(iii) $\forall K \inf_{G\supseteq K} \sup_{\mu\in\mathscr{P}} |\mu|(G\setminus K) = 0$ and $\forall A\in\mathscr{B} \inf_{K\subseteq A} \sup_{\mu\in\mathscr{P}} |\mu|(A\setminus K) = 0$.

To prove this, one proves the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). The ideas involved in the proof are either contained in this section or else well-known from other sources.

I do not know if the compactness results for *nets* on $\mathfrak{M}_+(\ldots)$ generalize to nets on $\mathfrak{M}(\ldots)$.

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STUDIA MATHEMATICA, T. XXXVI. (1970)

On isometries of normed linear spaces

by

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1. Introduction. Fixman [2] showed that it was not possible to develop a general spectral theory for invertible isometries on an arbitrary Banach space. The purpose of this note is to obtain a simple characterization of the isometries on a normed linear space (real or complex), in terms of semi-inner products. This is then used to show that eigenvectors corresponding to distinct eigenvalues of an isometry are "orthogonal", and to establish some facts about the point spectrum of isometries. In some minor respects, invertible isometries have spectral behavior like unitary operators on Hilbert space. Also, conditions are found that are necessary and sufficient for a given operator on a normed linear space to be equivalent to an isometry on an equivalent normed linear space.

Lumer [8] has shown that in any normed linear space X, one can construct a semi-inner-product $[\cdot,\cdot]$ (there may be more than one), i.e., a mapping from $X\times X$ into C such that

(i) $[x, x] = \|x\|^2$,

(ii) $[ax+by, z] = a[x, z]+b[y, z]$,

(iii) $|[x, y]|^2 \leqslant [x, x]\cdot[y, y]$.

Giles has shown ([3], p. 437) that it is always possible to choose a semi-inner-product such that

(iv) $[x, ay] = \bar{a}[x, y]$.

We shall assume for the rest of the paper that all semi-inner-products satisfy (iv).

We shall follow James [6] in saying that x is *orthogonal* to y if $\|x\| \leqslant \|x+ay\|$ for all scalars a.

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2. A characterization of isometries.

THEOREM 1. *Let X be a normed linear space (real or complex) and let U be an operator mapping X into itself. Then U is an isometry if and only if there is a semi-inner-product $[\cdot,\cdot]$ such that $[Ux, Uy] = [x, y]$ for all x and y.*

Proof. The proof is similar to a proof given by Sz.-Nagy [9] for a related theorem (see Theorem 2, below) in the case of Hilbert space. The sufficiency is clear. To see the necessity, let $[\cdot,\cdot]$ be any semi-inner-product on X. Then observe that $\{[U^n x, U^n y]\}_{n=1}^{\infty}$ is a bounded sequence of complex numbers for each fixed x and y, (since $|[U^n x, U^n y]| \leqslant \|U^n x\|\,\|U^n y\| = \|x\|\,\|y\|$). Let Lim be a Banach limit on l^∞, i.e., a linear functional of norm 1 such that $\mathrm{Lim}\{1,1,1,\ldots\} = 1$ and $\mathrm{Lim}\{x_n\} = \mathrm{Lim}\{x_{n+1}\}$ (see [1]). Now define

$$[[x, y]] = \mathrm{Lim}\{[U^n x, U^n y]\}.$$

It is easy to see that $[[\cdot,\cdot]]$ is a semi-inner-product that generates the given norm. Further, the translation invariance of Lim gives

$$[[Ux, Uy]] = \mathrm{Lim}\{[U^{n+1}x, U^{n+1}y]\} = \mathrm{Lim}\{[U^n x, U^n y]\} = [[x, y]].$$

Stampfli [10] has called an invertible operator U on a Banach space X *iso-abelian* if there exists a semi-inner-product $[\cdot,\cdot]$ such that $[Ux, y] = [x, U^{-1}y]$.

COROLLARY 1. *U is iso-abelian if and only if it is an invertible isometry.*

Proof. Iso-abelian operators are isometric, as Stampfli observes. Conversely, if U is isometric choose a semi-inner product that is preserved by U. Then, if $Uz = y$, $[Ux, y] = [Ux, Uz] = [x, z] = [x, U^{-1}y]$.

COROLLARY 2. *If X is smooth, then U is an isometry if and only if it preserves the semi-inner-product on X.*

Proof. If X is smooth, then it has a unique semi-inner-product. Corollary 2 appears in [7].

THEOREM 2. *Let T be an operator on a normed linear space X. Then T is similar to an isometry on an equivalent normed linear space if and only if there are constants δ and M such that $\delta\|x\| \leqslant \|T^n x\|$ and $\|T^n\| \leqslant M$ for all positive integers n and $x \in X$.*

Proof. Clearly, every operator similar to an isometry satisfies the conditions. To see the converse, define $|||x||| = \mathrm{Lim}\{\|T^n x\|\}$, where Lim is any Banach limit. It is easy to see that $|||\ |||$ defines a norm equivalent to $\|\ \|$. Hence if we let S denote T considered as an operator on the vector space X with norm $|||\ |||$, it follows that S is an isometry (because of the translation invariance of Lim), and that S and T are similar.

The case of Theorem 2 where T is invertible is well-known (see [9] and [4]), with proofs similar to ours.

3. Point spectrum of isometries.

THEOREM 3. *If U is an isometry, e and f are eigenvectors corresponding to distinct eigenvalues of U, and if $[\cdot,\cdot]$ is any semi-inner-product preserved by U, then $[e, f] = [f, e] = 0$.*

Proof. Let $Ue = \alpha e$, $Uf = \beta f$. Then $|\alpha| = |\beta| = 1$, and $\alpha\bar{\beta} = \alpha/\beta$. Now $[e, f] = [Ue, Uf] = (\alpha/\beta)[e, f]$. Since $\alpha/\beta \neq 1$, $[e, f] = 0$.

COROLLARY 3. *Eigenvectors corresponding to distinct eigenvalues of an isometry are mutually orthogonal.*

Proof. Giles [3], p. 438, observes that $[y, x] = 0$ for any semi-inner-product implies that $\|x+\lambda y\| \geqslant \|x\|$ for all scalars λ. Hence the result follows immediately from Theorems 1 and 3.

THEOREM 4. *A finite-dimensional eigenspace of an isometry has a complement invariant under the isometry.*

Proof. The proof is by induction on the dimension of the eigenspace. Let U be an isometry and let $E_\lambda = \{x\colon Ux = \lambda x\}$.

If the dimension of E_λ is 1, let x_0 be a unit vector in E_λ. Choose a semi-inner product preserved by U and let $M = \{x\colon [x, x_0] = 0\}$. Then M is the nullspace of a continuous linear functional and hence is a complement of E_λ. Now if $x \in M$, then $[Ux, Ux_0] = [Ux, \lambda x_0] = \bar{\lambda}[Ux, x_0]$, and also $[Ux, Ux_0] = [x, x_0] = 0$. Hence $Ux \in M$.

Assume that the result is known when the dimension of the eigenspace is n, and let E_λ have dimension $n+1$. Choose a unit vector x_0 in E_λ. If $N = \{x\colon [x, x_0] = 0\}$, then N is a complement of the subspace spanned by x_0, and N is invariant under U (as in the case $n = 1$). Clearly, the eigenspace of $U|_N$ corresponding to λ is $E_\lambda \cap N$, and has dimension n. By the inductive hypothesis $E_\lambda \cap N$ has a complement M in N which is invariant under U. Then M is a complement of E_λ in the original space.

The following corollary is known [4]:

COROLLARY 4. *An isometry on a finite-dimensional normed linear space is diagonable (i.e., the space has a basis consisting of eigenvectors of the isometry).*

Proof. It follows immediately from Theorem 4 that whenever U is an isometry on a finite-dimensional space X, then X is a direct sum of eigenspaces of U.

Stampfli [10] shows that the following result holds for adjoint-abelian operators, and observes that his proof goes through for iso-abelian operators too.

THEOREM 5. *If U is an invertible isometry and λ is an isolated point of the spectrum of U, then λ is an eigenvalue and the eigenspace of U corresponding to λ has an invariant complement.*

Proof. This follows from Stampfli's work and our Corollary 1. Theorems 4 and 5 suggest the possibility that every eigenspace of an isometry has an invariant complement. This holds for isometries of Hilbert space, since every isometry of a Hilbert space is the direct sum of a unitary operator and a unilateral shift (see [5]), but is apparently unknown for isometries of arbitrary Banach spaces.

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A characterization of multiplication by the independent variable on $\mathscr{L}^p$

by

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1. Introduction. One way of viewing the spectral theorem for Hermitian operators on complex Hilbert spaces is: every Hermitian operator is unitarily equivalent to a multiplication operator. This formulation of the spectral theorem has been popularized by Halmos [2]. The essence of the spectral theorem is then the statement that an operator A is a Hermitian operator with a cyclic vector if and only if there is a compact subset $\mathscr{S}$ of $\boldsymbol{R}$ and a finite measure μ on $\mathscr{S}$ such that A is unitarily equivalent to multiplication by the independent variable on $\mathscr{L}^2(\mathscr{S},\mu)$. The proof of this assertion in the case where A is an operator on a real Hilbert space can proceed exactly as the proof of the complex case in [2] once it is known that $\|q(A)\| = \sup_{t\in\sigma(A)}|q(t)|$ for all (real) polynomials q.

In this note we consider the problem of characterizing the operator M_x defined on $\mathscr{L}^p(\mathscr{S},\mu)$ (where $\mathscr{S}$ is a compact subset of $\boldsymbol{R}$ and $1\leqslant p<\infty$) by

$$(M_x f)(x) = xf(x) \quad \text{ for } f\in\mathscr{L}^p.$$

That is, we find a necessary and sufficient condition that an operator A on a Banach space be isometrically equivalent to M_x on $\mathscr{L}^p(\mathscr{S},\mu)$. Our proof will be very similar to the proof of the case $p=2$ presented in [2].

We give a similar characterization of multiplication by z on $\mathscr{L}^p(\mathscr{S},\mu)$ where $\mathscr{S}$ is a compact subset of the complex plane.

2. Properties of M_x. Let μ be a finite Borel measure on $\boldsymbol{R}$ with compact support $\mathscr{S}$ and fix $p, 1\leqslant p<\infty$. We consider some properties of the operator M_x on $\mathscr{L}^p(\mathscr{S},\mu)$. We consider the real and complex cases simultaneously unless otherwise specified.

Clearly $\sigma(M_x)=\mathscr{S}$, and $\|q(M_x)\| = \sup_{x\in\mathscr{S}}|q(x)|$ for all polynomials q. This means that the map $q\to q(M_x)$ is an isometry from the polynomials (with sup norm) into the algebra of bounded operators on $\mathscr{L}^p$ (with