STUDIA MATHEMATICA, T. XXXVI. (1970)

Large squares and sets of analyticity in tensor algebras

by

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1. Introduction. If X and Y are compact Hausdorff spaces, $V(X\times Y) = C(X)\hat{\otimes} C(Y)$ will denote the projective tensor product of the Banach algebras $C(X)$ and $C(Y)$. $V(X\times Y)$ is a regular symmetric Banach algebra under the projective norm: for a detailed discussion of its definitions and elementary properties see [11]; a summary of them is to be found in [10].

If E is a closed subset of $X\times Y$, we define the closed ideal $I(E)$ by

$$I(E) = \{f\in V(X\times Y)\colon f(z) = 0 \text{ when } z\in E\}.$$

$V(E) = V(X\times Y)/I(E)$ is the algebra of restrictions to E of functions of $V(X\times Y)$. If $V(E)\cong C(E)$, we call E a *V-Helson set.*

Let φ be a continuous complex-valued function defined on the interval $[-1,1]$ of the real line. φ is said to *operate* on the algebra $V(E)$ if $\varphi\circ f\in V(E)$ whenever $f\in V(E)$ has range in $[-1,1]$. E is called a *set of analyticity* (for the algebra $V(X\times Y)$) if any function operating on $V(E)$ can be extended to an analytic function in a neighbourhood of $[-1,1]$, i.e. E is a set of analyticity if "only the analytic functions" operate on $V(E)$.

For a compact abelian group G, we define $A(G)$ to be, as usual, the algebra of Gelfand (Fourier) transforms of $L^1(\hat{G})$, where $\hat{G}$ is the dual group of G. If E is a compact subset of G, $I(E)$ and $A(E)$ are defined similarly to the respective cases above. If $A(E)\cong C(E)$, E is called a *Helson set.* E is a *set of analyticity* (for the algebra $A(G)$) if only the analytic functions operate on $A(E)$.

The dichotomy conjecture (cf. [4], [6]) is that every compact subset of a compact abelian group which is not a Helson set should be a set of analyticity. We can pose the same question in the context of the tensor algebra $V(X\times Y)$ and there too the answer is not known, however the following combinatorial characterization of countable V-Helson sets suggests combinatorial sufficient conditions for sets to be sets of analyticity.

If X and Y are compact Hausdorff spaces and n is a positive integer, we call *n-squares* those subsets of $X\times Y$ of the form $S_n = X_n\times Y_n$, where $X_n\subset X$, $Y_n\subset Y$ and $|X_n| = |Y_n| = n$.

THEOREM 1.1 (Varopoulos [11], 6.4]). *A countable closed subset $E \subset X \times Y$ is a V-Helson set if and only if there exists a positive integer λ such that $|E \cap S_n| < \lambda n$ for all n-squares S_n and for all positive integers n.*

In this paper, we consider the case of the intersection of a sequence of n-squares $\{S_n\}_{n=1}^{\infty}$ with $E \subset X \times Y$, showing first that if $|E \cap S_n| = n^2$, then E is a set of analyticity. Then, by a combinatorial refinement we show that it is sufficient to have $|E \cap S_n| \geqslant n^{2-\varepsilon_n}$, where $\varepsilon_n \to 0$ as $n \to \infty$. For the remaining results we use the methods developed by Katznelson and Malliavin [5, 6] to show that if $n|E \cap S_n|^{-1} \to 0$, then E is almost surely (with respect to a certain probability space) a set of analyticity. We also give similar results for a more specialised class of sets.

In conclusion, I would like to thank Dr. N. Th. Varopoulos for his extremely stimulating help and guidance. I want also to thank the S. R. C. and N. A. T. O. for their invaluable financial support.

2. LEMMA 2.1. *Let X, Y be compact Hausdorff spaces and let E be a closed subset of $X \times Y$. Let E contain an n-square for every positive integer n.*

Then E is a set of analyticity for the tensor algebra $V(X \times Y)$.

Proof. By $\boldsymbol{T}$ we shall denote the group of complex numbers with unit modulus, and by $\boldsymbol{Z}(n)$ the cyclic group of order n. We shall identify $\boldsymbol{Z}(n)$ with its embedding in $\boldsymbol{T}$ as the group of n^{th} roots of unity.

It is known [2] that there exists a constant $a > 0$ such that for any positive real number $R > 0$ there exists a real function $f \in A(\boldsymbol{T})$ with

$$\|f\|_{A(\boldsymbol{T})} < R \quad \text{and} \quad \|e^{if}\|_{A(\boldsymbol{T})} > e^{aR}.$$

Let $\varphi \in A^*(\boldsymbol{T})$ be such that $\|\varphi\|_{A^*} = 1$ and $|\langle e^{if}, \varphi\rangle| > e^{aR}$. By the method of Herz [7], there exists a sequence $(\varphi_n)_{n=1}^{\infty}$ of pseudomeasures $\varphi_n \in A^*(\boldsymbol{Z}(n))$ such that $\|\varphi_n\| \leqslant 1$ and $\varphi_n \to \varphi$ in the weak star topology. Hence there exists n_0 such that for all $n > n_0$, $|\langle e^{if}, \varphi_n\rangle| > e^{aR}$ and thus $\|e^{if}|_{\boldsymbol{Z}(n)}\|_{A(\boldsymbol{Z}(n))} > e^{aR}$. So, fixing $n > n_0$, we have a real function $f_1 = f|_{\boldsymbol{Z}(n)} \in A(\boldsymbol{Z}(n))$ such that

$$\|f_1\|_A < R \quad \text{and} \quad \|e^{if_1}\|_A > e^{aR}.$$

Now let $S_n = X_n \times Y_n$ be an n-square in E. To transfer f_1 from $A(\boldsymbol{Z}(n))$ to $V(S_n)$, we identify S_n with the set $\boldsymbol{Z}(n) \times \boldsymbol{Z}(n)$ and we consider the map $M\colon A(\boldsymbol{Z}(n)) \to V(S_n)$ defined by

$$Mh(x, y) = h(x+y)$$

for any $h \in A(\boldsymbol{Z}(n))$ and $x, y \in \boldsymbol{Z}(n)$. This map is an isometric isomorphism sidentifying $A(\boldsymbol{Z}(n))$ with a closed subalgebra of $V(S_n)$ ([11], 8.1), o we have a real function $g_1 \in V(S_n)$ satisfying

$$\|g_1\|_{V(S_n)} < R \quad \text{and} \quad \|e^{ig_1}\|_{V(S_n)} > e^{aR}.$$

Hence there is a function $g \in V(E)$ taking only real values and having

$$\|g\|_{V(E)} < 2R \quad \text{and} \quad \|e^{ig}\|_{V(E)} > e^{aR}$$

which is enough to show that E is a set of analyticity [3].

Comparing this result with theorem 1.1, it is natural to ask whether we can glean information from the type of intersection of a sequence of n-squares with a particular set.

We shall say that a sequence $\{S_n\}_{n=1}^{\infty}$ of n-squares has *incidence* $d(n)$ in a subset E of $X \times Y$ if $d(n)$ is a function on the positive integers such that

$$|(E \cap S_n)| \geqslant d(n)$$

for an infinity of values of n.

THEOREM 2.2. *Let X, Y be compact Hausdorff spaces, let E be a closed subset of $X \times Y$ and let $\{S_n\}_{n=1}^{\infty}$ be a sequence of n-squares in $X \times Y$ having incidence $n^{2-\varepsilon_n}$ in E, where $\varepsilon_n \to 0$ as $n \to 0$.*

Then E is a set of analyticity for the tensor algebra $V(X \times Y)$.

This theorem immediately follows from lemma 2.1 and the following proposition, due to Kővari, Sós and Turàn [8]:

PROPOSITION 2.3. *Let X, Y be arbitrary sets, let E be a subset of $X \times Y$ and let $\{S_n\}_{n=1}^{\infty}$ be a sequence of n-squares having incidence $n^{2-\varepsilon_n}$ in E, where $\varepsilon_n \to 0$ as $n \to \infty$.*

Then we can find a sequence of n-squares $\{T_n\}_{n=1}^{\infty}$ such that $T_n \subset E$ for each n.

Proof. Without loss of generality we shall assume that n^{ε_n} is an integer. To avoid unnecessary complication we shall suppose also that $|(S_n \cap E)| \geqslant n^{2-\varepsilon_n}$ for every $n \geqslant 1$.

By *rows* (respectively *columns*) of S_n we shall understand sets of the form $x_0 \times Y_n$ (respectively $X_n \times y_0$), where $x_0 \in X_n$ $(y_0 \in Y_n)$. Let r be a positive integer and suppose n_0 is such that $\varepsilon_n < 1/r$ for $n \geqslant n_0$. For some such n we select a subset $S_n' \subset S_n$ consisting of $t = rn^{\varepsilon_n}$ rows of S_n and containing at least rn points of E. Let s be the number of columns of S_n' containing at least r points of E. Then, calculating an upper bound for the number of points in S_n', we obtain $st + (n-s)(r-1) \geqslant rn$ and hence $st \geqslant n$ for $r > 1$ (the case $r = 1$ is, of course, trivial). In any column of S_n' containing at least r points of E, r points of E can be arranged in $\binom{t}{r}$ ways, so if $s > r\binom{t}{r}$, we shall have obtained an r-square $T_r \subset E$. To do this we observe that

$$r\binom{t}{r} < t^r = r^r n^{r\varepsilon_n} \quad \text{and} \quad s \geqslant (n^{1-\varepsilon_n})/r,$$

so if n is large enough, we obtain the desired result.

3. We follow Katznelson and Malliavin [5] in introducing the definitions below.

Let F be a finite-dimensional vector space over R and let $\| \|_1, \| \|_2$ be two norms on F such that $\| \|_2 \leqslant \| \|_1$. Let $\| \|_1^*$ be the dual norm on the dual F^* of $(F, \| \|_1)$. A *set of majoration* of $\| \|_1$ with respect to $\| \|_2$ is a subset S of the unit ball of $(F, \| \|_2)$ such that

$$\|\mu\|_1^* \leqslant 2 \sup_S |\langle \mu, f\rangle|$$

for all $\mu \in F^*$. The *scale* of $\| \|_1$ compared with $\| \|_2$ is the cardinal of a smallest possible set of majoration. If X, Y are two compact spaces and E is a finite subset of $X\times Y$, the *arithmetic diameter* $d(E)$ of E is the scale of $\| \|_{V(E)}$ compared with $\| \|_{C(E)}$ (this latter definition being applicable also when considering $V(E)$, $C(E)$ as complex vector spaces).

Lemma 3.1. *Let X, Y be finite sets having $|X| = |Y| = n$, a positive integer. Then $d(E) \leqslant 2^{4n}$ for any subset E of $X\times Y$.*

Proof. Let μ be a measure on E. Then

$$\|\mu\|_{A^*(E)} = \sup |\langle \mu, f\otimes g\rangle|,$$

where $f\in C(X)$, $g\in C(Y)$ and $\|f\|_\infty = \|g\|_\infty = 1$.

Put

$$F = \{f\in C(X)\colon f(x) = e^{2\pi ir/4};\ r = 0, 1, 2, 3;\ x\in X\},$$
$$G = \{g\in C(Y)\colon g(y) = e^{2\pi ir/4};\ r = 0, 1, 2, 3;\ y\in Y\}.$$

Then $S = \{f\otimes g\colon f\in F, g\in G\}$ is a set of majoration. To show this, let $f'\in C(X)$, $g'\in C(Y)$, $\|f'\|_\infty = \|g'\|_\infty = 1$. Consider

$$\langle \mu, f'\otimes g'\rangle = \sum_{x,y} \mu(x, y) f'(x) g'(y) = \varrho e^{i\theta}, \quad \text{say}.$$

For each x, replace $f'(x)$ by $f(x) = \exp\big(2\pi i r(x)/4\big)$, where $r(x) \in \{0, 1, 2, 3\}$ in such a way that

$$\Big|\arg\Big\{f(x) \sum_y \mu(x, y) g'(y)\Big\} - \theta\Big| \leqslant \pi/4.$$

It is then clear that

$$|\langle \mu, f\otimes g'\rangle| \geqslant \frac{1}{\sqrt{2}}\, \varrho.$$

Similarly, replacing g' by $g\in G$, we obtain

$$\|\mu\|_{A^*(E)} = \sup |\langle \mu, f'\otimes g'\rangle| \leqslant 2 \sup_S |\langle \mu, f\otimes g\rangle|.$$

Theorem 3.2. *Let X, Y be compact Hausdorff spaces and let $\{S_n\}_{n=1}^\infty$ be a collection of n-squares of $X\times Y$. Let δ_x be the Dirac measure of a point $x\in S_n$ and let $\Delta_n = \{\delta_x\}_{x\in S_n}$. Let also θ_n be a random measure equidistributed in Δ_n and put*

$$\xi_n = \frac{1}{p_n} \sum_{j=1}^{p_n} \theta_{nj},$$

where the θ_{nj} are p_n independent copies of θ_n. ξ_n and ξ_m are to be independent when $n \neq m$. Put $E_n = \operatorname{supp} \xi_n$.

Then if $np_n^{-1} \to 0$ as $n\to\infty$, the set $H = \overline{\bigcup_{n=1}^\infty E_n}$ is almost surely a set of analyticity.

We start by proving two lemmas.

Lemma 3.3. *Let m and t be relatively prime positive integers and put $n = mt$. Let μ and ν denote the Haar measures on $\boldsymbol{Z}(m)$, $\boldsymbol{Z}(n)$ respectively. Then to any real function $f\in A\big(\boldsymbol{Z}(m)\big)$ there corresponds a real function, $g\in A\big(\boldsymbol{Z}(n)\big)$ with*

$$\|f\|_A = \|g\|_A \quad \text{and} \quad \|\mu e^{if}\|_{A^*} = \|\nu e^{ig}\|_{A^*}.$$

Proof. Since $(m, t) = 1$, we can write $\boldsymbol{Z}(n) = \boldsymbol{Z}(m)\times\boldsymbol{Z}(t)$ and $\boldsymbol{Z}(n)\hat{} = \boldsymbol{Z}(m)\hat{}\times\boldsymbol{Z}(t)\hat{}$. Let p be the corresponding quotient map $p\colon \boldsymbol{Z}(n)\to\boldsymbol{Z}(m)$. Given $f\in A\big(\boldsymbol{Z}(m)\big)$, we define $g = \check{p}f\in A\big(\boldsymbol{Z}(n)\big)$ by $g(z) = f(pz)$ for each $z\in\boldsymbol{Z}(n)$. Clearly the map $\check{p}$ is a monomorphism and if f is a real function so is g.

Any character $\chi\in\boldsymbol{Z}(n)\hat{}$ can be written in the form $\chi = (\chi_m, \chi_t)$, where χ_m, χ_t are elements of $\boldsymbol{Z}(m)\hat{}$, $\boldsymbol{Z}(t)\hat{}$ respectively. Similarly, we write $z = (x, y)\in\boldsymbol{Z}(n)$, where $x\in\boldsymbol{Z}(m)$ and $y\in\boldsymbol{Z}(t)$. Then we have

$$\begin{aligned}
\hat{g}(\chi) &= (1/n) \sum_{\boldsymbol{Z}(n)} g(z)\bar{\chi}(z) \\
&= (1/t) \sum_{\boldsymbol{Z}(t)} \Big(1/m \sum_{\boldsymbol{Z}(m)} f(x)\bar{\chi}_m(x)\Big)\, \bar{\chi}_t(y) \\
&= \begin{cases} \hat{f}(\chi_m) & \text{if } \chi_t \equiv 1, \\ 0 & \text{otherwise}. \end{cases}
\end{aligned}$$

Hence $\|f\|_A = \|g\|_A$ and similarly, $\|\mu e^{if}\|_{A^*} = \|\nu e^{ig}\|_{A^*}$.

In the next lemma, $C > 0$ and $\alpha > 0$ are absolute constants, the values of which are irrelevant for our purposes.

LEMMA 3.4. *Let t be a positive integer. We can find a positive integer n such that if R is any integer with $C < R \leqslant t$, then there exists a real function $f \in A(\boldsymbol{Z}(n))$ satisfying*

$$\|f\|_A < R \quad \text{and} \quad \|\nu e^{if}\|_{A^*} < e^{-aR},$$

where ν is the Haar measure on $\boldsymbol{Z}(n)$.

Proof. By [5] there are positive real numbers k_1, k_2 such that if $m > k_1$ is a positive integer, there exists a real function $f \in A(\boldsymbol{Z}(m))$ with

$$\|f\|_A < k_2 \log m \quad \text{and} \quad \|\mu e^{if}\|_{A^*} < m^{-1/4},$$

where μ is the Haar measure on $\boldsymbol{Z}(m)$. Writing $R = [k_2 \log m]+1$, we obtain

$$\|f\|_A < R \quad \text{and} \quad \|\mu e^{if}\|_{A^*} < e^{-aR} \tag{3.5}$$

for some $a > 0$. It follows that if R is a large enough positive integer, there is an integer m' such that for any integer m with $m' \leqslant m \leqslant 2m'$, we can find a real function $f \in A(\boldsymbol{Z}(m))$ satisfying (3.5). By Bertrand's postulate (see e.g. [1]), we can find a prime, m_R say, such that $m' \leqslant m_R \leqslant 2m'$. If we put $n = \text{l.c.m}\,\{m_R\}_{C<R\leqslant t}$, it follows from lemma 3.3 that for any R with $C \leqslant R \leqslant t$, we can find $f \in A(\boldsymbol{Z}(n))$ satisfying the inequalities of lemma 3.4.

Proof of theorem 3.2. For any positive integer n we identify S_n once and for all with $\boldsymbol{Z}(n)\times\boldsymbol{Z}(n)$. The map $M\colon A(\boldsymbol{Z}(n)) \to V(S_n)$ of lemma 2.1 identifies $A(\boldsymbol{Z}(n))$ isometrically with a closed subalgebra, $A'(S_n)$, of $V(S_n)$. $A'(S_n)$ consists of those functions g of $V(S_n)$ which satisfy $g(x,y) = g(z,t)$ whenever $x+y = z+t$ ([11], 8.1).

Let t be a positive integer. By lemma 3.4 and the remarks above, there is a positive integer $n = n_t$ such that for any integer R with $C < R \leqslant t$ there is a real function $f = f_{R,t} \in A'(S_n)$ such that

$$\|f\|_{V(S_n)} < R \quad \text{and} \quad \|e^{if}\nu_n\|_{A'(S_n)^*} < e^{-aR},$$

where ν_n is the equidistributed positive measure of total mass 1 on S_n, and C and a are the constants of lemma 3.4.

Moreover,

$$\|e^{if}\nu_n\|_{V^*(S_n)} < e^{-aR}$$

for if not, there exists $g \in V(S_n)$ with

$$\|g\|_V \leqslant 1 \quad \text{and} \quad |\langle g, e^{if}\nu_n\rangle| \geqslant e^{-aR}.$$

Consider

$$\tilde{g} = \frac{1}{n}\sum_{x\in\boldsymbol{Z}(n)} g_x,$$

where g_x is defined by

$$g_x(y,z) = g(y+x, z-x)$$

for all $y, z \in \boldsymbol{Z}(n)$. Clearly $\tilde{g} \in A'(S_n)$,

$$\|\tilde{g}\|_V \leqslant 1 \quad \text{and} \quad |\langle \tilde{g}, e^{if}\nu_n\rangle| \geqslant e^{-aR},$$

which is a contradiction. So we have a real function $f \in V(S_n)$ with

$$\|f\|_V < R \quad \text{and} \quad \|e^{if}\nu_n\|_{V^*} < e^{-aR}.$$

Let ξ_n be the random measure of the statement of the theorem. Let $\{k_1, \ldots, k_s\}$ be a set of majoration of $\|\ \|_{V(S_n)}$ with respect to $\|\ \|_\infty$, where s is the arithmetic diameter of S_n. Then we have

$$\|e^{if}\xi_n\|_{V^*} \leqslant 2 \sup_{r=1,\ldots,s} |\langle k_r, e^{if}\xi_n\rangle|. \tag{3.6}$$

Let Z^r be the random variable defined by

$$Z^r(x,y) = k_r(x,y)\exp\big(if(x,y)\big)$$

as (x,y) is chosen at random in S_n. Then we have

$$|\mathscr{E}(Z^r)| \leqslant \|e^{if}\nu_n\|_{V^*} < e^{-aR}$$

and

$$\langle k_r, e^{if}\xi_n\rangle = \frac{1}{p_n}\sum_{q=1}^{p_n} Z_q^r,$$

where Z_q^r is a copy of Z^r. We shall use the following lemma (for a proof see [9]):

LEMMA 3.7. *Let Z be a complex-valued random variable with $|Z| \leqslant 1$ and $\mathscr{E}(Z) = a$. Let $Z_1, \ldots, Z_p$ be p independent copies of Z and let*

$$Z^* = p^{-1}(Z_1+\ldots+Z_p).$$

Then, for any $\varepsilon > 0$,

$$P\{|Z^*-a| > \varepsilon\} < 4e^{-\beta p\varepsilon^2}$$

for some $\beta > 0$.

From this, and the inequalities above, we obtain

$$P\{|\langle k_r, e^{if}\xi_n\rangle - \mathscr{E}(Z^r)| > \varepsilon \text{ for some } r \in [1, 2, \ldots s]\} < 4se^{-\beta p_n\varepsilon^2} \leqslant 4^{2n+1}e^{-\beta p_n\varepsilon^2}.$$

Hence

$$P\{\|e^{if}\xi_n\|_{V^*} > 2e^{-aR}+2\varepsilon\} \leqslant \exp\big(k(2n+1)-\beta p_n\varepsilon^2\big).$$

Writing f_n for $f|_{E_n}$, we have

$$\langle e^{if}\xi_n, e^{-if_n}\rangle = 1$$

and, extending f_n to a real function $g \in V(H)$ such that $\|g\|_{V(H)} < 2R$, we get

$$P\{\|e^{ig}\|_{V(H)} < (2e^{-aR}+2\varepsilon)^{-1}\} < \exp\big(k(2n+1)-\beta p_n \varepsilon^2\big).$$

We now choose positive numbers ε_n such that $\varepsilon_n \to 0$ and $n/p_n\varepsilon_n \to 0$ as $n\to\infty$ and we consider the functions $g_{R,t} \in V(H)$ corresponding to $f_{R,t} \in V(S_{n_t})$. Since

$$P\{\|e^{ig_{R,t}}\|_V < (2e^{-aR}+2\varepsilon_{n_t})^{-1}\} < \exp\big(k(2n_t+1)-\beta p_{n_t}\varepsilon_{n_t}^2\big),$$

it is clear that, given $\delta > 0$, we can find, for each $R > C$, an integer t_R such that

$$P\{\|e^{ig_R}\|_V \leqslant \tfrac{1}{3}e^{aR}\} < \delta/2^R$$

(where $g_R = g_{R,t_R} \in V(H)$). But $\|g_R\|_V < 2R$, so we can deduce (by the criterion of [3]) that

$$P\{H \text{ is not a set of analyticity}\} < \delta.$$

Hence H is almost surely a set of analyticity.

4. For some particular subsets of $X\times Y$ we can get direct transpositions of the results of Malliavin and Katznelson [5], [6].

Definition 4.1. Let X, Y be sets and let $S_n = X_n\times Y_n$ be an n-square in $X\times Y$. If G is an abelian group of order n and X_n, Y_n are identified with G, the *G-fibres* of S_n are the equivalence classes of points of S_n corresponding to the relation

$$(x, y) \sim (z, t) \Leftrightarrow x+y = z+t.$$

A subset E of S_n is called a *G-diagonal* subset if

$$(x, y)\in E \Leftrightarrow (z, t)\in E$$

for all $x, y, z, t \in G$ having $x+y = z+t$. (These definitions are, of course, dependent on the particular identifications of X_n, Y_n with G.)

THEOREM 4.2. *Let X, Y be compact Hausdorff spaces and let H be a closed subset of $X\times Y$. Suppose H contains a set of the form $\bigcup_{n=1}^{\infty} E_n$, where each E_n is a $Z(n)$-diagonal subset of an n-square S_n.*

Then if

$$\overline{\lim_{n\to\infty}}|E_n| > n^{1+\varepsilon}$$

for some $\varepsilon > 0$, H is a set of analyticity.

Proof. Let $F_n = \{z\in Z(n) : z = x+y \text{ for some } (x, y)\in E_n\}$. Clearly $|F_n| = |E_n|/n$ and thus

$$\overline{\lim_{n\to\infty}}|F_n| > n^{\varepsilon}.$$

It then follows from [5], that, given $R > 0$, we can find a real function $f\in A(F_n)$ for some n, having

$$\|f\|_{A(F_n)} < R \quad \text{and} \quad \|e^{if}\|_{A(F_n)} > e^{aR}$$

for some positive constant a. Applying the mapping $M\colon A(Z(n)) \to V(S_n)$ we obtain the required result.

THEOREM 4.3. *Let X, Y be compact Hausdorff spaces and let H be a closed subset of $X\times Y$. Let H contain a set of the form $\bigcup_{n=1}^{\infty} E_n$, where E_n is the union of $|E_n|/n$ $Z(n)$-fibres of S_n chosen at random.*

Then if

$$(n\log n)|E_n|^{-1} \to 0$$

as $n\to\infty$, H is almost surely a set of analyticity.

Proof. As above, using [6], Theorem 1, in place of [5].

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Reçu par la Rédaction le 27. 10. 1969