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Orlicz spaces based on families of measures

by

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Pitcher [8] introduced a condition on sets of probability measures (on a fixed measurable space) which is more general than the existence of a dominating measure, yet strong enough to give good results in certain statistical inference problems. In this connection, the Banach space E_p was defined; a generalization of this space, called $E_\Phi(\theta, G)$, is considered in the present paper (cf. [11]).

Since $E_\Phi(\theta, G)$ is also a generalization of Orlicz space, the reader is referred to [5] for background on Orlicz spaces. The first section of this paper is devoted to the definition of $E_\Phi(\theta, G)$, the proof that it is a Banach space, and discussion of its dependence on the parameters. In the next section are considered reflexivity of $E_\Phi(\theta, G)$ and "compactness" (of the unit ball in a certain weak topology) as an extension of the concept introduced in [8]. The paper concludes with some results concerned with convexity properties–rotundity and uniform rotundity of $E_\Phi(\theta, G)$.

1. Definition and general properties. Let (Ω, Σ) be a measurable space, and M a set of probability measures on (Ω, Σ). If f and g are Σ-measurable functions and $\mu \in M$, $f = g\,[\mu]$ means that $f = g$ μ-almost everywhere; f and g will be identified if $f = g\,[\mu]$ for all $\mu \in M$.

1.1. Definition. A set $A \subset \Omega$ is *M-null* if for every $\mu \in M$, A has μ-outer measure zero. Σ is *M-complete* if it contains every M-null set.

For any (Ω, Σ, M) which is mentioned hereafter, Σ will be assumed to be M-complete. This, of course, guarantees that if f is Σ-measurable, and g differs from f only on an M-null set, then g is Σ-measurable (and is identified with f).

Fixing (Ω, Σ, M), let Φ and θ be Young's functions, that is, symmetric non-negative convex functions on the line vanishing at the origin, and let $G: M \to (0, \infty)$. If S is the power set of M, and for any $A \in S$,

$$m_G(A) = \sum_{\mu \in A} G(\mu)$$

($= \infty$ for A uncountable), then clearly (M, S, m_G) is a measure space. Now consider the two Orlicz spaces

$$l_\theta^G \equiv L_\theta(M, S, m_G), \qquad L_\Phi(\mu) \equiv L_\Phi(\Omega, \Sigma, \mu),$$

where μ is any measure in M. For each $\mu \in M$, $L_\Phi(\mu)$ will be endowed with the Orlicz norm

$$\|f\|_\Phi^\mu = \sup\Big\{\Big|\int_\Omega fg\,d\mu\Big| : \int_\Omega \Psi(g)\,d\mu \leqslant 1\Big\},$$

where Ψ is (and will continue to denote) the Young function complementary to Φ, i.e.

$$\Psi(x) = \sup_{y\geqslant 0}\{|x|y - \Phi(y)\}.$$

On l_θ^G, the norm

$$N_\theta^G(F) = \inf\Big\{K>0 : \int_M \theta\Big(\frac{F}{K}\Big)dm_G \equiv \sum_{\mu\in M}\theta\Big(\frac{F(\mu)}{K}\Big)G(\mu) \leqslant 1\Big\}$$

will be used. It is known ([5], p. 70, 78) that with the respective norms, l_θ^G and $L_\Phi(\mu)$ are Banach spaces.

If $f \in \bigcap_{\mu\in M} L_\Phi(\mu)$ (i.e., for all $\mu\in M$, the μ-equivalence class $[f]_\mu \in L_\Phi(\mu)$), then $F_f(\mu) = \|f\|_\Phi^\mu$ defines a function $F_f\colon M\to[0,\infty)$. If $f, g\in \bigcap_{\mu\in M} L_\Phi(\mu)$ and α is real, then clearly

$$F_{f+g} \leqslant F_f + F_g \quad \text{and} \quad F_{\alpha f} = |\alpha| F_f.$$

1.2. Definition. $E_\Phi(\theta, G) = \{f\in \bigcap_{\mu\in M} L_\Phi(\mu) : F_f \in l_\theta^G\}$.

1.3. Lemma. *If $f\in E_\Phi(\theta, G)$, then*

$$\|f\|_\Phi^\mu \leqslant \theta^{-1}\Big(\frac{1}{G(\mu)}\Big) N_\theta^G(F_f)$$

for all $\mu\in M$, where as usual, for $a\geqslant 0$, $\theta^{-1}(a) = \sup\{x\geqslant 0 : \theta(x)\leqslant a\}$.

Proof. Since

$$N_\theta^G(F_f) = \inf\Big\{K>0 : \sum_{\mu\in M}\theta\Big(\frac{F_f(\mu)}{K}\Big)G(\mu)\leqslant 1\Big\},$$

it follows that for every $\varepsilon > 0$ and every $\mu_0\in M$,

$$\theta\Big(\frac{F_f(\mu_0)}{N_\theta^G(F_f)+\varepsilon}\Big)G(\mu_0) \leqslant \sum_{\mu\in M}\theta\Big(\frac{F_f(\mu)}{N_\theta^G(F_f)+\varepsilon}\Big)G(\mu)\leqslant 1,$$

so that

$$\|f\|_\Phi^{\mu_0} = F_f(\mu_0) \leqslant [N_\theta^G(F_f)+\varepsilon]\,\theta^{-1}\Big(\frac{1}{G(\mu_0)}\Big).$$

Since ε is arbitrary, the result follows, q.e.d.

1.4. Theorem. *If $N_\Phi^{\theta,G}(f) \equiv N_\theta^G(F_f)$, then $N_\Phi^{\theta,G}$ is a norm on $E_\Phi(\theta, G)$ under which this space is complete; i.e., $E_\Phi(\theta, G)$ is a Banach space.*

Proof. If $f, g\in E_\Phi(\theta, G)$ and α is real,

$$N_\theta^G(F_{f+g}) \leqslant N_\theta^G(F_f+F_g) \leqslant N_\theta^G(F_f) + N_\theta^G(F_g) < \infty$$

and

$$N_\theta^G(F_{\alpha f}) = N_\theta^G(|\alpha| F_f) = |\alpha| N_\theta^G(F_f) < \infty$$

so that $E_\Phi(\theta, G)$ is a linear space. In fact, this shows that $N_\Phi^{\theta,G}$ is a seminorm on $E_\Phi(\theta, G)$. To show that $N_\Phi^{\theta,G}$ is actually a norm, suppose $N_\Phi^{\theta,G}(f) = 0$, i.e. $N_\theta^G(F_f) = 0$. Since N_θ^G is a norm, $F_f\equiv 0$, i.e., for all $\mu\in M$, $\|f\|_\Phi^\mu = 0$. But this implies $f = 0$ $[\mu]$ for all $\mu \in M$, so that $f = 0$.

Now suppose $\{f_n\}\subset E_\Phi(\theta, G)$ and $N_\Phi^{\theta,G}(f_n - f_m)\to 0$ as $n, m\to\infty$; it can be assumed, by extracting a subsequence if necessary, that

$$\sum_{n=1}^{\infty} N_\Phi^{\theta,G}(f_n - f_{n+1}) < \infty.$$

By 1.3,

$$\|f_n - f_{n+1}\|_\Phi^\mu \leqslant \theta^{-1}\Big(\frac{1}{G(\mu)}\Big) N_\Phi^{\theta,G}(f_n - f_{n+1})$$

for each $\mu\in M$, which implies that

$$\sum_{n=1}^{\infty}\|f_n - f_{n+1}\|_\Phi^\mu \leqslant \theta^{-1}\Big(\frac{1}{G(\mu)}\Big)\sum_{n=1}^{\infty} N_\Phi^{\theta,G}(f_n - f_{n+1}) < \infty.$$

The Hölder inequality ([5], p. 80) implies that, for each $\mu\in M$,

$$\int_\Omega |f_n - f_{n+1}|\,d\mu \leqslant \|f_n - f_{n+1}\|_\Phi^\mu N_\Psi^\mu(\chi_\Omega) = \frac{1}{\Psi^{-1}(1)}\|f_n - f_{n+1}\|_\Phi^\mu.$$

Thus if $p_k(\mu)$ is so large that

$$\sum_{n=p_k}^{\infty}\|f_n - f_{n+1}\|_\Phi^\mu < \Psi^{-1}(1)\,4^{-k},$$

then

$$\sum_{n=p_k}^{\infty}\int_\Omega |f_n - f_{n+1}|\,d\mu < 4^{-k},$$

so that for $m > p_k(\mu)$,

$$\mu\{\omega : |f_{p_k}(\omega) - f_{p_k+1}(\omega)| + \ldots + |f_{m-1}(\omega) - f_m(\omega)| \geqslant 2^{-k}\} < 2^{-k}.$$

If

$$E_{k,m} = \{\omega\colon |f_{p_k}(\omega)-f_{p_k+1}(\omega)|+\ldots+|f_{m-1}(\omega)-f_m(\omega)| \geqslant 2^{-k}\}$$

and

$$E_\mu = \bigcap_{k=1}^{\infty} \bigcup_{m=p_k+1}^{\infty} E_{k,m},$$

then $\mu(E_\mu) = 0$, and it is easily seen that $\{f_n\}$ is pointwise Cauchy on $\Omega - E_\mu$. Hence there is an f such that $f_n \to f$ except on an M-null set (namely, $\bigcap_{\mu\in M} E_\mu$), and it follows by M-completeness that f is Σ-measurable.

However, for each $\mu \in M$, $L_\Phi(\mu)$ is complete under $\|\cdot\|_\Phi^\mu$, so there is a function $f_\mu \in L_\Phi(\mu)$ with $\|f_n - f_\mu\|_\Phi^\mu \to 0$ as $n\to\infty$. This implies

$$\int_\Omega |f_n - f_\mu|\, d\mu \to 0,$$

which in turn implies that $f_n \to f_\mu$ in μ-measure, so that there exists a subsequence $\{f_{n_k}\}$ with $f_{n_k} \to f_\mu$, μ-almost everywhere, as $k\to\infty$. It must therefore be $f_\mu = f[\mu]$, whence $f \in \bigcap_{\mu\in M} L_\Phi(\mu)$ and

$$\|f_n - f\|_\Phi^\mu \to 0.$$

Since $0 \leqslant F_n \uparrow F$ (pointwise) implies $N_\theta^G(F_n) \uparrow N_\theta^G(F)$, and for all $\mu \in M$,

$$F_{(f_n-f)}(\mu) = \|f_n - f\|_\Phi^\mu = \lim_{m\to\infty} \|f_n - f_m\|_\Phi^\mu = \lim_{m\to\infty} F_{(f_n-f_m)}(\mu) \leqslant \sum_{k=n}^{\infty} F_{(f_k-f_{k+1})}(\mu),$$

one obtains

$$N_\Phi^{\theta,G}(f_n - f) = N_\theta^G(F_{(f_n-f)}) \leqslant N_\theta^G\Big(\sum_{k=n}^{\infty} F_{(f_k-f_{k+1})}\Big)$$
$$= \lim_{m\to\infty} N_\theta^G\Big(\sum_{k=n}^{m} F_{(f_k-f_{k+1})}\Big) \leqslant \lim_{m\to\infty} \sum_{k=n}^{m} N_\theta^G(F_{(f_k-f_{k+1})})$$
$$= \sum_{k=n}^{\infty} N_\theta^G(F_{(f_k-f_{k+1})}) = \sum_{k=n}^{\infty} N_\Phi^{\theta,G}(f_k - f_{k+1}).$$

But the last expression goes to zero as $n\to\infty$, i.e. $f \in E_\Phi(\theta, G)$ and $N_\Phi^{\theta,G}(f_n - f) \to 0$, q.e.d.

$E_\Phi(\theta,G)$ will be called the *Orlicz space based on M relative to* (Φ, θ, G). Clearly, if $M = \{\mu_0\}$, then $E_\Phi(\theta, G) = L_\Phi(\mu_0)$, the ordinary Orlicz space.

$B_\Phi(\theta, G)$ will denote the unit ball $\{f\colon N_\Phi^{\theta,G}(f) \leqslant 1\}$. It is easily seen that $|f| \leqslant |g|$ implies $N_\Phi^{\theta,G}(f) \leqslant N_\Phi^{\theta,G}(g)$; hence every ball $aB_\Phi(\theta, G)$ is solid.

1.5. THEOREM. (a) *If $\Phi_1(x) \leqslant \Phi_2(cx)$ for all $x \geqslant x_0$ $(c > 0,\ x_0 \geqslant 0)$, then*

$$N_{\Phi_1}^{\theta,G}(\cdot) \leqslant c\big[1 + \Psi_2(\varphi_2(cx_0))\big]\, N_{\Phi_2}^{\theta,G}(\cdot),$$

where Ψ_2 is the complementary function to Φ_2 and where φ_2 is the left derivative of Φ_2. Hence $E_{\Phi_2}(\theta, G) \subset E_{\Phi_1}(\theta, G)$, whenever $\Psi_2(\varphi_2(cx_0)) < \infty$.

(b) *If $\theta_1(x) \leqslant \theta_2(cx)$ for all $x \geqslant 0$ $(c > 0)$, then $N_\Phi^{\theta_1,G}(\cdot) \leqslant cN_\Phi^{\theta_2,G}(\cdot)$. Hence $E_\Phi(\theta_2, G) \subset E_\Phi(\theta_1, G)$.*

(c) *If $G_1 \leqslant cG_2$, then $N_\Phi^{\theta,G_1}(\cdot) \leqslant \max(c,1)\, N_\Phi^{\theta,G_2}(\cdot)$, so that $E_\Phi(\theta, G_2) \subset E_\Phi(\theta, G_1)$.*

Proof. (a) According to [5], p. 113, for every $\mu \in M$,

$$\|\cdot\|_{\Phi_1}^\mu \leqslant c\big[1 + \mu(\Omega)\,\Psi_2(\varphi_2(cx_0))\big]\, \|\cdot\|_{\Phi_2}^\mu.$$

Since $\mu(\Omega) = 1$, applying N_θ^G to both sides gives the desired result.

(b) If $\theta_1(x) \leqslant \theta_2(cx)$, then

$$N_\Phi^{\theta_1,G}(f) = \inf\Big\{K > 0\colon \sum_{\mu\in M} \theta_1\Big(\frac{\|f\|_\Phi^\mu}{K}\Big) G(\mu) \leqslant 1\Big\}$$
$$\leqslant \inf\Big\{K > 0\colon \sum_{\mu\in M} \theta_2\Big(\frac{c\|f\|_\Phi^\mu}{K}\Big) G(\mu) \leqslant 1\Big\} = cN_\Phi^{\theta_2,G}(f).$$

(c) $G_1(\mu) \leqslant cG_2(\mu)$ for all $\mu \in M$ implies, for $a = \max(c, 1)$, $b = N_\Phi^{\theta,G_2}(f)$,

$$\sum_{\mu\in M} \theta\Big(\frac{\|f\|_\Phi^\mu}{ab}\Big) G_1(\mu) \leqslant c \sum_{\mu\in M} \theta\Big(\frac{\|f\|_\Phi^\mu}{ab}\Big) G_2(\mu)$$
$$\leqslant \frac{c}{a} \sum_{\mu\in M} \theta\Big(\frac{\|f\|_\Phi^\mu}{b}\Big) G_2(\mu) \leqslant \frac{c}{a} \leqslant 1,$$

and so $N_\Phi^{\theta,G_1}(f) \leqslant ab = \max(c,1)\, N_\Phi^{\theta,G_2}(f)$, q.e.d.

When

$$\theta(x) = \begin{cases} 0, & |x| \leqslant 1, \\ \infty, & |x| > 1, \end{cases}$$

l_θ^G consists of the bounded functions, in which case $E_\Phi(\theta, G)$ consists of the functions f in $\bigcap_{\mu\in M} L_\Phi(\mu)$ such that

$$\sup\{\|f\|_\Phi^\mu\colon \mu \in M\} < \infty;$$

in fact, $N_\Phi^{\theta,G}(f) = \sup\{\|f\|_\Phi^\mu\colon \mu\in M\}$. This space $E_\Phi(\theta, G)$ (which does not depend on G) will be denoted merely by E_Φ (norm by $N_\Phi(\cdot)$, unit ball by B_Φ). In particular, if $\Phi(x) = |x|^p$, $1 \leqslant p < \infty$, then $E_\Phi = E_p$ of [8].

1.6. PROPOSITION. *If either θ is discontinuous or $1/G$ is bounded, then $E_\Phi(\theta, G) \subset E_\Phi$.*

Proof. Observe that θ is discontinuous iff θ^{-1} is bounded, and that in any case θ^{-1} is finite and non-decreasing. It follows easily that the above hypothesis is equivalent to: $\theta^{-1}(1/G(\cdot))$ is bounded. This, together with 1.3, implies that

$$\sup\{\|f\|_\Phi^\mu\colon \mu \in M\} \leqslant \beta N_\Phi^{\theta,G}(f),$$

where

$$\beta = \sup\left\{\theta^{-1}\left(\frac{1}{G(\mu)}\right):\ \mu\in M\right\} < \infty;$$

i.e. $E_\Phi(\theta, G) \subset E_\Phi$, q.e.d.

Two special cases of Φ should be discussed here, as they clarify the situations in what follows. Define

$$\Phi_\infty(x) = \begin{cases} 0, & |x| \leqslant 1, \\ \infty, & |x| > 1. \end{cases}$$

The complementary function to Φ_∞ is given by

$$\Phi_1(x) = |x| \quad \text{for all } x.$$

Of course, $L_{\Phi_\infty} = L_\infty$ and $L_{\Phi_1} = L_1$.

If Φ is discontinuous, then by [12], p. 82, $L_\Phi(\mu) = L_\infty(\mu)$ with $a\|\cdot\|_\infty^\mu \leqslant \|\cdot\|_\Phi^\mu \leqslant b\|\cdot\|_\infty^\mu$ for all $\mu\in M$ and some $a, b > 0$ which (since $\mu(\Omega) = 1$ for all μ) do not depend on μ. Hence $E_\Phi(\theta, G) = E_\infty(\theta, G)$ with $aN_\infty^{\theta,G}(\cdot) \leqslant N_\Phi^{\theta,G}(\cdot) \leqslant bN_\infty^{\theta,G}(\cdot)$. For this reason, *when Φ is discontinuous, it can (and will) be replaced by Φ_∞*. If $\Phi(x)/|x|$ is bounded, then $L_\Phi(\mu) = L_1(\mu)$ ([12], p. 82) with $c\|\cdot\|_1^\mu \leqslant \|\cdot\|_\Phi^\mu \leqslant d\|\cdot\|_1^\mu$ for all $\mu\in M$ and some $c, d > 0$ independent of μ. Hence $E_\Phi(\theta, G) = E_1(\theta, G)$ with $cN_1^{\theta,G}(\cdot) \leqslant N_\Phi^{\theta,G}(\cdot) \leqslant dN_1^{\theta,G}(\cdot)$. Thus any Φ with $\Phi(x)/|x|$ bounded can (and will) be replaced by Φ_1.

Note that 1.5 (a) is of no consequence when $\Psi_2(\varphi_2(cx_0)) = \infty$; but since by the equality in Young's inequality, $\Psi_2(\varphi_2(cx_0)) = cx_0\varphi_2(cx_0) - \Phi_2(cx_0)$, this can happen only if Φ_2 is discontinuous. The preceding remarks show that, in this case, $a\|\cdot\|_\infty^\mu \leqslant \|\cdot\|_{\Phi_2}^\mu \leqslant b\|\cdot\|_\infty^\mu$. Also, by [12], p. 82, $L_\infty(\mu) \subset L_\Phi(\mu)$ for *any* Φ and any $\mu\in M$, with $\|\cdot\|_\Phi^\mu \leqslant c\|\cdot\|_\infty^\mu$ for some $c > 0$ independent of μ. Therefore

$$\|\cdot\|_\Phi^\mu \leqslant c\|\cdot\|_\infty^\mu \leqslant \frac{c}{a}\|\cdot\|_{\Phi_2}^\mu, \qquad N_\Phi^{\theta,G}(\cdot) \leqslant cN_\infty^{\theta,G}(\cdot) \leqslant \frac{c}{a} N_{\Phi_2}^{\theta,G}(\cdot),$$

and

$$E_{\Phi_2}(\theta, G) \subset E_\Phi(\theta, G)$$

(compare with 1.5 (a)).

1.7. LEMMA. $\|f\|_\Phi^\mu = \inf\limits_{K>0} \frac{1}{K}\left(1 + \int_\Omega \Phi(Kf)\,d\mu\right)$.

Proof. This has been proved ([5], p. 92) for the case Φ continuous,

$$\lim_{x\to 0}\frac{\Phi(x)}{|x|} = 0, \qquad \lim_{x\to\infty}\frac{\Phi(x)}{|x|} = \infty.$$

If Φ is discontinuous, i.e., (by convention) $\Phi = \Phi_\infty$, then

$$\begin{aligned}\inf_{K>0}\frac{1}{K}\left(1 + \int_\Omega \Phi(Kf)\,d\mu\right) &= \inf_{K>0}\frac{1}{K}\left(1 + \int_{|f|>1/K}\infty\,d\mu\right) \\ &= \inf\left\{\frac{1}{K}:\ \mu\left[|f| > \frac{1}{K}\right] = 0\right\} = \mu\text{-ess sup}\,|f| \\ &= \|f\|_\infty^\mu = \|f\|_\Phi^\mu.\end{aligned}$$

If Φ is continuous,

$$\lim_{x\to\infty}\frac{\Phi(x)}{|x|} = \infty,$$

but

$$\lim_{x\to 0}\frac{\Phi(x)}{|x|} = a > 0,$$

then define

$$\tilde\Phi(x) = \Phi(x) - a|x|$$

(clearly a Young function), so that $\tilde\Phi$ is continuous,

$$\lim_{x\to\infty}\frac{\tilde\Phi(x)}{|x|} = \infty, \qquad \lim_{x\to 0}\frac{\tilde\Phi(x)}{|x|} = a - a = 0.$$

Now

$$\begin{aligned}\inf_{K>0}\frac{1}{K}\left(1 + \int_\Omega \Phi(Kf)\,d\mu\right) &= \inf_{K>0}\frac{1}{K}\left(1 + \int_\Omega \tilde\Phi(Kf)\,d\mu + aK\int_\Omega |f|\,d\mu\right) \\ &= \inf_{K>0}\frac{1}{K}\left(1 + \int_\Omega \tilde\Phi(Kf)\,d\mu\right) + a\int_\Omega |f|\,d\mu \\ &= \|f\|_{\tilde\Phi}^\mu + a\int_\Omega |f|\,d\mu.\end{aligned}$$

An easy computation shows that $\Psi(x) = \tilde\Psi(|x| - a)$, so that also

$$\begin{aligned}\|f\|_\Phi^\mu &= \sup\left\{\int_\Omega |fg|\,d\mu:\ \int_\Omega \Psi(g)\,d\mu \leqslant 1\right\} = \sup\left\{\int_\Omega |fg|\,d\mu:\ \int_\Omega \tilde\Psi(|g| - a)\,d\mu \leqslant 1\right\} \\ &= \sup\left\{\int_\Omega |f|(|g| + a)\,d\mu:\ \int_\Omega \tilde\Psi(g)\,d\mu \leqslant 1\right\} = \|f\|_{\tilde\Phi}^\mu + a\int_\Omega |f|\,d\mu.\end{aligned}$$

Finally, if

$$\lim_{x\to\infty}\frac{\Phi(x)}{|x|} < \infty,$$

then $\Phi(x)/|x|$ is bounded, i.e. (by convention) $\Phi = \Phi_1$. Then

$$\inf_{K>0}\frac{1}{K}\Big(1+\int_\Omega \Phi(Kf)\,d\mu\Big) = \inf_{K>0}\frac{1}{K}\Big(1+K\int_\Omega |f|\,d\mu\Big)$$
$$= \inf_{K>0}\frac{1}{K}+\int_\Omega |f|\,d\mu = \int_\Omega |f|\,d\mu$$
$$= \|f\|_1^\mu = \|f\|_\Phi^\mu, \qquad \text{q.e.d.}$$

1.8. Lemma. *If $\mu, \mu' \in M$, $0 \leqslant \alpha \leqslant 1$ and $\nu = \alpha\mu+(1-\alpha)\mu'$, then*

$$\alpha\|f\|_\Phi^\mu+(1-\alpha)\|f\|_\Phi^{\mu'} \leqslant \|f\|_\Phi^\nu \leqslant \|f\|_\Phi^\mu+\|f\|_\Phi^{\mu'}$$

for any $f \in L_\Phi(\mu) \cap L_\Phi(\mu')$.

Proof. By definition,

$$\|f\|_\Phi^\nu = \sup\Big\{\alpha\int_\Omega |fg|\,d\mu+(1-\alpha)\int_\Omega |fg|\,d\mu' : \alpha\int_\Omega \Psi(g)\,d\mu+(1-\alpha)\int_\Omega \Psi(g)\,d\mu' \leqslant 1\Big\}$$
$$\leqslant \alpha\sup\Big\{\int_\Omega |fg|\,d\mu : \alpha\int_\Omega \Psi(g)\,d\mu \leqslant 1\Big\}+$$
$$+(1-\alpha)\sup\Big\{\int_\Omega |fg|\,d\mu' : (1-\alpha)\int_\Omega \Psi(g)\,d\mu' \leqslant 1\Big\}$$
$$\leqslant \sup\Big\{\int_\Omega |f(\alpha g)|\,d\mu : \int_\Omega \Psi(\alpha g)\,d\mu \leqslant 1\Big\}+$$
$$+\sup\Big\{\int_\Omega \big|f\big((1-\alpha)g\big)\big|\,d\mu' : \int_\Omega \Psi\big((1-\alpha)g\big)\,d\mu' \leqslant 1\Big\} = \|f\|_\Phi^\mu+\|f\|_\Phi^{\mu'}.$$

On the other hand, using 1.7,

$$\|f\|_\Phi^\nu = \inf_{K>0}\frac{1}{K}\Big\{\alpha\Big(1+\int_\Omega \Phi(Kf)\,d\mu\Big)+(1-\alpha)\Big(1+\int_\Omega \Phi(Kf)\,d\mu'\Big)\Big\}$$
$$\geqslant \alpha\inf_{K>0}\frac{1}{K}\Big(1+\int_\Omega \Phi(Kf)\,d\mu\Big)+(1-\alpha)\inf_{K>0}\frac{1}{K}\Big(1+\int_\Omega \Phi(Kf)\,d\mu'\Big)$$
$$= \alpha\|f\|_\Phi^\mu+(1-\alpha)\|f\|_\Phi^{\mu'}, \quad \text{q.e.d.}$$

1.9. Proposition. *If M contains at least two distinct measures and is convex (i.e., $\mu, \mu' \in M$, $0 \leqslant \alpha \leqslant 1$ imply $\alpha\mu+(1-\alpha)\mu' \in M$, so that M has infinitely many distinct members), then*

$$N_\Phi^{\theta,G}(f) = \inf\{K>0 : \sup_{\mu\in M}\|f\|_\Phi^\mu \leqslant K\theta^{-1}(0)\}.$$

Consequently, $E_\Phi(\theta, G) = E_\Phi$ whenever $\theta^{-1}(0) > 0$, and $E_\Phi(\theta, G) = \{0\}$ when $\theta^{-1}(0) = 0$.

Proof. Note that

$$\sup_{\mu\in M}\|f\|_\Phi^\mu \leqslant K\theta^{-1}(0)$$

iff $\theta(\|f\|_\Phi^\mu/K) = 0$ for all $\mu \in M$, i.e. $\theta\big(F_f(\cdot)/K\big) \equiv 0$. It will suffice to show that $\theta\big(F_f(\cdot)/K\big) \not\equiv 0$ if $K < N_\Phi^{\theta,G}(f)$ and $\theta\big(F_f(\cdot)/K\big) \equiv 0$ if $K > N_\Phi^{\theta,G}(f)$. Thus, suppose $\theta\big(F_f(\cdot)/K\big) \equiv 0$; it follows that

$$\sum_{\mu\in M}\theta\Big(\frac{F_f(\mu)}{K}\Big)G(\mu) = 0 \leqslant 1,$$

so that $K \geqslant N_\Phi^{\theta,G}(f)$. Conversely, let $\theta\big(F_f(\mu_0)/K\big) > 0$ for some $\mu_0 \in M$, and fix $\beta \in (0,1)$. Then for $\mu' \in M$, $\mu' \neq \mu_0$, and any $\alpha \in [\beta, 1]$, use of 1.8 gives

$$\theta\Big(\frac{F_f\big(\alpha\mu_0+(1-\alpha)\mu'\big)}{\beta K}\Big) \geqslant \theta\Big(\frac{F_f\big(\alpha\mu_0+(1-\alpha)\mu'\big)}{\alpha K}\Big) \geqslant \theta\Big(\frac{\alpha F_f(\mu_0)}{\alpha K}\Big) > 0.$$

Since infinitely many measures in M, of the form $\alpha\mu_0+(1-\alpha)\mu'$, are obtained by varying α in $[\beta, 1]$, it is seen that

$$\sum_{\mu\in M}\theta\Big(\frac{F_f(\mu)}{\beta K}\Big)G(\mu) = \infty.$$

Hence $N_\Phi^{\theta,G}(f) \geqslant \beta K$ for all $\beta \in (0,1)$, which implies $K \leqslant N_\Phi^{\theta,G}(f)$.

If $\theta^{-1}(0) > 0$, then obviously

$$N_\Phi^{\theta,G}(f) = \frac{1}{\theta^{-1}(0)}\sup_{\mu\in M}\|f\|_\Phi^\mu,$$

so that $E_\Phi(\theta, G) = E_\Phi$. If $\theta^{-1}(0) = 0$, it is clear that $N_\Phi^{\theta,G}(f) = 0$, i.e. $E_\Phi(\theta, G) = \{0\}$, q.e.d.

2. Compactness and reflexivity. In this section a compactness condition for the set of measures M will be introduced, and then various properties of $E_\Phi(\theta, G)$, including reflexivity, will be investigated.

2.1. Lemma. *If $\mu \in M$ and $h \in L_\Psi(\mu)$, then $l(h,\mu): E_\Phi(\theta, G) \to R$ defined by $l(h,\mu)(f) = \int_\Omega fh\,d\mu$ is a continuous linear functional, and*

$$\|l(h,\mu)\| \leqslant \theta^{-1}\Big(\frac{1}{G(\mu)}\Big)N_\Psi^\mu(h).$$

Proof. $l(h,\mu)$ is obviously linear. By Hölder's inequality and 1.3,

$$|l(h,\mu)(f)| = \Big|\int_\Omega fh\,d\mu\Big| \leqslant \|f\|_\Phi^\mu N_\Psi^\mu(h) \leqslant \Big[\theta^{-1}\Big(\frac{1}{G(\mu)}\Big)N_\Psi^\mu(h)\Big]N_\Phi^{\theta,G}(f).$$

Hence

$$\|l(h,\mu)\| = \sup\{|l(h,\mu)(f)| : N_\Phi^{\theta,G}(f) \leqslant 1\} \leqslant \theta^{-1}\Big(\frac{1}{G(\mu)}\Big)N_\Psi^\mu(h), \qquad \text{q.e.d.}$$

2.2. Definition. $\mathscr{E}_\Psi$ denotes the (real) linear space spanned by the functionals $l(h, \mu)$ for $\mu \in M$ and $h \in K_\Psi(\mu)$, where $K_\Psi(\mu)$ is the closed subspace determined by the bounded functions in $L_\Psi(\mu)$.

2.3. Theorem. *$\mathscr{E}_\Psi$ is a total subspace of $(E_\Phi(\theta, G))^*$, the adjoint space of $E_\Phi(\theta, G)$. Thus $E_\Phi(\theta, G)$ is a Hausdorff space under the $\mathscr{E}_\Psi$-topology of $E_\Phi(\theta, G)$.*

Proof. By 2.1, $\mathscr{E}_\Psi \subset (E_\Phi(\theta, G))^*$. If $l(h, \mu)(f) = 0$ for all $\mu \in M$, $h \in K_\Psi(\mu)$, then, in particular, $\int_\Omega f\varkappa_A \, d\mu = 0$ for all $\mu \in M$ and $A \in \Sigma$ which implies $f = 0$ $[\mu]$ for all $\mu \in M$, i.e., $f = 0$. In other words, $\mathscr{E}_\Psi$ is a total subspace. Since any topology induced by a total subspace of the dual space is Hausdorff, the second statement follows, q.e.d.

2.4. Definition. (Ω, Σ, M) is said to be *(θ, G)-compact* if and only if for some Young's function Φ, the unit ball $B_\Phi(\theta, G)$ of $E_\Phi(\theta, G)$ is $\mathscr{E}_\Psi$-compact.

It will be shown that under a mild restriction on (θ, G), every (Ω, Σ, M) where M is a *dominated* set of measures, is (θ, G)-compact. A special case is defined by Pitcher [8], when θ is a two-valued (0 and ∞) Young function. Pitcher also demonstrates that the compactness condition is really more general than domination.

For any $\mu \in M$, $c \geqslant 0$, $cB_\Phi(\mu)$ will denote the set

$$\{f \in L_\Phi(\mu) \colon \|f\|_\Phi^\mu \leqslant c\}.$$

Writing c_μ for $\theta^{-1}(1/G(\mu))$, 1.3 takes the form

$$B_\Phi(\theta, G) \subset \bigcap_{\mu \in M} c_\mu B_\Phi(\mu).$$

2.5. Definition. $\Delta_\Phi^{\theta,G}$, or simply Δ, is the diagonal map of $B_\Phi(\theta, G)$ into $\prod_{\mu \in M} c_\mu B_\Phi(\mu)$, i.e., $\Delta_\Phi^{\theta,G}(f) = \{f_\mu\}_{\mu \in M}$, where $f_\mu = f[\mu]$.

In the following, for each $\mu \in M$, the topology considered on $c_\mu B_\Phi(\mu)$ is the weak topology induced by the $l(h, \mu)$, $h \in K_\Psi(\mu)$; $\prod_{\mu \in M} c_\mu B_\Phi(\mu)$ is then endowed with the corresponding product topology.

2.6. Lemma. *If $B_\Phi(\theta, G)$ is $\mathscr{E}_\Psi$-compact, then $\Delta(B_\Phi(\theta, G))$ is closed in the product topology of $\prod_{\mu \in M} c_\mu B_\Phi(\mu)$. The converse holds providing Ψ is continuous.*

Proof. Clearly, Δ is a one-to-one map. Consider the topology induced on $\Delta(B_\Phi(\theta, G))$ by defining $U \subset \Delta(B_\Phi(\theta, G))$ to be open if and only if $\Delta^{-1}(U)$ is $\mathscr{E}_\Psi$-open. In this topology, a base at 0 consists of sets of the form

$$\{\{f_\mu\}_{\mu \in M} \colon f_\mu = f[\mu], f \in B_\Phi(\theta, G), |l(h_i, \mu_i)(f)| < \varepsilon, i = 1, \ldots, n\},$$

where $h_i \in K_\Psi(\mu_i)$. But these sets are just the intersections with $\Delta(B_\Phi(\theta, G))$ of the sets in the base at 0 of the product topology, i.e.,

$$\{\{f_\mu\}_{\mu \in M} \colon |l(h_i, \mu_i)(f_{\mu_i})| < \varepsilon, i = 1, \ldots, n\}.$$

Hence the given topology is the restriction to $\Delta(B_\Phi(\theta, G))$ of the product topology. So if $B_\Phi(\theta, G)$ is $\mathscr{E}_\Psi$-compact, then $\Delta(B_\Phi(\theta, G))$ is compact in the product topology (Δ being continuous); but since each $c_\mu B_\Phi(\mu)$ is Hausdorff, so is $\prod_{\mu \in M} c_\mu B_\Phi(\mu)$, and $\Delta(B_\Phi(\theta, G))$ is closed.

If Ψ is continuous, then for $\mu \in M$ every continuous linear functional on $K_\Psi(\mu)$ is of the form ([5], p. 128)

$$l^*(f, \mu)(h) = \int_\Omega fh \, d\mu, \quad f \in L_\Phi(\mu),$$

and the correspondence $f \to l^*(f, \mu)$ is an isometric isomorphism of $L_\Phi(\mu)$ with $(K_\Psi(\mu))^*$. Since $l^*(f, \mu)(h) = l(h, \mu)(f)$, it is seen that the given topology on $B_\Phi(\mu)$ is equivalent to the weak*-topology on the unit ball of $(K_\Psi(\mu))^*$, in which that unit ball is compact, by Alaoglu's Theorem. Thus $c_\mu B_\Phi(\mu)$ is compact and, by Tychonoff's Theorem, so is $\prod_{\mu \in M} c_\mu B_\Phi(\mu)$. It follows that if $\Delta(B_\Phi(\theta, G))$ is closed in the product topology, then it is compact, and (since Δ^{-1} is continuous) $B_\Phi(\theta, G)$ is $\mathscr{E}_\Psi$-compact, q.e.d.

The requirement that Ψ be continuous, for the converse part of the preceding lemma, is essential, for the following reason. When Ψ is discontinuous, then, as has been noted, $K_\Psi(\mu) = L_\infty(\mu)$ for each $\mu \in M$, and ([4], p. 296) the dual space $(L_\infty(\mu))^*$ is isometrically isomorphic with the space $ba(\mu)$ of finitely additive (real-valued) set functions on the μ-completion of Σ which are of bounded variation and vanish on μ-null sets. The $L_\infty(\mu)$-topology on $ba(\mu)$ is thus the weak*-topology (hence Hausdorff). $B_1(\mu)$, the unit ball of $L_1(\mu)$, is a subset of $ba(\mu)$ (under isometric isomorphism); in order for $B_1(\mu)$ to be $L_\infty(\mu)$-compact, it must therefore be an $L_\infty(\mu)$-closed subset of $ba(\mu)$. But an example given by Doob [3], p. 631, shows this need not be true.

2.7. Lemma. *If $\{\mu_1, \ldots, \mu_n\} \subset M$, $f \in \bigcap_{i=1}^n L_\Phi(\mu_i)$, and if for all $\varepsilon > 0$ and $h_i \in K_\Psi(\mu_i)$, $i = 1, \ldots, n$, there is a $g \in B_\Phi(\theta, G)$ with*

$$\Big|\int_\Omega (f-g) h_i \, d\mu_i\Big| < \varepsilon, \quad i = 1, \ldots, n,$$

then

$$\sum_{i=1}^n \theta(\|f\|_\Phi^{\mu_i}) G(\mu_i) \leqslant 1.$$

Proof. Choose δ in $(0,1)$; then by definition, for each $i = 1, \ldots, n$, there is an $h_i^\delta \in K_\Psi(\mu_i)$ with $\int_\Omega \Psi(h_i^\delta)\,d\mu_i \leqslant 1$, satisfying

$$\Big|\int_\Omega f h_i^\delta d\mu_i\Big| > \delta \|f\|_\Phi^{\mu_i},$$

since ([5], p. 87) the latter space is norm-determining. Now given $\varepsilon > 0$, there is by hypothesis, a $g \in B_\Phi(\theta, G)$ with

$$\Big|\int_\Omega (f-g)\, h_i^\delta d\mu_i\Big| < \varepsilon, \qquad i = 1, \ldots, n.$$

Combining the above,

$$\delta\|f\|_\Phi^{\mu_i} < \Big|\int_\Omega f h_i^\delta d\mu_i\Big| \leqslant \Big|\int_\Omega (f-g)\, h_i^\delta d\mu_i\Big| + \Big|\int_\Omega g h_i^\delta d\mu_i\Big| < \varepsilon + \|g\|_\Phi^{\mu_i}.$$

Then

$$\sum_{i=1}^n \theta(\delta^2\|f\|_\Phi^{\mu_i})\, G(\mu_i) \leqslant \sum_{i=1}^n \theta(\delta\|g\|_\Phi^{\mu_i} + \delta\varepsilon) G(\mu_i).$$

But since $g \in B_\Phi(\theta, G)$, Lemma 1.3 implies that

$$\delta\|g\|_\Phi^{\mu_i} \leqslant \delta\theta^{-1}\left(\frac{1}{G(\mu_i)}\right) \leqslant \delta\alpha,$$

where

$$\alpha \equiv \max_{1\leqslant i\leqslant n} \theta^{-1}\left(\frac{1}{G(\mu_i)}\right) \leqslant \beta = \sup\{x \geqslant 0 : \theta(x) < \infty\}$$

(the "jump point" of θ, if it jumps), and so $\delta\alpha < \beta$. Since δ does not depend on ε, one can take $\varepsilon < \beta/\delta - \alpha$. θ is convex and continuous on $(0, \beta)$, so $\theta(\delta\|g\|_\Phi^{\mu_i} + \delta\varepsilon) - \theta(\delta\|g\|_\Phi^{\mu_i}) \leqslant \theta(\delta\alpha + \delta\varepsilon) - \theta(\delta\alpha)$, and

$$\sum_{i=1}^n \theta(\delta^2\|f\|_\Phi^{\mu_i}) G(\mu_i) \leqslant \sum_{i=1}^n \theta(\delta\|g\|_\Phi^{\mu_i}) G(\mu_i) + [\theta(\delta\alpha + \delta\varepsilon) - \theta(\delta\alpha)] \sum_{i=1}^n G(\mu_i).$$

But

$$\sum_{i=1}^n \theta(\delta\|g\|_\Phi^{\mu_i}) G(\mu_i) \leqslant 1$$

since $g \in B_\Phi(\theta, G)$, and $\theta(\delta\alpha + \delta\varepsilon) - \theta(\delta\alpha) \to 0$ as $\varepsilon \to 0$, because δ does not depend on ε and θ is continuous on $[0, \beta)$. Therefore,

$$\sum_{i=1}^n \theta(\delta^2\|f\|_\Phi^{\mu_i}) G(\mu_i) \leqslant 1,$$

and letting $\delta \to 1$, the result follows, q.e.d.

2.8. THEOREM. *$B_\Phi(\theta, G)$ is $\mathscr{E}_\Psi$-closed.*

Proof. Follows easily from 2.7.

The next Lemma is important for a key result of this section.

2.9. LEMMA. *If $f_\mu \in L_\Phi(\mu)$ for each $\mu \in M$ and for every $\{\mu_1, \ldots, \mu_n\} \subset M$ there is an $f \in B_\Phi(\theta, G)$ with $f = f_{\mu_i}[\mu_i]$, $i = 1, \ldots, n$, then $\{f_\mu\}_{\mu\in M}$ is in the closure of $\Delta(B_\Phi(\theta, G))$ (in the product topology). Conversely, if $\{f_\mu\}_{\mu\in M}$ is in the closure of $\Delta(B_\Phi(\theta, G))$, then for every $\{\mu_1, \mu_2, \ldots\} \subset M$ there is a Σ-measurable f (possibly depending on the given countable collection) with $f = f_{\mu_i}[\mu_i]$, $i = 1, 2, \ldots,$ and*

$$\sum_{i=1}^\infty \theta(\|f\|_\Phi^{\mu_i}) G(\mu_i) \leqslant 1.$$

Proof. Given $\{\mu_1, \ldots, \mu_n\}$ and $f \in B_\Phi(\theta, G)$ as in the first statement, then for any $h_i \in K_\Psi(\mu_i)$, $i = 1, 2, \ldots, n$, clearly $l(h_i, \mu_i)(f - f_{\mu_i}) = 0$, so that $\{f\}_{\mu\in M}$ $(= \Delta(f))$ is in every neighborhood of $\{f_\mu\}_{\mu\in M}$ of the form

$$\{\{g_\mu\}_{\mu\in M} : |l(h_i, \mu_i)(g_{\mu_i} - f_{\mu_i})| < \varepsilon,\ i = 1, \ldots, n\},$$

where h_i and $\varepsilon > 0$ may vary. Since by selecting the various finite subsets of M, one obtains in this way *all* basis neighborhoods centered at $\{f_\mu\}_{\mu\in M}$, and each such neighborhood contains an element of $\Delta(B_\Phi(\theta, G))$, it follows that $\{f_\mu\}_{\mu\in M}$ is in the closure of $\Delta(B_\Phi(\theta, G))$.

Now suppose $\{f_\mu\}_{\mu\in M}$ is in the closure of $\Delta(B_\Phi(\theta, G))$, and let $\{\mu_1, \mu_2, \ldots\} \subset M$. Let $\nu = \sum_{i=1}^\infty 2^{-i}\mu_i$, and define

$$A_{n,m} = \left\{\omega : \frac{d\mu_n(\omega)}{d\nu} > 0,\ \frac{d\mu_m(\omega)}{d\nu} > 0\right\};$$

then

$$\frac{d\mu_n}{d\mu_m} = \frac{d\mu_n}{d\nu} \Big/ \frac{d\mu_m}{d\nu}$$

is finite and strictly positive on $A_{n,m}$. Let

$$C_{n,m} = \{\omega \in A_{n,m} : f_{\mu_n}(\omega) > f_{\mu_m}(\omega)\}$$

and, for $k = 1, 2, \ldots,$

$$C_{n,m}(k) = \left\{\omega \in C_{n,m} : 0 < \frac{d\mu_n(\omega)}{d\mu_m} \leqslant k\right\}.$$

Clearly,

$$\mu_m(C_{n,m}) = \lim_{k\to\infty} \mu_m(C_{n,m}(k)),$$

so if $\mu_m(C_{n,m}) > 0$, then, for some k_0, $\mu_m(C_{n,m}(k_0)) > 0$. If g denotes the indicator function of $C_{n,m}(k_0)$, then g and $g(d\mu_n/d\mu_m)$ are bounded Σ-measurable functions and hence belong to $K_\Psi(\mu)$ for all $\mu \in M$. Since $\{f_\mu\}_{\mu\in M}$ is in the closure of $\Delta(B_\Phi(\theta, G))$, for any $\varepsilon > 0$ there is an $f_\varepsilon \in B_\Phi(\theta, G)$

such that

$$\Big|\int_\Omega (f_{\mu_n}-f_\varepsilon)\,g\,d\mu_n\Big| < \varepsilon/2$$

and

$$\Big|\int_\Omega (f_{\mu_m}-f_\varepsilon)g\,d\mu_n\Big| = \Big|\int_\Omega (f_{\mu_m}-f_\varepsilon)g\frac{d\mu_n}{d\mu_m}\,d\mu_m\Big| < \varepsilon/2.$$

If follows that

$$\int_{C_{n,m}(k_0)} (f_{\mu_n}-f_{\mu_m})\frac{d\mu_n}{d\mu_m}\,d\mu_m = \Big|\int_\Omega (f_{\mu_n}-f_{\mu_m})\,g\,d\mu_n\Big| < \varepsilon,$$

and, since the left side is independent of ε,

$$\int_{C_{n,m}(k_0)} (f_{\mu_n}-f_{\mu_m})\frac{d\mu_n}{d\mu_m}\,d\mu_m = 0,$$

which is a contradiction since $\mu_m(C_{n,m}(k_0)) > 0$ and the integrand is strictly positive on $C_{n,m}(k_0)$. Thus $\mu_m(C_{n,m}) = 0$ and, similarly,

$$\mu_m\{\omega\in A_{n,m}\colon f_{\mu_n}(\omega) < f_{\mu_m}(\omega)\} = 0.$$

This implies that

$$\mu_m\{\omega\in A_{n,m}\colon f_{\mu_n}(\omega) \neq f_{\mu_m}(\omega)\} = 0.$$

Next, let

$$D_n = \Big\{\omega\colon \frac{d\mu_n(\omega)}{d\nu} > 0,\ \frac{d\mu_j(\omega)}{d\nu} = 0 \text{ for all } j<n\Big\},$$

and define

$$f = \sum_{n=1}^\infty \chi_{D_n} f_{\mu_n}.$$

Since f_{μ_n} are Σ-measurable, and clearly the $D_n\in\Sigma$ (and are disjoint), f is Σ-measurable. Also, note that

$$\{\omega\colon f(\omega)\neq f_{\mu_n}(\omega)\}$$
$$\subset \Big\{\omega\colon \frac{d\mu_n(\omega)}{d\nu} = 0\Big\} \cup \bigcup_{j=1}^{n-1}\Big\{\omega\in D_j\colon \frac{d\mu_n(\omega)}{d\nu} > 0,\ f_{\mu_j}(\omega)\neq f_{\mu_n}(\omega)\Big\}$$
$$\subset \Big\{\omega\colon \frac{d\mu_n(\omega)}{d\nu} = 0\Big\} \cup \bigcup_{j=1}^{n-1}\{\omega\in A_{j,n}\colon f_{\mu_j}(\omega)\neq f_{\mu_n}(\omega)\}.$$

But obviously

$$\mu_n\Big\{\omega\colon \frac{d\mu_n(\omega)}{d\nu} = 0\Big\} = 0,$$

and it has been shown that

$$\mu_n\{\omega\in A_{j,n}\colon f_{\mu_j}(\omega)\neq f_{\mu_n}(\omega)\} = 0;$$

thus $f = f_{\mu_n}[\mu_n]$, $n = 1, 2, \ldots$

Furthermore, $f\in\bigcap_{i=1}^\infty L_\Phi(\mu_i)$, since by definition $f_{\mu_i}\in L_\Phi(\mu_i)$, $i = 1, 2, \ldots$ Fixing n, $\varepsilon>0$, and $h_i\in K_\Psi(\mu_i)$, $i = 1,\ldots,n$, then since $\{f_\mu\}_{\mu\in M}$ is in the closure of $\Delta(B_\Phi(\theta, G))$, there is a $g\in B_\Phi(\theta, G)$ with

$$\Big|\int_\Omega (f-g)h_i\,d\mu_i\Big| = \Big|\int_\Omega (f_{\mu_i}-g)h_i\,d\mu_i\Big| < \varepsilon, \quad i = 1,\ldots,n.$$

Hence, by 2.7,

$$\sum_{i=1}^n \theta(\|f\|_\Phi^{\mu_i})G(\mu_i)\leqslant 1.$$

But n is arbitrary, so

$$\sum_{i=1}^\infty \theta(\|f\|_\Phi^{\mu_i})G(\mu_i)\leqslant 1, \quad \text{q.e.d.}$$

2.10. Corollary. *If $\{f_\mu\}_{\mu\in M}$ is in the closure of $\Delta(B_\Phi(\theta, G))$, then*

$$\sum_{\mu\in M}\theta(\|f_\mu\|_\Phi^\mu)G(\mu)\leqslant 1.$$

Proof. For any $\{\mu_1, \mu_2, \ldots\}\subset M$, one obtains by 2.9 an f with $f = f_{\mu_i}[\mu_i]$, $i = 1, 2, \ldots$, and

$$\sum_{i=1}^\infty \theta(\|f\|_\Phi^{\mu_i})G(\mu_i)\leqslant 1,$$

so that

$$\sum_{i=1}^\infty \theta(\|f_{\mu_i}\|_\Phi^{\mu_i})G(\mu_i)\leqslant 1.$$

But this holds for *any* countable subset, so the proof is done, q.e.d.

2.11. Proposition. *If M is countable and Ψ is continuous, then $B_\Phi(\theta, G)$ is $\mathscr{E}_\Psi$-compact.*

Proof. By 2.6, it suffices to show that $\Delta(B_\Phi(\theta, G))$ is closed. Suppose $\{f_\mu\}_{\mu\in M}$ is in the closure of $\Delta(B_\Phi(\theta, G))$; then by 2.9, since M is countable, there is an $f = f_\mu[\mu]$ for all $\mu\in M$, and

$$\sum_{\mu\in M}\theta(\|f\|_\Phi^\mu)G(\mu)\leqslant 1,$$

i.e., $f\in B_\Phi(\theta, G)$. Hence $\{f_\mu\}_{\mu\in M}\in\Delta(B_\Phi(\theta, G))$, and $\Delta(B_\Phi(\theta, G))$ is closed, q.e.d.

The first main result of this section can now be given in the following:

2.12. Theorem. *On $B_\infty(\theta, G)$, the $\mathscr{E}_\Psi$- and $\mathscr{E}_\infty$-topologies are equivalent for any Ψ. If (Ω, Σ, M) is (θ, G)-compact, then $B_\infty(\theta, G)$ is $\mathscr{E}_1$-compact. Conversely, if $B_\infty(\theta, G)$ is $\mathscr{E}_1$-compact and contains a (strictly) positive function, then $B_\Phi(\theta, G)$ is $\mathscr{E}_\Psi$-compact, whenever Ψ is continuous.*

Proof. Since $K_\Psi(\mu)$ contains all μ-essentially bounded measurable functions, the $\mathscr{E}_\Psi$-topology is stronger than the $\mathscr{E}_\infty$-topology. Now if $l(h, \mu) \in \mathscr{E}_\Psi$, let

$$(*) \qquad h^{(n)}(\omega) = \begin{cases} h(\omega), & |h(\omega)| \leqslant n, \\ 0, & |h(\omega)| > n; \end{cases}$$

then $l(h^{(n)}, \mu) \in \mathscr{E}_\infty$ for $n = 1, 2, \ldots$ If $f \in B_\infty(\theta, G)$, then, by 1.3,

$$|f| \leqslant \theta^{-1}\left(\frac{1}{G(\mu)}\right)[\mu],$$

so that

$$|l(h, \mu)(f) - l(h^{(n)}, \mu)(f)| = \left|\int_\Omega f(h - h^{(n)})\, d\mu\right|$$

$$\leqslant \int_\Omega |f|\,|h - h^{(n)}|\, d\mu \leqslant \theta^{-1}\left(\frac{1}{G(\mu)}\right) \int_{|h|>n} |h|\, d\mu,$$

which goes to 0, uniformly in f, as $n \to \infty$, since $K_\Psi(\mu) \subset L_\Psi(\mu) \subset L_1(\mu)$ ([12], p. 82). Hence, all $l(h, \mu)$, and therefore all elements of $\mathscr{E}_\Psi$, can be uniformly approximated on $B_\infty(\theta, G)$ by elements of $\mathscr{E}_\infty$, so that the topologies are in fact equivalent on $B_\infty(\theta, G)$.

Suppose (Ω, Σ, M) is (θ, G)-compact, i.e., for some Φ, $B_\Phi(\theta, G)$ is $\mathscr{E}_\Psi$-compact. The inequality $N_\Phi^{\theta,G}(\cdot) \leqslant cN_\infty^{\theta,G}(\cdot)$, for some $c > 0$, has been noted earlier; in other terms, $B_\infty(\theta, G) \subset cB_\Phi(\theta, G)$. But $cB_\Phi(\theta, G)$ is $\mathscr{E}_\Psi$-compact (since scalar multiplication is continuous) and, by 2.8, $B_\infty(\theta, G)$ is $\mathscr{E}_1$-closed, hence $\mathscr{E}_\Psi$-closed, by the first part of this theorem. Thus $B_\infty(\theta, G)$ is $\mathscr{E}_\Psi$- (and so $\mathscr{E}_1$-) compact.

Now assume that Ψ is continuous, and that $B_\infty(\theta, G)$ is $\mathscr{E}_1$-compact and contains a function $f_0 > 0$. Let $\{f_\mu\}_{\mu \in M}$ be in the closure of $\Delta_\Phi(B_\Phi(\theta, G))$. By 2.9, for all $\{\mu_1, \ldots, \mu_n\} \subset M$ there is an f' with $f' = f_{\mu_i}[\mu_i]$, $i = 1, \ldots, n$. If $f^{(N)}$ denotes the truncation of the function f as in $(*)$, then

$$\frac{1}{N} f_0 (f')^{(N)} = \frac{1}{N} f_0 f_{\mu_i}^{(N)} [\mu_i], \qquad i = 1, \ldots, n,$$

for every integer $N > 0$. But since $|(f')^{(N)}| \leqslant N$, it follows that

$$\frac{1}{N} f_0 (f')^{(N)} \in B_\infty(\theta, G)$$

by the fact that the ball is solid. Again by 2.9, this means that $\{N^{-1} f_0 f_\mu^{(N)}\}_{\mu \in M}$ is in the closure of $\Delta_\infty(B_\infty(\theta, G))$ for each N. Now, according to 2.6, $\Delta_\infty(B_\infty(\theta, G))$ is closed, so for each N there is a $g_N \in B_\infty(\theta, G)$ such that

$$g_N = \frac{1}{N} f_0 f_\mu^{(N)} [\mu] \quad \text{for all } \mu \in M.$$

Since for each $\mu \in M$, $f_\mu^{(N)} \to f_\mu$, and $f_\mu^{(N)} = (Ng_N/f_0)[\mu]$, the sequence $\{Ng_N/f_0\}$ must converge μ-almost everywhere to a measurable function f, which must then satisfy $f = f_\mu[\mu]$ for all $\mu \in M$. By 2.10, this implies

$$\sum_{\mu \in M} \theta(\|f\|_\Phi^\mu)\, G(\mu) \leqslant 1,$$

i.e., $f \in B_\Phi(\theta, G)$, and so $\{f_\mu\}_{\mu \in M} \in \Delta_\Phi(B_\Phi(\theta, G))$. Thus $\Delta_\Phi(B_\Phi(\theta, G))$ is closed; so 2.6 and the continuity of Ψ imply that $B_\Phi(\theta, G)$ is $\mathscr{E}_\Psi$-compact, q.e.d.

The condition in 2.12 that there exist $0 < f_0 \in B_\infty(\theta, G)$ is essentially the same as requiring that $\theta^{-1}(0) > 0$. More precisely,

2.13. Lemma. *If $\theta^{-1}(0) > 0$ (i.e., if $\theta(x_0) = 0$ for some $x_0 > 0$), then $B_\infty(\theta, G)$ contains a function $f_0 > 0$. If M is uncountable, then the converse holds.*

Proof. If $x_0 > 0$ and $\theta(x_0) = 0$, define $f_0 \equiv x_0$. Then $\|f_0\|_\infty^\mu = x_0$ for all $\mu \in M$ and

$$\sum_{\mu \in M} \theta(\|f_0\|_\infty^\mu)\, G(\mu) = \sum_{\mu \in M} \theta(x_0)\, G(\mu) = 0 \leqslant 1,$$

so $f_0 \in B_\infty(\theta, G)$.

Conversely, let M be uncountable, and suppose $0 < f_0 \in B_\infty(\theta, G)$. This means that

$$\sum_{\mu \in M} \theta(\|f_0\|_\infty^\mu)\, G(\mu) \leqslant 1,$$

and so there must be a $\mu_0 \in M$ such that $\theta(\|f_0\|_\infty^{\mu_0}) = 0$. But since $f_0 > 0$, $\|f_0\|_\infty^{\mu}$ is also > 0. Hence $\theta^{-1}(0) > 0$, q.e.d.

The following result will be found useful later on:

2.14. Lemma. *If (Ω, Σ, M) is (θ, G)-compact, $B_\infty(\theta, G)$ contains an $f_0 > 0$, and $\{f_\mu\}_{\mu \in M}$ are measurable functions such that for every $\{\mu_1, \ldots, \mu_n\} \subset M$ there is a measurable function f' with $f' = f_{\mu_i}[\mu_i]$, $i = 1, \ldots, n$, then there is a measurable f such that $f = f_\mu[\mu]$ for all $\mu \in M$.*

Proof. The proof here is essentially extracted from the proof of 2.12 (the part where $B_\infty(\theta, G)$ is assumed $\mathscr{E}_1$-compact). By the first part of 2.12, it follows from the above hypothesis that $B_\infty(\theta, G)$ is indeed $\mathscr{E}_1$-compact. Although $\{f_\mu\}_{\mu \in M}$ is not assumed here to be in the closure of anything, the assumption of the existence, for each $\{\mu_1, \ldots, \mu_n\}$, of an f' as above, sufficiently compensates for this. One now proceeds exactly as in 2.12 to obtain the desired function f, q.e.d.

In the sequel, different sets of measures M will be considered on (Ω, Σ); this will be indicated as, for example, $B_\Phi(\theta, G; M)$. If G is defined on M and $M' \subset M$, G will also stand for the restriction of G to M'. Also, $f = g[M]$ or $[f]_M = [g]_M$ will mean $f = g[\mu]$ for all $\mu \in M$.

2.15. THEOREM. *If $B_\Phi(\theta, G; M)$ is $\mathscr{E}_\Psi(M)$-compact, and $M' \subset M$ is such that for all $[g]_{M'} \in B_\Phi(\theta, G; M')$ there is an $f = g[M']$ with $[f]_M \in B_\Phi(\theta, G; M)$, then $B_\Phi(\theta, G; M')$ is $\mathscr{E}_\Psi(M')$-compact.*

Proof. Consider the map $[f]_M \to [f]_{M'}$ of $B_\Phi(\theta, G; M)$ into $B_\Phi(\theta, G; M')$. It is well-defined since $f_1 = f_2[M]$ implies $f_1 = f_2[M']$. It is continuous in the $\mathscr{E}_\Psi(M)$- and $\mathscr{E}_\Psi(M')$-topologies, since the inverse image of the open set

$$\{[f]_{M'}: |l(h_i, \mu_i)(f)| < \varepsilon,\ i = 1, \ldots, n\}, \qquad \mu_i \in M',\ h_i \in K_\Psi(\mu_i),$$

is

$$\{[f]_M: |l(h_i, \mu_i)(f)| < \varepsilon,\ i = 1, \ldots, n\}.$$

Furthermore, by hypothesis, the map is onto. But this means that $B_\Phi(\theta, G; M')$ is the continuous image of a compact set, and so is $\mathscr{E}_\Psi(M')$-compact, q.e.d.

2.16. THEOREM. *If Ψ is continuous, and M has a subset M' such that*

(a) $B_\Phi(\theta, G; M')$ is $\mathscr{E}_\Psi(M')$-compact;

(b) $[f]_M \in B_\Phi(\theta, G; M)$ implies $f = 0\,[M - M']$;

(c) $[g]_{M'} \in B_\Phi(\theta, G; M')$ implies that there is an $f = g[M']$ with $f = 0\,[M - M']$;

then $B_\Phi(\theta, G; M)$ is $\mathscr{E}_\Psi(M)$-compact.

Proof. Since Ψ is continuous, one may assume, by 2.6, $\Delta(B_\Phi(\theta, G; M'))$ is closed and show that $\Delta(B_\Phi(\theta, G; M))$ is closed. So suppose $\{f_\mu\}_{\mu\in M}$ is in the closure of $\Delta(B_\Phi(\theta, G; M))$; then for all $\{\mu_1, \ldots, \mu_n\} \subset M$, $\varepsilon > 0$, $h_i \in K_\Psi(\mu_i)$, $i = 1, \ldots, n$, there is an $[f']_M \in B_\Phi(\theta, G; M)$ with

$$\Big|\int_\Omega (f_{\mu_i} - f')\, h_i\, d\mu_i\Big| < \varepsilon, \qquad i = 1, \ldots, n.$$

By (b), $f' = 0\,[M - M']$, which implies that for all $\mu \in M - M'$, $\varepsilon > 0$, $h \in K_\Psi(\mu)$,

$$\Big|\int_\Omega f_\mu h\, d\mu\Big| < \varepsilon,$$

so that $f_\mu = 0\,[\mu]$ for each $\mu \in M - M'$. Furthermore, it is clear that $[f']_{M'} \in B_\Phi(\theta, G; M')$, since

$$\sum_{\mu\in M'} \theta(\|f'\|_\Phi^\mu)\, G(\mu) \leqslant \sum_{\mu\in M} \theta(\|f'\|_\Phi^\mu)\, G(\mu) \leqslant 1.$$

It follows that $\{f_\mu\}_{\mu\in M'}$ is in the closure of $\Delta(B_\Phi(\theta, G; M'))$, and since the latter is closed, there is a $[g]_{M'} \in B_\Phi(\theta, G; M')$ with $f_\mu = g[\mu]$ for all $\mu \in M'$. But by (c), it may be assumed that $g = 0\,[M - M']$, and so $g = f_\mu = 0\,[\mu]$ for all $\mu \in M - M'$. Thus it is seen that $[g]_M \in B_\Phi(\theta, G; M)$ and $g = f_\mu[\mu]$ for all $\mu \in M$. Hence $\{f_\mu\}_{\mu\in M}$ is in $\Delta(B_\Phi(\theta, G; M))$, the latter is closed, and $B_\Phi(\theta, G; M)$ is $\mathscr{E}_\Psi(M)$-compact, q.e.d.

2.17. THEOREM. *If M has a subset M' such that*

(a) (Ω, Σ, M') is (θ, G)-compact;

(b) for every $\mu \in M$ there exist $\{\mu_1, \mu_2, \ldots\} \subset M'$ with $\mu \ll \sum_{n=1}^{\infty} 2^{-n}\mu_n$;

(c) for some $K > 0$, $[f]_{M'} \in B_\infty(\theta, G; M')$ implies $|f| \leqslant K[M']$;

(d) $[f]_{M'} \in B_\infty(\theta, G; M')$ implies $[f]_M \in B_\infty(\theta, G; M)$;

then (Ω, Σ, M) is (θ, G)-compact.

Proof. Obviously, every M-null set is M'-null. Conversely, suppose A is M'-null and $\mu \in M$. Select $\{\mu_1, \mu_2, \ldots\} \subset M'$ as in (b); since A is μ_n-null for $n = 1, 2, \ldots$, it is also μ-null. Hence the M-null and M'-null sets coincide. Now (d) and its converse (which is always true) imply that $B_\infty(\theta, G; M')$ is identical with $B_\infty(\theta, G; M)$. By (a) and 2.12, $B_\infty(\theta, G; M)$ $(= B_\infty(\theta, G; M'))$ is $\mathscr{E}_\infty(M')$-compact. Thus it remains only to show that the $\mathscr{E}_\infty(M')$- and $\mathscr{E}_\infty(M)$-topologies on $B_\infty(\theta, G)$ $(= B_\infty(\theta, G; M))$ are equivalent. It suffices to prove that every element of $\mathscr{E}_\infty(M)$ is the limit of elements of $\mathscr{E}_\infty(M')$, uniformly on $B_\infty(\theta, G)$.

If $\mu \in M$, and $\{\mu_1, \mu_2, \ldots\}$ are such that $\mu \ll \sum_{n=1}^{\infty} 2^{-n}\mu_n$, then, by the Radon-Nikodym theorem, there exists a non-negative f_0 such that

$$\mu(A) = \int_A f_0\, d\Big(\sum_{n=1}^{\infty} 2^{-n}\mu_n\Big) = \sum_{n=1}^{\infty} \int_A f_n\, d\mu_n$$

for every $A \in \Sigma$, where $f_n = 2^{-n} f_0$. If $h \in L_\infty(\mu)$, then there is an $N \geqslant 0$ such that $\mu\{\omega: |h(\omega)| > N\} = 0$, so that, for $n = 1, 2, \ldots$,

$$\mu_n\{\omega: |h(\omega)| > N,\ f_0(\omega) > 0\} = 0.$$

But since $\min(f_n(\omega), k)|h(\omega)| > kN$ implies $|h(\omega)| > N$ and $f_0(\omega) > 0$,

$$\mu_n\{\omega: \min(f_n(\omega), k)|h(\omega)| > kN\} = 0, \qquad n = 1, 2, \ldots,\quad k = 1, 2, \ldots,$$

so $\min(f_n, k)\, h \in L_\infty(\mu_n)$. Hence, if $l(h, \mu) \in \mathscr{E}_\infty(M)$, then

$$\sum_{n=1}^{k} l(\min(f_n, k)\, h, \mu_n) \in \mathscr{E}_\infty(M').$$

If $f \in B_\infty(\theta, G)$, then, by (c), $|f| \leqslant K[M]$, so

$$\Big|l(h,\mu)(f) - \sum_{n=1}^{k} l\big(\min(f_n,k)h,\mu_n\big)(f)\Big|$$
$$= \Big|\sum_{n=1}^{\infty}\int_\Omega fhf_n\,d\mu_n - \sum_{n=1}^{k}\int_\Omega fh\min(f_n,k)\,d\mu_n\Big|$$
$$= \Big|\sum_{n=k+1}^{\infty}\int_\Omega fhf_n\,d\mu_n + \sum_{n=1}^{k}\int_\Omega fh\big(f_n - \min(f_n,k)\big)\,d\mu_n\Big|$$
$$\leqslant \sum_{n=k+1}^{\infty}\int_\Omega |fh|f_n\,d\mu_n + \sum_{n=1}^{k}\int_\Omega |fh|\big(f_n-\min(f_n,k)\big)\,d\mu_n$$
$$\leqslant KN\Big[\sum_{n=k+1}^{\infty}\int_\Omega f_n\,d\mu_n + \sum_{n=1}^{k}\int_\Omega\big(f_n-\min(f_n,k)\big)\,d\mu_n\Big].$$

Since $\sum_{n=1}^{\infty}\int_\Omega f_n\,d\mu_n = 1$, the first term here tends to 0 as $k\to\infty$. Now given $\varepsilon>0$, one can choose n_ε so large that

$$\sum_{n=n_\varepsilon+1}^{\infty}\int_\Omega f_n\,d\mu_n < \varepsilon/2;$$

then take k_ε so large that (for the fixed n_ε)

$$\sum_{n=1}^{n_\varepsilon}\int_{f_n>k_\varepsilon} f_n\,d\mu_n < \varepsilon/2.$$

It follows that

$$0 \leqslant \sum_{n=1}^{k_\varepsilon}\int_\Omega\big(f_n-\min(f_n,k_\varepsilon)\big)\,d\mu_n$$
$$\leqslant \sum_{n=1}^{\infty}\int_{f_n>k_\varepsilon} f_n\,d\mu_n$$
$$\leqslant \sum_{n=1}^{n_\varepsilon}\int_{f_n>k_\varepsilon} f_n\,d\mu_n + \sum_{n=n_\varepsilon+1}^{\infty}\int_\Omega f_n\,d\mu_n < \varepsilon,$$

and since ε is arbitrary,

$$\sum_{n=1}^{k} l\big(\min(f_n,k)h,\mu_n\big)\to l(h,\mu)$$

as $k\to\infty$ uniformly on $B_\infty(\theta, G)$, q.e.d.

2.18. COROLLARY. *If M is dominated by a probability measure μ_0 (i.e., $\mu \ll \mu_0$ for all $\mu \in M$), Ψ is continuous, and either*

(a) *$\theta^{-1}\big(1/G(\cdot)\big)$ is bounded*

or

(b) *$[g]_M \in B_\Phi(\theta, G; M)$ implies that there is an $f = g[M]$ with $[f]_{M\cup\{\mu\}_0} \in B_\Phi(\theta, \bar{G}; M \cup \{\mu_0\})$ for some extension $\bar{G}$ of G to $M\cup\{\mu_0\}$;*

then $B_\Phi(\theta, G; M)$ is $\mathscr{E}_\Psi(M)$-compact.

Proof. Let $\tilde{M} = M \cup \{\mu_0\}$.

(a) First consider

$$\theta_1(x) = \begin{cases} 0, & |x|\leqslant 1,\\ \infty, & |x|>1,\end{cases}$$

so that for any $M' \subset \tilde{M}$,

$$N_\infty^{\theta,G}(f; M') = \sup_{\mu\in M'}\|f\|_\infty^\mu.$$

By 2.11, $B_\infty(\theta_1, \tilde{G}; \{\mu_0\})$ is $\mathscr{E}_1(\{\mu_0\})$-compact. Clearly, every $\mu \in \tilde{M}$ is dominated by a countable subset of $\{\mu_0\}$. If $[f]_{\{\mu_0\}} \in B_\infty(\theta_1, \tilde{G}; \{\mu_0\})$, then $\|f\|_\infty^{\mu_0} \leqslant 1$, i.e., $|f| \leqslant 1[\mu_0]$; and, moreover, this implies by the domination that $|f| \leqslant 1[\tilde{M}]$, i.e., $[f]_{\tilde{M}} \in B_\infty(\theta_1, \tilde{G}; \tilde{M})$. Thus (a)-(d) of 2.17 are satisfied, so that $B_\infty(\theta_1, \tilde{G}; \tilde{M})$ is $\mathscr{E}_1(\tilde{M})$-compact. But now $M \subset \tilde{M}$, and given $[g]_M \in B_\infty(\theta_1, G; M)$ (i.e., $|g| \leqslant 1[M]$) there is an $f = g[M]$ with $[f]_{\tilde{M}} \in B_\infty(\theta_1, \tilde{G}; \tilde{M})$; let

$$f(\omega) = \begin{cases} g(\omega), & |g(\omega)|\leqslant 1,\\ 0, & |g(\omega)|>1.\end{cases}$$

It follows from 2.15 that $B_\infty(\theta_1, G; M)$ is $\mathscr{E}_1(M)$-compact. Since $B_\infty(\theta_1, G; M)$ contains the constant 1, 2.12 implies that $B_\Phi(\theta_1, G; M)$ is $\mathscr{E}_\Psi(M)$-compact.

In general, when $\theta^{-1}\big(1/G(\cdot)\big)$ is bounded, 1.6 shows that

$$B_\Phi(\theta, G; M) \subset cB_\Phi(\theta_1, G; M)$$

for some $c>0$. By 2.8, $B_\Phi(\theta, G; M)$ is an $\mathscr{E}_\Psi(M)$-closed subset of the $\mathscr{E}_\Psi(M)$-compact $cB_\Phi(\theta_1, G; M)$, hence is itself $\mathscr{E}_\Psi(M)$-compact. This concludes this part of the proof.

(b) If $\{f_\mu\}_{\mu\in\tilde{M}}$ is in the closure of $\Delta\big(B_\Phi(\theta, \tilde{G}; \tilde{M})\big)$, then by 2.9, for every $\mu \in M$ there is a measurable f' with

$$f' = f_\mu[\mu], \qquad f' = f_{\mu_0}[\mu_0].$$

But since $\mu \ll \mu_0$, the second equation implies $f' = f_{\mu_0}[\mu]$, so that

$$f_{\mu_0} = f_\mu[\mu] \quad \text{for all } \mu\in M.$$

Using 2.10, $f_{\mu_0} \in B_\Phi(\theta, \tilde{G}; \tilde{M})$, and so $\Delta\big(B_\Phi(\theta, \tilde{G}; \tilde{M})\big)$ is closed and, by 2.6, $B_\Phi(\theta, \tilde{G}; \tilde{M})$ is $\mathscr{E}_\Psi(\tilde{M})$-compact. The conclusion follows from this, the hypothesis, and 2.15, q.e.d.

Special properties of *reflexive* $E_\Phi(\theta, G)$'s will now be discussed. Compactness is more general than reflexivity, in the following sense:

2.19. PROPOSITION. *If $E_\Phi(\theta, G)$ is reflexive, then $B_\Phi(\theta, G)$ is $\mathscr{E}_\Psi$-compact.*

Proof. By 2.1, the $\mathscr{E}_\Psi$-topology on $E_\Phi(\theta, G)$ is weaker than the weak (i.e. $\sigma(E_\Phi(\theta, G), E_\Phi^*(\theta, G))$) topology. By a well-known theorem ([4], p. 425) $E_\Phi(\theta, G)$ is reflexive if and only if $B_\Phi(\theta, G)$ is weakly compact; but by the preceding remark, weak compactness implies $\mathscr{E}_\Psi$-compactness, q.e.d.

2.20. LEMMA. *If $E_\Phi(\theta, G)$ is reflexive, then $\mathscr{E}_\Psi$ is norm-dense in $E_\Phi^*(\theta, G)$, and all $l \in E_\Phi^*(\theta, G)$ are absolutely continuous, i.e., $\{f_n\} \subset E_\Phi(\theta, G)$, $f_n \downarrow 0$ imply $l(f_n) \to 0$.*

Proof. If $\bar{\mathscr{E}}_\Psi \neq E_\Phi^*(\theta, G)$ and $l_0 \in E_\Phi^*(\theta, G) - \bar{\mathscr{E}}_\Psi$, then by the Hahn-Banach theorem there is an $L \in E_\Phi^{**}(\theta, G)$ with $L(l_0) = 1$, $L(\bar{\mathscr{E}}_\Psi) = 0$. But since $E_\Phi(\theta, G)$ is reflexive, there is an $f \in E_\Phi(\theta, G)$ $(f \leftrightarrow L)$ with $l_0(f) = 1$ and $l(f) = 0$ for all $l \in \mathscr{E}_\Psi$. This, however, contradicts the fact (2.3) that $\mathscr{E}_\Psi$ is a *total* subspace of $E_\Phi^*(\theta, G)$. Hence $\bar{\mathscr{E}}_\Psi = E_\Phi^*(\theta, G)$.

Now note that each element of $\mathscr{E}_\Psi$ is absolutely continuous; for if $\{f_n\} \subset E_\Phi(\theta, G)$, $f_n \downarrow 0$, $\mu \in M$, and $h \in K_\Psi(\mu)$, then

$$|l(h, \mu)(f_n)| \leqslant \int_\Omega f_n |h| \, d\mu \to 0,$$

by the Dominated Convergence Theorem. If l is an arbitrary element of $E_\Phi^*(\theta, G)$ and $\varepsilon > 0$, then since $\bar{\mathscr{E}}_\Psi = E_\Phi^*(\theta, G)$, there is an $l_\varepsilon \in \mathscr{E}_\Psi$ with $\|l - l_\varepsilon\| < \varepsilon$. Hence if $\{f_n\} \subset E_\Phi(\theta, G)$, $f_n \downarrow 0$,

$$|l(f_n)| \leqslant |l_\varepsilon(f_n)| + \|l - l_\varepsilon\| N_\Phi^{\theta, G}(f_n) \leqslant |l_\varepsilon(f_n)| + \varepsilon N_\Phi^{\theta, G}(f_1);$$

but since $l_\varepsilon(f_n) \to 0$ and ε is arbitrary, it follows that $l(f_n) \to 0$, q.e.d.

In what follows in this section, θ will be restricted to the two-valued case; i.e., $E_\Phi(\theta, G) = E_\Phi$ (as previously defined) with

$$N_\Phi^{\theta, G}(f) = \sup_{\mu \in M} \|f\|_\Phi^\mu = N_\Phi(f).$$

The methods of proof in the following do not seem to extend to the more general case.

2.21. THEOREM. *If E_Φ is reflexive, $\Phi^{-1}(0) = \Psi^{-1}(0) = 0$, and $f \in E_1$, and if*

$$f_n(\omega) = \begin{cases} f(\omega), & |f(\omega)| \leqslant n, \\ 0, & |f(\omega)| > n, \end{cases}$$

then $N_1(f - f_n) \to 0$; i.e., bounded functions are dense in E_1. If, moreover, $0 \neq f \in E_\Phi$, then

$$\sup_{\mu \in M} \int_\Omega \Phi\left(\frac{f - f_n}{N_\Phi(f)}\right) d\mu \to 0;$$

and if, further, Φ satisfies the Δ_2-condition (i.e., there exist x_0 and $c > 0$ with $\Phi(2x) \leqslant c\Phi(x)$ for all $x \geqslant x_0$), then $N_\Phi(f - f_n) \to 0$; i.e., bounded functions are dense in E_Φ.

Proof. Since $N_1(f - f_n) \leqslant N_1(f^+ - f_n^+) + N_1(f^- - f_n^-)$, it may be assumed that $f \geqslant 0$. Note that $N_1(f - f_n) \downarrow \eta \geqslant 0$; suppose $\eta > 0$. Since

$$N_1(f - f_n) = \sup_{\mu \in M} \|f - f_n\|_1^\mu,$$

for each n there is a $\mu_n \in M$ with $\|f - f_n\|_1^{\mu_n} \geqslant \eta/2$. Define for $g \in E_\Phi$,

$$l_n(g) = \int_\Omega \Psi^{-1}(f - f_n) g \, d\mu_n, \quad n = 1, 2, \ldots$$

Then since

$$\int_\Omega \Psi\left(\frac{\Psi^{-1}(f - f_n)}{N_1(f) + 1}\right) d\mu_n \leqslant \frac{1}{N_1(f) + 1} \int_\Omega \Psi\big(\Psi^{-1}(f - f_n)\big) d\mu_n$$
$$\leqslant \frac{1}{N_1(f) + 1} \int_\Omega (f - f_n) \, d\mu_n \leqslant \frac{1}{N_1(f) + 1} N_1(f) \leqslant 1,$$

we have

$$|l_n(g)| \leqslant N_\Psi^{\mu_n}\big(\Psi^{-1}(f - f_n)\big) \|g\|_\Phi^{\mu_n} \leqslant \big(N_1(f) + 1\big) N_\Phi(g).$$

Hence $l_n \in E_\Phi^*$ (it is clearly a linear functional) and $\|l_n\| \leqslant N_1(f) + 1$ for all n, i.e., $l_n \in \big(N_1(f) + 1\big) B_\Phi^*$, where B_Φ^* is the closed unit ball in E_Φ^*. But by Alaoglu's Theorem, B_Φ^* (and therefore $\big(N_1(f) + 1\big) B_\Phi^*$) is compact in the weak*-topology of E_Φ^*, and so $\{l_n\}$ has a weak* cluster point l in $\big(N_1(f) + 1\big) B_\Phi^*$.

Now $\Phi^{-1}(f) \in E_\Phi$, since (by Young's inequality)

$$\|\Phi^{-1}(f)\|_\Phi^\mu \leqslant \int_\Omega \Phi\big(\Phi^{-1}(f)\big) d\mu + 1 \leqslant \int_\Omega f \, d\mu + 1,$$

so that

$$N_\Phi\big(\Phi^{-1}(f)\big) = \sup_{\mu \in M} \|\Phi^{-1}(f)\|_\Phi^\mu \leqslant \sup_{\mu \in M} \int_\Omega f \, d\mu + 1 = N_1(f) + 1 < \infty.$$

Using the fact that ([5], p. 13) $\Phi^{-1}(x)\Psi^{-1}(x) \geqslant x$ for all $x \geqslant 0$, we have

$$l\big(\Phi^{-1}(f)\big) = \lim_{k \to \infty} l_{n_k}\big(\Phi^{-1}(f)\big) = \lim_{k \to \infty} \int_\Omega \Phi^{-1}(f) \Psi^{-1}(f - f_{n_k}) \, d\mu_{n_k}$$
$$\geqslant \varliminf_{k \to \infty} \int_\Omega \Phi^{-1}(f - f_{n_k}) \Psi^{-1}(f - f_{n_k}) \, d\mu_{n_k} \geqslant \varliminf_{k \to \infty} \int_\Omega (f - f_{n_k}) \, d\mu_{n_k} \geqslant \eta/2 > 0,$$

by the way the μ_n were chosen; $\{l_{n_k}\}$ is the subsequence corresponding to $\Phi^{-1}(f)$ by the weak*-compactness of $\big(N_1(f) + 1\big) B_\Phi^*$. However, note that for $n_k > m$ and any $\omega \in \Omega$, either $f(\omega) - f_{n_k}(\omega) = 0$ or $f_{n_k}(\omega) = 0$, the latter implying $f_m(\omega) = 0$; thus $\Phi^{-1}(f_m)\,\Psi^{-1}(f-f_{n_k}) = 0$ whenever $n_k > m$. Hence for each m there is a subsequence $\{f_{n_k}^{(m)}\}$ with

$$l(\Phi^{-1}(f_m)) = \lim_{k\to\infty} \int_\Omega \Phi^{-1}(f_m)\,\Psi^{-1}(f-f_{n_k}^{(m)})\,d\mu_{n_k}^{(m)} = 0.$$

But since $\Phi^{-1}(0) = 0$,

$$\Phi^{-1}(f) - \Phi^{-1}(f_m) = \left\{\begin{matrix} 0, & |f| \leqslant m \\ \Phi^{-1}(f), & |f| > m \end{matrix}\right\} = \Phi^{-1}(f-f_m) \downarrow 0,$$

so by the absolute continuity of l (2.20),

$$l(\Phi^{-1}(f)) = \lim_{m\to\infty} l(\Phi^{-1}(f_m)) = 0,$$

which is a contradiction. Therefore $\eta = 0$ and $N_1(f-f_n) \downarrow 0$.

If in fact, $0 \neq f \in E_\Phi$, then $\Phi(f/N_\Phi(f)) \in E_1$ since

$$\sup_{\mu\in M} \int_\Omega \Phi\left(\frac{f}{N_\Phi(f)}\right) d\mu \leqslant \sup_{\mu\in M} \int_\Omega \Phi\left(\frac{f}{\|f\|_\Phi^\mu}\right) d\mu \leqslant 1.$$

It follows from the first part of this theorem that

$$\sup_{\mu\in M} \int_\Omega \Phi\left(\frac{f-f_n}{N_\Phi(f)}\right) d\mu = N_1\left[\Phi\left(\frac{f}{N_\Phi(f)}\right) - \Phi\left(\frac{f_n}{N_\Phi(f)}\right)\right] \downarrow 0.$$

Finally, suppose $\Phi(2x) \leqslant c\Phi(x)$ for all $x \geqslant x_0$. It follows that for any $a > 0$, there is a $K > 0$ such that ([5], p. 23)

$$\Phi(ax) \leqslant K\Phi(x) \quad \text{for all } x \geqslant x_0.$$

This clearly implies that

$$\Phi\left(\frac{ax}{N_\Phi(f)}\right) \leqslant K\Phi\left(\frac{x}{N_\Phi(f)}\right) + \Phi(ax_0) \quad \text{for all } x \geqslant 0.$$

Let $\varepsilon > 0$ be given, and set $a = N_\Phi(f)/\varepsilon$; then, by the above,

$$\sup_{\mu\in M} \int_\Omega \Phi\left(\frac{f}{\varepsilon}\right) d\mu \leqslant K \sup_{\mu\in M} \int_\Omega \Phi\left(\frac{f}{N_\Phi(f)}\right) d\mu + \Phi\left(\frac{N_\Phi(f)x_0}{\varepsilon}\right) < \infty.$$

Thus, as before,

$$\sup_{\mu\in M} \int_\Omega \Phi\left(\frac{f-f_n}{\varepsilon}\right) d\mu \downarrow 0,$$

and

$$\lim_{n\to\infty} \sup_{\mu\in M} \left\|\frac{1}{\varepsilon}(f-f_n)\right\|_\Phi^\mu \leqslant \lim_{n\to\infty} \sup_{\mu\in M} \int_\Omega \Phi\left(\frac{1}{\varepsilon}(f-f_n)\right) d\mu + 1 = 1,$$

so $\lim_{n\to\infty} N_\Phi(f-f_n) \leqslant \varepsilon$, i.e., $N_\Phi(f-f_n) \downarrow 0$, q.e.d.

2.22. Definition. Denote by Q_Φ the weak closure of the set $\{l(1,\mu)\colon \mu\in M_c\}$ in E_Φ^*, where M_c is the convex hull of M.

2.23. Lemma. *If E_Φ is reflexive, then Q_Φ is weakly compact.*

Proof. By reflexivity, B_Φ^* is weakly compact. Since Q_Φ is weakly closed, it is sufficient to show that Q_Φ is bounded. But if $\mu\in M$, then using the Hölder inequality and [5], p. 79,

$$\|l(1,\mu)\| = \sup\Big\{\Big|\int_\Omega f\,d\mu\Big|\colon N_\Phi(f)\leqslant 1\Big\} \leqslant \sup\Big\{\int_\Omega |f|\,d\mu\colon \|f\|_\Phi^\mu \leqslant 1\Big\}$$
$$\leqslant N_\Psi^\mu(1) = \frac{1}{\Psi^{-1}(1)} < \infty.$$

If $\mu = a_1\mu_1+\ldots+a_n\mu_n \in M_c$, then

$$\|l(1,\mu)\| = \|a_1 l(1,\mu_1)+\ldots+a_n l(1,\mu_n)\| \leqslant \sum_{i=1}^n a_i\|l(1,\mu_i)\| \leqslant \frac{1}{\Psi^{-1}(1)},$$

q.e.d.

2.24. Lemma. *If E_Φ is reflexive and $l\in Q_\Phi$, then l can be represented as*

$$l(f) = \int_\Omega f\,d\nu, \quad f\in E_\Phi,$$

for some probability measure ν. If $M'_\Phi = \{\nu\colon l(1,\nu)\in Q_\Phi\}$, then $E_\Phi(M'_\Phi) = E_\Phi(M)$; in fact, $N_\Phi(f; M'_\Phi) = N_\Phi(f; M)$ for all $f\in E_\Phi(M)$.

Proof. Suppose E_Φ is reflexive and $l\in Q_\Phi$; then, given $f\in E_\Phi$ and $\varepsilon > 0$, there is a $\mu\in M_c$ with

$$|l(f) - l(1,\mu)(f)| < \varepsilon.$$

If, in particular, f is a fixed non-negative function, then

$$l(1,\mu)(f) = \int_\Omega f\,d\mu \geqslant 0,$$

so that $l(f) \geqslant 0$. Since $l\in E_\Phi^*$ and E_Φ is reflexive, l is absolutely continuous by 2.20. Also $l(1) = 1$, since for any μ,

$$l(1,\mu)(1) = \int_\Omega d\mu = 1.$$

Since E_Φ is clearly a vector lattice ($f\in E_\Phi$ implies $|f|\in E_\Phi$), the Daniell extension theorem ([6], p. 21) implies that there is a probability measure ν with

$$l(f) = \int_\Omega f\,d\nu \quad \text{for all } f\in E_\Phi.$$

Since $M \subset M'_\Phi$, the inequality

$$N_\Phi(f; M'_\Phi) \geqslant N_\Phi(f; M)$$

obviously holds for all $f \in E_\Phi(M)$. To complete the proof, let $\nu \in M'_\Phi$ and $\varepsilon > 0$, and first assume f is bounded. According to [5], p. 86, there is a bounded h^ε with

$$\int_\Omega \Psi(h^\varepsilon)\,d\nu \leqslant 1 \quad \text{and} \quad \int_\Omega |fh^\varepsilon|\,d\nu > \|f\|_\Phi^\nu - \varepsilon.$$

If Ψ is continuous, then clearly $\Psi(h^\varepsilon)$ is bounded; but even for discontinuous Ψ, h^ε can be truncated so that $\Psi(h^\varepsilon)$ is bounded. Then $|fh^\varepsilon|$ and $\Psi(h^\varepsilon) \in E_\Phi(M)$. Since $l(1,\nu)$ is in the weak (= weak* here) closure of $\{l(1,\mu): \mu \in M_c\}$, there is a sequence $\{\mu_n\} \subset M_c$ with

$$\int_\Omega |fh^\varepsilon|\,d\mu_n \to \int_\Omega |fh^\varepsilon|\,d\nu \quad \text{and} \quad \int_\Omega \Psi(h^\varepsilon)\,d\mu_n \to \int_\Omega \Psi(h^\varepsilon)\,d\nu.$$

Thus, for large enough n,

$$\int_\Omega |fh^\varepsilon|\,d\mu_n > \|f\|_\Phi^\nu - \varepsilon \quad \text{and} \quad \int_\Omega \Psi(h^\varepsilon)\,d\mu_n \leqslant 1+\varepsilon.$$

Letting $\overline{h^\varepsilon} = h^\varepsilon/(1+\varepsilon)$, one obtains (using convexity of Ψ)

$$\int_\Omega |f\overline{h^\varepsilon}|\,d\mu_n > \frac{\|f\|_\Phi^\nu - \varepsilon}{1+\varepsilon} \quad \text{and} \quad \int_\Omega \Psi(\overline{h^\varepsilon})\,d\mu_n \leqslant \frac{1+\varepsilon}{1+\varepsilon} = 1.$$

It follows that

$$N_\Phi(f; M_c) \geqslant \|f\|_\Phi^{\mu_n} > \frac{\|f\|_\Phi^\nu - \varepsilon}{1+\varepsilon},$$

and since ε is independent of ν and f,

$$N_\Phi(f; M_c) \geqslant \|f\|_\Phi^\nu.$$

For general $f \in E_\Phi(M)$, let

$$f_m = \begin{cases} f, & |f| \leqslant m, \\ 0, & |f| > m; \end{cases}$$

then $N_\Phi(f; M_c) \geqslant N_\Phi(f_m; M_c) \geqslant \|f_m\|_\Phi^\nu$, and using the fact ([5], p. 91) that $\|f_m\|_\Phi^\nu \uparrow \|f\|_\Phi^\nu$ as $m \to \infty$,

$$N_\Phi(f; M_c) \geqslant \sup_{\nu \in M'_\Phi} \lim_{m\to\infty} \|f_m\|_\Phi^\nu = N_\Phi(f; M'_\Phi).$$

But it easily follows from 1.9 that $N_\Phi(f; M_c) = N_\Phi(f; M)$, and therefore $N_\Phi(f; M'_\Phi) = N_\Phi(f; M)$, q.e.d.

2.25. THEOREM. *If $E_\Phi(M)$ is reflexive, then M'_Φ is dominated.*

Proof. The proof is essentially identical to the corresponding one given by Pitcher [8].

2.26. THEOREM. *If Φ satisfies the Δ_2-condition, $\Phi^{-1}(0) = \Psi^{-1}(0) = 0$, and E_Φ is reflexive, then for each $f \in E_\Phi$ there is a $\mu \in M'_\Phi$ with*

$$\|f\|_\Phi^\mu = N_\Phi(f).$$

Proof. It will first be shown that if $l(1,\mu)$ is a weak limit point of $\{l(1,\mu_n)\}$ in E_Φ^*, then

$$\|g\|_\Phi^\mu \geqslant \overline{\lim_{n\to\infty}} \|g\|_\Phi^{\mu_n}$$

for all bounded g.

Define

$$h(K) = \frac{1}{K}\Big(1+\int_\Omega \Phi(Kg)\,d\mu\Big) \quad \text{for all } K > 0.$$

Since g is bounded and Φ must be continuous (by the Δ_2-condition), it follows that each $\Phi(Kg)$ is bounded, hence $\in E_\Phi$. Given $\varepsilon > 0$, choose $K_\varepsilon > 0$ such that $h(K_\varepsilon) < \inf_{K>0} h(K) + \varepsilon$. Then (by assumption) for every $\varepsilon > 0$ there exists a subsequence $\{\mu_{n_j}(\varepsilon)\}$ of $\{\mu_n\}$ such that

$$\lim_{j\to\infty} \int_\Omega \Phi(K_\varepsilon g)\,d\mu_{n_j}(\varepsilon) = \int_\Omega \Phi(K_\varepsilon g)\,d\mu.$$

Define

$$h_j(\varepsilon) = \frac{1}{K_\varepsilon}\Big(1+\int_\Omega \Phi(K_\varepsilon g)\,d\mu_{n_j}(\varepsilon)\Big).$$

Then, using 1.7,

$$\overline{\lim_{n\to\infty}} \|g\|_\Phi^{\mu_n} \leqslant \overline{\lim_{j\to\infty}} \|g\|_\Phi^{\mu_{n_j}(\varepsilon)} \leqslant \lim_{j\to\infty} h_j(\varepsilon) = h(K_\varepsilon) < \inf_{K>0} h(K) + \varepsilon = \|g\|_\Phi^\mu + \varepsilon.$$

Since the left end does not involve ε, the assertion is proved.

Now let $\{\mu_n\} \subset M$ with $\|f\|_\Phi^{\mu_n} \uparrow N_\Phi(f)$, and let $\mu \in M'_\Phi$ be such that $l(1,\mu)$ is a weak limit point of $\{l(1,\mu_n)\}$ (this exists by 2.23). If f_m is the usual bounded approximation to $f \in E_\Phi$, then using the first part of this proof and 2.21,

$$\begin{aligned} \|f\|_\Phi^\mu &= \lim_{m\to\infty} \|f_m\|_\Phi^\mu \geqslant \underline{\lim_{m\to\infty}}\ \overline{\lim_{n\to\infty}} \|f_m\|_\Phi^{\mu_n} \\ &\geqslant \underline{\lim_{m\to\infty}}\ \overline{\lim_{n\to\infty}} \big(\|f\|_\Phi^{\mu_n} - \|f-f_m\|_\Phi^{\mu_n}\big) \\ &\geqslant \underline{\lim_{m\to\infty}}\ \overline{\lim_{n\to\infty}} \big(\|f\|_\Phi^{\mu_n} - N_\Phi(f-f_m)\big) \\ &= \lim_{n\to\infty} \|f\|_\Phi^{\mu_n} - \lim_{m\to\infty} N_\Phi(f-f_m) = N_\Phi(f). \end{aligned}$$

Equality holds since, by 2.24, $\|f\|_\Phi^\mu \leqslant N_\Phi(f; M'_\Phi) = N_\Phi(f; M)$, q.e.d.

Returning now to the more general $E_\Phi(\theta, G)$, a sufficient condition that $E_\Phi(\theta, G)$ be reflexive will be established; this condition will involve only Φ, θ, and (possibly) G, but *not* the family of measures. In what follows, Z will denote the Young function complementary to θ. Use will be made of an auxiliary space $\hat{E}_\Phi(\theta, G)$, defined as follows:

2.27. Definition. $\hat{E}_\Phi(\theta, G) = \{\hat{f} \in \prod_{\mu\in M} L_\Phi(\mu) \colon F_{\hat{f}} \in l_\theta^G\}$, where $F_{\hat{f}}(\mu) \equiv \|\hat{f}(\mu)\|_\Phi^\mu$.

All the considerations used in proving that $E_\Phi(\theta, G)$ is a Banach space carry over to $\hat{E}_\Phi(\theta, G)$, with obvious alterations, e.g., F_f is replaced by $F_{\hat{f}}$ and $N_\Phi^{\theta,G}(f)$ by the norm $\hat{N}_\Phi^{\theta,G}(\hat{f}) = N_\theta^G(F_{\hat{f}})$, the zero element is the $\hat{f}$ such that $\hat{f}(\mu) = 0\,[\mu]$ for all $\mu \in M$, etc. Hence $\hat{E}_\Phi(\theta, G)$ is a Banach space (under $\hat{N}_\Phi^{\theta,G}(\cdot)$) containing $E_\Phi(\theta, G)$, and note that $N_\Phi^{\theta,G}(\cdot) = \hat{N}_\Phi^{\theta,G}(\cdot)$ on $E_\Phi(\theta, G)$. It follows that $E_\Phi(\theta, G)$ is a closed subspace of $\hat{E}_\Phi(\theta, G)$.

2.28. Theorem. *Suppose Φ satisfies the Δ_2-condition, and either*

(a) *$\theta(2x)/\theta(x)$ is bounded on $(0, \infty)$*

or

(b) *$\theta(2x)/\theta(x)$ is bounded on every $(0, a]$, $a < \infty$, and $1/G$ is bounded.*

Then $(\hat{E}_\Phi(\theta, G))^*$ and $\hat{E}_\Psi(Z, G)$ are linearly and topologically isomorphic.

Proof. Note that either (a) or (b) implies

$$l_\theta^G = \{F \colon \sum_{\mu\in M} \theta(F(\mu))G(\mu) < \infty\} \equiv \tilde{l}_\theta^G.$$

For any $\mu \in M$ and $f_\mu \in L_\Phi(\mu)$, define $\hat{f}_\mu \in \hat{E}_\Phi(\theta, G)$ by

$$\hat{f}_\mu(\mu') = \begin{cases} [f_\mu]_\mu & \text{if } \mu' = \mu, \\ [0]_{\mu'} & \text{if } \mu' \neq \mu. \end{cases}$$

If $l \in (\hat{E}_\Phi(\theta, G))^*$, then for $f_\mu \in L_\Phi(\mu)$, define

$$l_\mu(f_\mu) = \frac{l(\hat{f}_\mu)}{G(\mu)}.$$

l_μ is clearly a linear functional on $L_\Phi(\mu)$; and, furthermore,

$$\hat{N}_\Phi^{\theta,G}(\hat{f}_\mu) = \inf\left\{K > 0 \colon \theta\left(\frac{\|f_\mu\|_\Phi^\mu}{K}\right)G(\mu) \leqslant 1\right\} = \frac{\|f_\mu\|_\Phi^\mu}{\theta^{-1}(1/G(\mu))},$$

so l_μ is continuous (since l is continuous), i.e., $l_\mu \in (L_\Phi(\mu))^*$. But since Φ satisfies the Δ_2-condition, there is ([5], p. 128) a (unique) $\hat{l}(\mu) \in L_\Psi(\mu)$ with

$$(*)\qquad \frac{l(\hat{f}_\mu)}{G(\mu)} = l_\mu(f_\mu) = \int_\Omega f_\mu(\omega)\hat{l}(\mu)(\omega)\,d\mu(\omega)$$

for all $f_\mu \in L_\Phi(\mu)$.

Now ([5], p. 135)

$$N_\Psi^\mu(\hat{l}(\mu)) = \sup\left\{\left|\int_\Omega f_\mu(\omega)\hat{l}(\mu)(\omega)\,d\mu(\omega)\right| \colon \|f_\mu\|_\Phi^\mu \leqslant 1\right\},$$

so one can choose, for each $\mu \in M$, an $f_\mu \in L_\Phi(\mu)$ with

$$\|f_\mu\|_\Phi^\mu \leqslant 1, \qquad \int_\Omega f_\mu(\omega)\hat{l}(\mu)(\omega)\,d\mu(\omega) \geqslant \frac{1}{2}N_\Psi^\mu(\hat{l}(\mu)).$$

Fix these f_μ, and for any $F \in l_\theta^G$ $(= \tilde{l}_\theta^G)$, set

$$f_\mu^F = |F(\mu)|f_\mu \quad \text{for all } \mu \in M.$$

Then $f_\mu^F \in L_\Phi(\mu)$, and since by hypothesis $\theta^{-1}(0) = 0$, it follows that $F(\mu)$, whence f_μ^F, can be non-zero for at most countably many μ's.

It will now be shown that the linear map $l \to \vec{l}$, defined above, is a continuous operator on $(\hat{E}_\Phi(\theta, G))^*$ into $E_\Psi(Z, G)$. Thus let $l \in (\hat{E}_\Phi(\theta, G))^*$, and $F \in l_\theta^G$ with $N_\theta^G(F) \leqslant 1$. Then

$$(1)\qquad \sum_{\mu\in M} |F(\mu)|\,\|\hat{l}(\mu)\|_\Psi^\mu G(\mu)$$
$$\leqslant 2\sum_{\mu\in M} |F(\mu)|\,N_\Psi^\mu(\hat{l}(\mu))G(\mu) \leqslant 4\sum_{\mu\in M} |F(\mu)|\left(\int_\Omega f_\mu(\omega)\hat{l}(\mu)(\omega)\,d\mu(\omega)\right)G(\mu)$$
$$= 4\sum_{\mu\in M}\left(\int_\Omega f_\mu^F(\omega)\hat{l}(\mu)(\omega)\,d\mu(\omega)\right)G(\mu) = 4\sum_{\mu\in M} l(\widehat{f_\mu^F}),$$

the last equality coming from $(*)$. If $\{\mu_1, \mu_2, \ldots\}$ are the measures for which $f_\mu^F \neq 0$, then clearly the latter sum is just

$$(2)\qquad \sum_{k=1}^\infty l(\widehat{f_{\mu_k}^F}) = \lim_{n\to\infty} l\Big(\sum_{k=1}^n \widehat{f_{\mu_k}^F}\Big).$$

But for any $\mu \in M$,

$$\sum_{k=1}^n \widehat{f_{\mu_k}^F}(\mu) = \begin{cases} [f_{\mu_k}^F]_{\mu_k} & \text{if } \mu = \mu_k \text{ and } k = 1, \ldots, n, \\ [0]_\mu & \text{otherwise}. \end{cases}$$

Since

$$\sum_{k=1}^\infty \theta(\|f_{\mu_k}^F\|_\Phi^{\mu_k})G(\mu_k) \leqslant \sum_{k=1}^\infty \theta(F(\mu_k))G(\mu_k) \leqslant 1,$$

we have

$$\sum_{\mu\in M} \theta\Big(\|f_\mu^F - \sum_{k=1}^n \widehat{f_{\mu_k}^F}(\mu)\|_\Phi^\mu\Big)G(\mu) = \sum_{k=n+1}^\infty \theta(\|f_{\mu_k}^F\|_\Phi^{\mu_k})G(\mu_k) \to 0$$

as $n\to\infty$. Denoting for now $h(\mu)=f_\mu^F$, the fact that $l_0^G=\tilde{l}_0^G$ implies

$$\lim_{n\to\infty}\hat{N}_\Phi^{\theta,G}\big(h-\sum_{k=1}^{n}\widehat{h(\mu_k)}\big)=0.$$

Thus by the continuity of l, and the fact that $\hat{N}_\Phi^{\theta,G}(h)\leqslant 1$,

$$\lim_{n\to\infty} l\big(\sum_{k=1}^{n}\hat{f}_{\mu_k}^F\big)=l(h)\leqslant\|l\|, \tag{3}$$

so combining (1)-(3),

$$\hat{N}_\Psi^{Z,G}(\hat{l})=N_Z^G(\|\hat{l}(\cdot)\|_\Psi^{(\cdot)})\leqslant\sup\{\sum_{\mu\in M}|F(\mu)|\,\|\hat{l}(\mu)\|_\Psi^\mu G(\mu):N_\theta^G(F)\leqslant 1\}\leqslant 4\|l\|,$$

which proves the continuity of $l\to\hat{l}$.

Given $\hat{g}\in\hat{E}_\Psi(Z,G)$, let the linear functional $\lambda(\hat{g})$ on $\hat{E}_\Phi(\theta,G)$ be defined by

$$(\lambda(\hat{g}))(\hat{f})=\sum_{\mu\in M}\big(\int_\Omega\hat{f}(\mu)\hat{g}(\mu)\,d\mu\big)G(\mu),\quad \hat{f}\in\hat{E}_\Phi(\theta,G).$$

(This makes sense because $\hat{f}(\mu)=0$ except for countably many $\mu\in M$.) Then if $\hat{N}_\Phi^{\theta,G}(\hat{f})\leqslant 1$, using Hölder inequalities twice,

$$\begin{aligned}|(\lambda(\hat{g}))(\hat{f})|&\leqslant\sum_{\mu\in M}\big(\int_\Omega|\hat{f}(\mu)\hat{g}(\mu)|\,d\mu\big)G(\mu)\leqslant\sum_{\mu\in M}\|\hat{f}(\mu)\|_\Phi^\mu\|\hat{g}(\mu)\|_\Psi^\mu G(\mu)\\&\leqslant 2N_\theta^G(\|\hat{f}(\cdot)\|_\Phi^{(\cdot)})N_Z^G(\|\hat{g}(\cdot)\|_\Psi^{(\cdot)})=2\hat{N}_\Phi^{\theta,G}(\hat{f})\hat{N}_\Psi^{Z,G}(\hat{g})\leqslant 2\hat{N}_\Psi^{Z,G}(\hat{g}).\end{aligned}$$

Thus λ is a continuous linear map of $\hat{E}_\Psi(Z,G)$ into $(\hat{E}_\Phi(\theta,G))^*$ with $\|\lambda(\hat{g})\|\leqslant 2\hat{N}_\Psi^{Z,G}(\hat{g})$.

It remains to show that $l\to\hat{l}$ and λ are mutually inverse. So if $\hat{g}=\hat{l}\in\hat{E}_\Psi(Z,G)$ $(l\in(\hat{E}_\Phi(\theta,G))^*)$, then for any $\hat{f}\in\hat{E}_\Phi(\theta,G)$,

$$\begin{aligned}(\lambda(\hat{g}))(\hat{f})&=\sum_{\mu\in M}\big(\int_\Omega\hat{f}(\mu)\hat{l}(\mu)\,d\mu\big)G(\mu)\\&=(\text{as in (1)-(3) above, letting } f_\mu=\hat{f}(\mu)/|F(\mu)|)\ l(\hat{f});\end{aligned}$$

in other words, $\lambda(\hat{g})=l$. On the other hand, if $l=\lambda(\hat{g})\in(\hat{E}_\Phi(\theta,G))^*$ $(\hat{g}\in\hat{E}_\Psi(Z,G))$, then for every $\mu\in M$ and $f_\mu\in L_\Phi(\mu)$, (*) implies

$$\begin{aligned}\int_\Omega f_\mu\hat{l}(\mu)\,d\mu&=\frac{(\lambda(\hat{g}))(\hat{f}_\mu)}{G(\mu)}=\frac{1}{G(\mu)}\sum_{\mu'\in M}\big(\int_\Omega\hat{f}_\mu(\mu')\hat{g}(\mu')\,d\mu'\big)G(\mu')\\&=\frac{1}{G(\mu)}\big(\int_\Omega f_\mu\hat{g}(\mu)\,d\mu\big)G(\mu)=\int_\Omega f_\mu\hat{g}(\mu)\,d\mu,\end{aligned}$$

so that $\hat{l}=\hat{g}$. Therefore, $l\to\hat{l}$ is a linear, topological isomorphism of $(\hat{E}_\Phi(\theta,G))^*$ onto $\hat{E}_\Psi(Z,G)$, q.e.d.

2.29. COROLLARY. *Suppose both Φ and Ψ satisfy the Δ_2-condition, and θ and Z each satisfies either* (a) *or* (b) *of the theorem. Then $E_\Phi(\theta,G)$ is reflexive. Moreover,* $(E_\Phi(\theta,G))^*\cong E_\Psi(Z,G)$.

Proof. By 2.28, $(\hat{E}_\Phi(\theta,G))^*\cong\hat{E}_\Psi(Z,G)$ and $(\hat{E}_\Psi(Z,G))^*\cong\hat{E}_\Phi(\theta,G)$, the isomorphisms being given in the proof of 2.28. To show that $\hat{E}_\Phi(\theta,G)$ is reflexive, it need only be proved that the composite isomorphism $\hat{E}_\Phi(\theta,G)\cong(\hat{E}_\Psi(Z,G))^*\cong(\hat{E}_\Phi(\theta,G))^{**}$ coincides with the natural embedding of $\hat{E}_\Phi(\theta,G)$ in $(\hat{E}_\Phi(\theta,G))^{**}$.

Thus, for any $\hat{f}\in\hat{E}_\Phi(\theta,G)$, the corresponding $\lambda(\hat{f})\in(\hat{E}_\Psi(Z,G))^*$ is given by

$$(\lambda(\hat{f}))(\hat{l})=\sum_{\mu\in M}\big(\int_\Omega\hat{f}(\mu)\hat{l}(\mu)\,d\mu\big)G(\mu)\quad\text{for all } \hat{l}\in\hat{E}_\Psi(Z,G).$$

But the correspondence between $l\in(\hat{E}_\Phi(\theta,G))^*$ and $\hat{l}\in\hat{E}_\Psi(Z,G)$ implies (writing $f_\mu=\hat{f}(\mu)$), in the notation of the preceding proof,

$$\int_\Omega\hat{f}(\mu)\hat{l}(\mu)\,d\mu=\frac{l(\hat{f}_\mu)}{G(\mu)}\quad\text{for all } \mu\in M.$$

Denoting by $L_{\hat{f}}$ the image of $\lambda(\hat{f})$ in $(\hat{E}_\Phi(\theta,G))^{**}$ (under the isomorphism induced by the correspondence $l\to\hat{l}$), one obtains, by combining the above,

$$L_{\hat{f}}(l)=(\lambda(\hat{f}))(\hat{l})=\sum_{\mu\in M}l(\hat{f}_\mu).$$

However, as noted in the proof of the theorem,

$$\sum_{\mu\in M}l(\hat{f}_\mu)=l(\hat{f}),\quad\text{i.e.,}\quad L_{\hat{f}}(l)=l(\hat{f}),$$

which shows that $L_{\hat{f}}$ is just the natural image of $\hat{f}$. Hence $\hat{E}_\Phi(\theta,G)$ is reflexive; but $E_\Phi(\theta,G)$, being a closed subspace of $\hat{E}_\Phi(\theta,G)$, is then also reflexive ([4], p. 67).

Finally, note that $E_\Phi(\theta,G)\precapprox(E_\Psi(Z,G))^*$ and $E_\Psi(Z,G)\precapprox(E_\Phi(\theta,G))^*$ (where $\precapprox$ denotes a topological, linear embedding), since it is easily seen that the mapping λ, defined in the proof of 2.28, maps $E_\Phi(\theta,G)$ into $(E_\Psi(Z,G))^*$ and $E_\Psi(Z,G)$ into $(E_\Phi(\theta,G))^*$. Then, using the Hahn-Banach theorem and the reflexivity of $E_\Psi(Z,G)$ (which follows by the symmetry of the hypotheses),

$$E_\Psi(Z,G)\precapprox(E_\Phi(\theta,G))^*\precapprox(E_\Psi(Z,G))^{**}\cong E_\Psi(Z,G).$$

Hence $(E_\Phi(\theta,G))^*\cong E_\Psi(Z,G)$, q.e.d.

3. Convexity properties. Theorems describing sufficient conditions that $E_\Phi(\theta,G)$ have the properties of rotundity (also called strict convexity) and of uniform rotundity (uniform convexity) are developed in this section. These convexity properties, unlike reflexivity, are not topological properties: in fact, every Orlicz space on a probability space is isomorphic to a strictly convex Orlicz space ([10], Th. 1). Therefore, in this connection, a statement of the particular norm involved is essential.

3.1. Definition. A normed linear space $\mathfrak{X}$, with norm $\|\cdot\|$, is called *rotund* if $x, y \in \mathfrak{X}$, $\|x\| = \|y\| = 1$, $x \neq y$ imply $\|x+y\| < 2$. This clearly is equivalent to: $\|x\| = \|y\|$ and $x \neq y$ imply $\|x+y\| < 2\,\|x\|$.

Suppose

$$\theta(x) = \int_0^{|x|} \vartheta(t)\,dt$$

and $\vartheta(1) > 0$. Define the *normalized* Young's function

$$\bar{\theta}(x) = \frac{1}{\vartheta(1)}\,\theta(x).$$

A new norm $\overline{N}_\theta^G$ equivalent to N_θ^G ([13], p. 173) on l_θ^G is defined by

$$\overline{N}_\theta^G(F) = \inf\left\{K > 0\colon \sum_{\mu \in M} \theta\left(\frac{F(\mu)}{K}\right) G(\mu) \leqslant \theta(1)\right\}.$$

Since $\overline{N}_{\bar{\theta}}^G = \overline{N}_\theta^G$, θ can (and will) be assumed to be normalized, a property which has many advantages for computational simplicity. Now define

$$\overline{N}_\Phi^{\theta,G}(f) = \overline{N}_\theta^G(F_f), \qquad F_f(\mu) = \|f\|_\Phi^\mu, \qquad f \in E_\Phi(\theta, G).$$

It is clear from the preceding that $N_\Phi^{\theta,G}$ and $\overline{N}_\Phi^{\theta,G}$ are equivalent norms on $E_\Phi(\theta, G)$.

3.2. Lemma. *If ϑ is continuous and strictly increasing, then l_θ^G is rotund under $\overline{N}_\theta^G$.*

Proof. Note that since points of M have finite positive m_G-measure, m_G has the finite subset property (i.e., every set of positive m_G-measure contains a set of finite, positive m_G-measure). Since ϑ is strictly increasing, $\vartheta(1) > 0$, so that θ may be assumed normalized. Hence the hypotheses of Theorem 4 in [9] are met, and by that result, l_θ^G is rotund under $\overline{N}_\theta^G$, q.e.d.

Any increasing function will be called *continuous in the extended sense* if it has no jump discontinuities.

3.3. Lemma (Milnes [7]). *If μ is a probability measure, then $L_\Phi(\mu)$ is rotund under $\|\cdot\|_\Phi^\mu$ whenever Ψ and ψ $(=\Psi')$ are continuous in the extended sense.*

3.4. Theorem. *If ϑ is continuous and strictly increasing, and ψ and Ψ are continuous in the extended sense, then $E_\Phi(\theta, G)$ is rotund under $\overline{N}_\Phi^{\theta,G}$.*

Proof. Suppose $\overline{N}_\Phi^{\theta,G}(f) = \overline{N}_\Phi^{\theta,G}(g) = 1$ and $\overline{N}_\Phi^{\theta,G}(f+g) = 2$, i.e., if $\bar{f}(\mu) = \|f\|_\Phi^\mu$, $\bar{g}(\mu) = \|g\|_\Phi^\mu$, $\bar{h}(\mu) = \|f+g\|_\Phi^\mu$ for all $\mu \in M$, then

$$\overline{N}_\theta^G(\bar{f}) = \overline{N}_\theta^G(\bar{g}) = 1, \qquad \overline{N}_\theta^G(\bar{f}+\bar{g}) \geqslant \overline{N}_\theta^G(\bar{h}) = 2.$$

Hence, by 3.2, $\bar{f} = \bar{g}$, i.e., $\|f\|_\Phi^\mu = \|g\|_\Phi^\mu$ for all $\mu \in M$. Now

$$(*) \qquad \bar{h}(\mu) = \|f+g\|_\Phi^\mu \leqslant \|f\|_\Phi^\mu + \|g\|_\Phi^\mu = 2\|f\|_\Phi^\mu = 2\bar{f}(\mu),$$

and there is *strict* inequality by 3.3, whenever $f \neq g\,[\mu]$. Suppose there is some $\mu_0 \in M$ with $f \neq g\,[\mu_0]$; then since θ is strictly increasing, it holds for any $K > 0$ that (by $(*)$)

$$(**) \qquad \theta\left(\frac{\bar{h}(\mu_0)}{K}\right) < \theta\left(\frac{2\bar{f}(\mu_0)}{K}\right).$$

Obviously, for any $\mu \in M$ and $K > 0$,

$$\theta\left(\frac{\bar{h}(\mu)}{K}\right) \leqslant \theta\left(\frac{2\bar{f}(\mu)}{K}\right).$$

Thus, by $(**)$ and the fact (as in the proof of Theorem 4 in [9]) that

$$\sum_{\mu \in M} \theta\left(\frac{F(\mu)}{\overline{N}_\theta^G(F)}\right) G(\mu) = \theta(1),$$

one obtains

$$\sum_{\mu \in M} \theta\big(\bar{f}(\mu)\big) G(\mu) > \sum_{\mu \in M} \theta\left(\frac{\bar{h}(\mu)}{2}\right) G(\mu) = \theta(1),$$

i.e., $\overline{N}_\theta^G(\bar{f}) > 1$, a contradiction. Therefore, it must be $f = g\,[\mu]$ for every $\mu \in M$, i.e., $f = g$, q.e.d.

3.5. Definition. A Banach space $\mathfrak{X}$, with norm $\|\cdot\|$, is called *uniformly rotund* if for every $0 < \varepsilon \leqslant 2$ there is a $0 < \delta(\varepsilon) < 1$ such that whenever $x, y \in \mathfrak{X}$, $\|x\| = \|y\| = 1$ and $\|x-y\| \geqslant \varepsilon$, then $\|x+y\| < 2\big(1-\delta(\varepsilon)\big)$.

$\delta(\varepsilon)$ can always be assumed to be a non-decreasing function of ε, such that $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$; merely substitute δ_1 for δ, where

$$\delta_1(\varepsilon) = \frac{\varepsilon}{3} \sup_{0 < \varepsilon' \leqslant \varepsilon} \delta(\varepsilon').$$

3.6. Definition. Any function $\delta(\cdot)$ as above is called a *modulus of uniform rotundity* (m.u.r.) for the given space (with the given norm).

Obviously, every uniformly rotund space is rotund and, moreover, it is known ([2], p. 113) that uniform rotundity implies reflexivity.

It has been proved by Milnes [7] that, under certain conditions on Φ, the spaces $L_\Phi(\mu)$ are uniformly rotund. However, for the present work, a seemingly stronger result will be needed – namely, that the modulus of uniform rotundity of such a space may be taken to be independent of the particular probability measure μ involved. Milnes makes no mention of the modulus, and certain constants appearing in his proof do apparently depend on μ. But a careful reworking of the proof (which will be omitted) shows that the m.u.r. does indeed not depend on μ; in the following

$$\Phi(x) = \int_0^{|x|} \varphi(t)\,dt \quad \text{and} \quad \Psi(x) = \int_0^{|x|} \psi(t)\,dt.$$

3.7. THEOREM. *Suppose $\mu(\Omega) = 1$, ψ is continuous, $\Phi(2x) \leqslant N\Phi(x)$ for all $x \geqslant 0$, and for every $\alpha \in (0, 1)$,*

$$\lim_{x\to\infty} \frac{\varphi(x)}{\varphi\big((1-\alpha)x\big)} > 1.$$

Then $L_\Phi(\mu)$ is uniformly rotund under $\|\cdot\|_\Phi^\mu$, and the m.u.r. is independent of μ (for details, see [11]).

3.8. LEMMA. *Suppose*

$$\theta(x) = \int_0^{|x|} \vartheta(t)\,dt,$$

where ϑ is continuous. If θ satisfies the Δ_2-condition (for all values of its argument), and if for every $0 < \varepsilon < 1$, there is a constant $K_\varepsilon > 1$ such that $\vartheta\big((1+\varepsilon)x\big) \geqslant K_\varepsilon \vartheta(x)$ for all $x > 0$, then l_θ^G is uniformly rotund under the norm $\overline{N}_\theta^G$ (introduced previously).

Proof. Since $\theta(2x) \leqslant c\theta(x)$ for all $x \geqslant 0$, it follows that $\vartheta(1) > 0$, and so θ can be assumed to be normalized (in the sense already discussed). Also, as before, m_G has the finite subset property. Thus all conditions of Theorem 5 in [9] are satisfied, and so l_θ^G is uniformly rotund under $\overline{N}_\theta^G$, q.e.d.

3.9. THEOREM. *If*

$$\theta(x) = \int_0^{|x|} \vartheta(t)\,dt, \quad \Phi(x) = \int_0^{|x|} \varphi(t)\,dt, \quad \Psi(x) = \int_0^{|x|} \psi(t)\,dt,$$

and

(a) $\theta(2x) \leqslant c\theta(x)$, $\Phi(2x) \leqslant c\Phi(x)$ *for all* $x \geqslant 0$;

(b) ϑ *and* φ *are continuous*;

(c) *for all* $0 < \varepsilon < 1$,

$$\inf_{x>0} \frac{\vartheta\big((1+\varepsilon)x\big)}{\vartheta(x)} > 1 \quad \text{and} \quad \lim_{x\to\infty} \frac{\varphi\big((1+\varepsilon)x\big)}{\varphi(x)} > 1;$$

then $E_\Phi(\theta, G)$ is uniformly rotund under $\tilde{N}_\Phi^{\theta,G}$.

Proof. The space l_θ^G is a "proper function space", i.e., a normed linear space of real-valued functions such that if $F \in l_\theta^G$ and $0 \leqslant F_1 \leqslant F$, then $F_1 \in l_\theta^G$ and $\overline{N}_\theta^G(F_1) \leqslant \overline{N}_\theta^G(F)$. Comparing the hypotheses here with those of 3.7 and 3.8, it is seen that l_θ^G is uniformly rotund under $\overline{N}_\theta^G$, and all $L_\Phi(\mu)$, $\mu \in M$, are uniformly rotund under $\|\cdot\|_\Phi^\mu$, with m.u.r. independent of μ. But these facts, applied to the space $\hat{E}_\Phi(\theta, G)$ (cf. 2.27) with the norm $\tilde{N}_\Phi^{\theta,G}$, given by $\tilde{N}_\Phi^{\theta,G}(\hat{f}) = \overline{N}_\theta^G(\|\hat{f}(\cdot)\|_\Phi^{(\cdot)})$ imply by Theorem 3 in [1] that $\hat{E}_\Phi(\theta, G)$ is uniformly rotund under the given norm.

It is a trivial consequence of 3.5 that every subspace of a uniformly rotund space is uniformly rotund (under the restriction of the norm). Since the restriction of $\tilde{N}_\Phi^{\theta,G}$ to the subspace $E_\Phi(\theta, G)$ is $\tilde{N}_\Phi^{\theta,G}$, the proof is complete, q.e.d.

References

[1] M. M. Day, *Uniform convexity, III*, Bull. Amer. Math. Soc. 49 (1943), p. 745–750.
[2] — *Normed linear spaces*, Berlin 1958.
[3] J. L. Doob, *Stochastic processes*, New York 1953.
[4] N. Dunford and J. T. Schwartz, *Linear operators*, Part I, New York 1958.
[5] M. A. Krasnosel'skiĭ and Ya. B. Rutickiĭ, *Convex functions and Orlicz spaces* (translation), Groningen 1961.
[6] P. A. Meyer, *Probability and potentials*, Waltham, Mass., 1966.
[7] H. W. Milnes, *Convexity of Orlicz spaces*, Pac. J. Math. 7 (1957), p. 1451–1483.
[8] T. S. Pitcher, *A more general property than domination for sets of probability measures*, ibidem 15 (1965), p. 597–611.
[9] M. M. Rao, *Smoothness of Orlicz spaces, II*, Indag. Math. 27 (1965), p.681–690.
[10] — *Almost every Orlicz space is isomorphic to a strictly convex Orlicz space*, Proc. AMS 19 (1968), p. 377–379.
[11] R. L. Rosenberg, *Compactness in Orlicz spaces based on sets of probability measures*, Thesis, Carnegie—Mellon University, 1968.
[12] A. C. Zaanen, *Linear analysis*, Amsterdam 1953.
[13] A. Zygmund, *Trigonometric series*, Vol. I (second edition), Cambridge 1959.

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