasing in the interval in question. If $a \geqslant \beta^2/2(\gamma-\beta)$, then $w(t) \geqslant 0$ in $[0,1]$ and

$$\Phi(a) = a+\frac{1}{2}\beta+\frac{\gamma-\beta}{6}.$$

Thus, the function $\Phi(a)$ is decreasing in $(-\infty, -(\gamma+3\beta)/8)$ and increasing in $(-(\gamma+3\beta)/8, \infty)$. Hence,

$$\inf_{-\infty<a<\infty} \Phi(a) = \Phi\left(-\frac{\gamma+3\beta}{8}\right) = \frac{\gamma+\beta}{8}. \tag{97}$$

Because of $w'(t) \leqslant 0$ in $[0, t_0]$ and $w'(t) \geqslant 0$ in $[t_0, 1]$, we have

$$\int_0^1 |w'(t)|^p\,dt = \frac{1}{p+1}\,\frac{\gamma^{p+1}+(-\beta)^{p+1}}{\gamma-\beta}. \tag{98}$$

Let

$$g(x) = (p+1)\left(\frac{3}{2}\right)^p(1-x)^p(1+x)-x^{p+1}-1;$$

hence

$$g'(x) = -(p+1)\left[\left(\frac{3}{2}\right)^p(1-x)^{p-1}(p-1+px+x)+x^p\right] < 0 \quad \text{for } 0 \leqslant x \leqslant 1.$$

Since $g(\frac{1}{3}) = (p+1)\frac{4}{3}-1-(\frac{1}{3})^{p+1} > 0$, we have $g(x) \geqslant 0$ in $[0, \frac{1}{3}]$. By the substitution $x = -\beta/\gamma$, we obtain

$$\frac{1}{p+1}\,\frac{\gamma^{p+1}+(-\beta)^{p+1}}{\gamma-\beta} \leqslant \left[\frac{3}{2}(\gamma+\beta)\right]^p,$$

and in virtue of (97) and (98), we have

$$\int_0^1 |w'(t)|^p dt \leqslant \left[\frac{3}{2}(\gamma+\beta)\right]^p \leqslant \left(12\int_0^1 |w(t)|\,dt\right)^p.$$

This proves our assertion in this case.

3° Let $\frac{1}{4} < t_0 \leqslant \frac{1}{2}$. As previously, we obtain

$$\Phi(a) = -\left(a+\frac{1}{2}\beta+\frac{\gamma-\beta}{6}\right) \quad \text{for } a \leqslant -\frac{\gamma+\beta}{2},$$

$$\Phi = \frac{1}{2}\beta+\frac{\gamma-\beta}{6}-\beta\tau-\frac{1}{2}(\gamma-\beta)\tau^2+\beta\tau^2+\frac{2}{3}(\gamma-\beta)\tau^3$$

$$\text{for } -\frac{\gamma+\beta}{2} < a \leqslant 0, \quad \text{where } 2t_0 \leqslant \tau = \frac{-\beta+\sqrt{\Delta}}{\gamma-\beta} < 1,$$

$$\Phi = \frac{1}{2}\beta+\frac{\gamma-\beta}{6}+\frac{1}{2}[\beta t_0+(\gamma-\beta)x^2](4x-1)-4\beta t_0 x-\frac{2}{3}(\gamma-\beta)x(3t_0^2+x^2)$$

$$\text{for } 0 < a < \frac{\beta^2}{2(\gamma-\beta)}, \quad \text{where } 0 < x = \frac{\sqrt{\Delta}}{\gamma-\beta} < t_0,$$

$$\Phi = a+\frac{1}{2}\beta+\frac{\gamma-\beta}{6} \quad \text{for } a \geqslant \frac{\beta^2}{2(\gamma-\beta)}.$$

In this case the function Φ attains its minimum at $x = \frac{1}{4}$, and

(99) $$\inf_{-\infty<a<\infty} \Phi(a) = \frac{5\gamma^2+5\beta^2+6\beta\gamma}{32(\gamma-\beta)}.$$

Let $h(x) = (1+x^{p+1})(1+x)^{p-1}-(p+1)2^p$ for $0 \leqslant x \leqslant 1$. Because of $h(1) = -p\cdot 2^p < 0$ and

$$h'(x) = (p+1)x^p(1+x)^{p-1}+(1+x^{p+1})(p-1)(1+x)^{p-2} > 0,$$

we have $h(x) < 0$ in $[0, 1]$. By the substitution $x = -\beta/\gamma$, we have

(100) $$\frac{1}{p+1}\,\frac{\gamma^{p+1}+(-\beta)^{p+1}}{\gamma-\beta} \leqslant \left(\frac{2\gamma^2}{\gamma-\beta}\right)^p.$$

For every x we have $5x^2-6x+5 \geqslant \frac{16}{5}$, or

$$\frac{5x^2-6x+5}{1+x} \geqslant \frac{16}{5}\,\frac{1}{1+x} \quad \text{for } x \geqslant 0.$$

By the substitution $x = -\beta/\gamma$, we have

(101) $$\frac{5\gamma^2+5\beta^2+6\beta\gamma}{\gamma-\beta} \geqslant \frac{16}{5}\,\frac{\gamma^2}{\gamma-\beta}.$$

By means of (100), (101) and (99), we obtain

$$\int_0^1 |w'(t)|^p dt = \frac{1}{p+1}\,\frac{\gamma^{p+1}+(-\beta)^{p+1}}{\gamma-\beta} \leqslant \left(\frac{2\gamma^2}{\gamma-\beta}\right)^p$$
$$\leqslant \left(2\cdot\frac{5}{16}\,\frac{5\gamma^2+5\beta^2+6\beta\gamma}{\gamma-\beta}\right)^p \leqslant \left(20\int_0^1 |w(t)|\,dt\right)^p.$$

In the remaining cases, i.e. $\frac{1}{2} < t_0 \leqslant \frac{3}{4}$, $\frac{3}{4} < t_0 < 1$, $t_0 \geqslant 1$, inequality (94) is also true.

THEOREM. *If $\varphi \in A_{n,l}$, $p \geqslant 1$, then $\|\varphi'\|_p \leqslant 40n\|\varphi\|_p$ and $\|\varphi'\| \leqslant 40n\|\varphi\|$.*

Proof. By means of the latter Lemma and the Hölder inequality we obtain

$$\int_0^1 |\varphi'(t)|^p dt = \sum_{k=0}^{n+l-1} \int_{\xi_k}^{\xi_{k+1}} |w_k'(t)|^p dt$$
$$\leqslant \sum_{k=0}^{n+l-1} \frac{1}{(\xi_{k+1}-\xi_k)^{2p-1}} \left(20\int_{\xi_k}^{\xi_{k+1}} |w_k(t)|\,dt\right)^p$$
$$\leqslant \sum_{k=0}^{n+l-1} \left(\frac{20}{\xi_{k+1}-\xi_k}\right)^p \int_{\xi_k}^{\xi_{k+1}} |w_k(t)|^p dt$$
$$\leqslant \max\left(\frac{20}{\xi_{k+1}-\xi_k}\right)^p \cdot \sum_{k=0}^{n+l-1} \int_{\xi_k}^{\xi_{k+1}} |w_k(t)|^p dt$$
$$= (40n)^p \int_0^1 |\varphi(t)|^p dt.$$

The second inequality is obtained immediately by taking $p \to \infty$.

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ADAM MICKIEWICZ UNIVERSITY, POZNAŃ
UNIWERSYTET IM. A. MICKIEWICZA, POZNAŃ

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