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On subharmonicity inequalities involving solutions of generalized Cauchy-Riemann equations

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Suppose $F = (F_1, F_2, \ldots, F_k)$ is a system of C^1 real-valued functions defined in a domain $U \subset R^n$ ($=$ n-dimensional Euclidean space) satisfying the partial differential equations

$$\sum_{=1}^{n} A_j \frac{\partial F}{\partial x_j} = 0, \tag{1.1}$$

where A_j is a $l \times k$ constant matrix and $\partial F/\partial x_j$ is the (column) vector having components $\partial F_i/\partial x_j$, $i = 1, 2, \ldots, k$. We say that the system of partial differential equations (1.1) is a *generalized Cauchy-Riemann* (GCR) *system* if each solution $F = (F_1, F_2, \ldots, F_k)$ has harmonic components F_i, $i = 1, 2, \ldots, k$. When $k = l = n = 2$, a linear change of variables reduces such a system to the ordinary Cauchy-Riemann equations.

Several systems of partial differential equations that generalize, in one way or the other, the Cauchy-Riemann equations have been studied by Stein and Weiss [4], [5] and Calderón and Zygmund [2] in connection with various extensions of the theory of H^p-spaces. Each of these systems is a particular example of a GCR-system. The basic fact, common to all solutions of these equations, enabling one to develop the theory of H^p-spaces is the existence of a positive $p < 1$ such that

$$|F|^p = \Big(\sum_{i=1}^{k} |F_i|^2\Big)^{p/2}$$

is subharmonic (see [4]). A. P. Calderón observed that the existence of such a p is the consequence of the ellipticity of system (1.1). More precisely, system (1.1) is called *elliptic* provided

$$\sum_{j=1}^{n} \lambda_j A_j v = 0 \tag{1.2}$$

for a k-dimensional (column) vector v and an n-tuple $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ only if either v or λ is zero. It was pointed out to us by E. M. Stein that every GCR-system is necessarily elliptic. If this were not the case, the existence of non-zero v and λ satisfying (1.2) enables us to construct the non-harmonic solution

$$F(x) = \{\exp\big(\sum_{j=1}^{n} \lambda_j x_j\big)\} v$$

of (1.1). It is these facts that motivate the definition of the generalized Cauchy-Riemann systems given above.

Let $v = (v_1, v_2, \ldots, v_k)$, $u^{(j)} = (u_1^{(j)}, u_2^{(j)}, \ldots, u_k^{(j)})$, $j = 1, 2, \ldots, n$, denote elements of R^k,

$$u^{(j)} \cdot v = \sum_{l=1}^{k} u_l^{(j)} v_l, \quad \text{and} \quad |v|^2 = \sum_{l=1}^{k} v_l^2.$$

The subharmonicity result mentioned above is a consequence of the basic inequality (due to A. P. Calderón)

$$\max_{|v|=1} \sum_{j=1}^{n} (u^{(j)} \cdot v)^2 \leqslant a \sum_{j=1}^{n} |u^{(j)}|^2, \tag{1.3}$$

where

$$\sum_{j=1}^{n} A_j u^{(j)} = 0 \tag{1.4}$$

and a is a number less than 1 depending only on the matrices $A_1, \ldots, A_n$.

It is our purpose to show that, in addition, (1.3) implies other subharmonicity inequalities. More precisely we shall prove:

THEOREM. *Suppose F is a solution of the generalized Cauchy-Riemann system* (1.1) *and $F^\lambda = (F_1, F_2, \ldots, F_{k-1}, \lambda F_k)$; then the following functions are subharmonic*:

(a) $|F|^p$, *where* $p \geqslant 2 - 1/a$;

(b) $A|F^0|^p - |F|^p$, *where* $1 < p \leqslant 2$ *and* $A \geqslant (p-1)^{-1}(1-a)^{-1}$;

(c) $A|F^0|^2 |F^\lambda|^{p-2} - |F^\lambda|^p$, *where* $2 \leqslant p < \infty$, $A \geqslant p(p-1)/(1-a)$ *and* $0 < \lambda^2 \leqslant \min\{(1-a)/4(p-2), 1\}$ [1].

The best possible a has been calculated for many of the systems discussed in the above mentioned articles.

The subharmonicity of the functions in (b) and (c) was proved by Kuran [3] for a special case (the solutions of the M. Riesz equations) in order to obtain an extension to n dimensions of the proof by P. Stein of the M. Riesz inequality for conjugate functions (see Chap. VII, p. 261 of [6]). This extension is a consequence of the mean value inequality satisfied by the subharmonic functions occurring in (b) and (c). (Once this part of the theorem is established, the method of Kuran applies without change.)[2]

[1] This theorem can be stated more generally by introducing, instead of F^λ, the vector $G^\lambda = F - (1-\lambda)(F \cdot v)v$, where $v = (v_1, \ldots, v_k)$ is a unit vector.

We first establish (1.3). Clearly, we can assume that

$$\sum_{j=1}^{n} |u^{(j)}|^2 = 1.$$

By the Cauchy-Schwarz inequality

$$\sum_{j=1}^{n} (u^{(j)} v)^2 \leqslant \sum_{j=1}^{n} |u^{(j)}|^2 |v|^2 = \sum_{j=1}^{n} |u^{(j)}|^2 = 1.$$

Thus, if there does not exist $a < 1$ for which (1.3) is valid, by compactness there exist v and $u^{(1)}, \ldots, u^{(n)}$ satisfying (1.4), as well as

$$|v|^2 = 1 = \sum_{j=1}^{n} |u^{(j)}|^2, \quad \text{for which} \sum_{j=1}^{n} (u^{(j)} \cdot v)^2 = 1.$$

Hence, since $(u^{(j)} \cdot v)^2 \leqslant |u^{(j)}|^2 |v|^2$, we must have $(u^{(j)} \cdot v)^2 = |u^{(j)}|^2 |v|^2$ for $j = 1, 2, \ldots, n$. Thus, by the equality case of the Cauchy-Schwarz inequality, there exists $\lambda = (\lambda_1, \ldots, \lambda_n)$ such that $u^{(j)} = \lambda_j v$. This contradicts the ellipticity condition (1.2) since we then must have

$$0 = \sum_{j=1}^{n} A_j u^{(j)} = \sum_{j=1}^{n} \lambda_j A_j v \quad \text{with } |\lambda| |v| = 1 \neq 0.$$

We recall that a continuous function s on a domain $U \subset R^n$ is subharmonic if for each point $x \in U$ there exists $r_x > 0$ such that

$$s(x) \leqslant \frac{1}{\omega_{n-1} r^{n-1}} \int_{|y-x|=r} s(y)\, dy \tag{1.5}$$

for $0 \leqslant r < r_x$, where ω_{n-1} is the (surface) measure of the unit sphere Σ_{n-1} in R^n. If s is of class C^2, it is well known that inequality (1.5) is equivalent to

$$(\Delta s)(x) \geqslant 0 \tag{1.6}$$

for all $x \in U$, where

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$$

is the Laplace differential operator.

[2] In an unpublished work E. Lester has obtained results stronger than those in part (b) in the sense that, instead of F^0, a vector in which more than one of the components of F are removed can be used and the subharmonicity of the corresponding function is still valid.

If $s = |F|^p$, then our function is of class C^2 in the subdomain $V = U - \mathscr{E}$, where $\mathscr{E} = \{x \in U\colon F(x) = 0\}$. Thus, we have the mean-value inequality (1.5) at each $x \in V$, provided we show $\Delta s(x) \geqslant 0$ on V. But the fact that $s \geqslant 0$ clearly implies the validity of (1.5) for $x \in \mathscr{E}$ as well.

If we restrict $s = |F|^p$ on V, then a simple computation (making use of the harmonicity of F) yields

$$(1.7)\qquad \Delta s = p|F|^{p-2}\left\{\sum_{j=1}^{n}\left[(p-2)\left(\frac{F}{|F|}\cdot\frac{\partial F}{\partial x_j}\right)^2 + \left|\frac{\partial F}{\partial x_j}\right|^2\right]\right\}.$$

Applying (1.3) to $v = F/|F|$ and $u^{(j)} = \partial F/\partial x_j$, we see that the term in curly brackets is non-negative for $p \geqslant 1 - 1/a$. Thus, part (a) of the theorem is proved.

Part (b) is somewhat more complicated for two reasons. First, with $s = A|F^0|^p - |F|^p$ we have a C^2-function only if we restrict its domain to $W = U - \mathscr{G}$, where $\mathscr{G} = \{x \in U\colon F^0(x) = 0\} \supset \mathscr{E}$. Second, we can no longer assume that s is non-negative.

The fact that $\Delta s \geqslant 0$ on W is also an easy consequence of (1.3). In fact,

$$\Delta s = \Delta(A|F^0|^p - |F|^p)$$

(by (1.7) and omitting a negative term)

$$\geqslant pA|F^0|^{p-2}\left\{\sum_{j=1}^{n}\left|\frac{\partial F^0}{\partial x_j}\right|^2 - (2-p)\sum_{j=1}^{n}\left(\frac{F^0}{|F^0|}\cdot\frac{\partial F^0}{\partial x_j}\right)^2\right\} - p|F|^{p-2}\sum_{j=1}^{n}\left|\frac{\partial F}{\partial x_j}\right|^2$$

(using the Cauchy-Schwarz inequality)

$$\geqslant pA(p-1)|F^0|^{p-2}\left\{\sum_{j=1}^{n}\left|\frac{\partial F^0}{\partial x_j}\right|^2\right\} - p|F|^{p-2}\sum_{j=1}^{n}\left|\frac{\partial F}{\partial x_j}\right|^2$$

(since, for $1 < p \leqslant 2$, $|F|^{p-2} \leqslant |F^0|^{p-2}$)

$$\geqslant p|F|^{p-2}\left\{A(p-1)\sum_{j=1}^{n}\left|\frac{\partial F^0}{\partial x_j}\right|^2 - \sum_{j=1}^{n}\left|\frac{\partial F}{\partial x_j}\right|^2\right\}$$

(since $A \geqslant (1-a)^{-1}(p-1)^{-1}$)

$$\geqslant p|F|^{p-2}\left\{(1-a)^{-1}\sum_{j=1}^{n}\left|\frac{\partial F^0}{\partial x_j}\right|^2 - \sum_{j=1}^{n}\left|\frac{\partial F}{\partial x_j}\right|^2\right\}$$

$$= p(1-a)^{-1}|F|^{p-2}\left\{a\sum_{j=1}^{n}\left|\frac{\partial F}{\partial x_j}\right|^2 - \sum_{j=1}^{n}\left|\frac{\partial F}{\partial x_j}\right|^2\right\}$$

(by (1.3) with $v = (0, 0, \ldots, 0, 1)$)

$$\geqslant 0.$$

Let us observe that when $p = 2$ the function occurring in part (b) is of class C^2 on U, thus we have the desired subharmonicity result. Hence, the mean value inequality (1.5) is valid for this function at each point $x \in U$. In particular,

$$(1.8)\qquad -|F(x)|^2 \leqslant \frac{1}{\omega_{n-1}r^{n-1}}\int_{|x-y|=r}\left[\frac{1}{1-a}\left|F^0(y)\right|^2 - |F(y)|^2\right]dy,$$

where $x \in \mathscr{G}$ and $\{y\colon |y-x| \leqslant r\} \subset U$.

Suppose $1 < p < 2$ and $A \geqslant (1-a)^{-1}(p-1)^{-1}$; then

$$\frac{1}{1-a} \leqslant \frac{1}{1-a}\cdot\frac{1}{p-1} \leqslant \left[\frac{1}{1-a}\cdot\frac{1}{p-1}\right]^{2/p} \leqslant A^{2/p}.$$

Consequently,

$$(1.9)\quad (|F^p| - A|F^0|^p)^{2/p} + \frac{1}{1-a}|F^0|^2 \leqslant (|F|^p - A|F^0|^p)^{2/p} + (A|F^0|^p)^{2/p}$$
$$\leqslant (|F|^p - A|F^0|^p + A|F^0|^p)^{2/p} = |F|^2.$$

Therefore, if $x \in \mathscr{G} - \mathscr{E}$ and r is small (so that $|F(y)| \geqslant A^{1/p}|F^0(y)|$ for $|y-x| \leqslant r$) we have, using Hölder's inequality, (1.9) and (1.8):

$$\frac{1}{\omega_{n-1}r^{n-1}}\int_{|y-x|=r} - s(y)\,dy = \frac{1}{\omega_{n-1}r^{n-1}}\int_{|y-x|=r}[-A|F^0(y)|^p + |F(y)|^p]\,dy$$
$$\leqslant \left\{\frac{1}{\omega_{n-1}r^{n-1}}\int_{|y-x|=r}[-A|F^0(y)|^p + |F(y)|^p]^{2/p}\,dy\right\}^{p/2}$$
$$\leqslant \left\{\frac{1}{\omega_{n-1}r^{n-1}}\int_{|y-x|=r}\left[-\frac{1}{1-a}|F^0(y)|^2 + |F(y)|^2\right]dy\right\}^{p/2}$$
$$\leqslant |F(x)|^p = -s(x).$$

That is, we have the mean value inequality for $s(x) = A|F^0(x)|^p - |F(x)|^p$ for $x \in \mathscr{G} - \mathscr{E}$.

We have shown, therefore, that $s = A|F^0|^p - |F|^p$ is subharmonic on the open set $V = U - \mathscr{E}$. The fact that s is subharmonic on U follows from the following result:

Lemma. *Suppose F is a solution of the generalized Cauchy-Riemann system* (1.1); *then the set $\mathscr{E} = \{x \in U\colon F(x) = 0\}$ is a polar set*(3).

In order to prove this lemma we use two facts concerning polar sets: first, a countable union of polar sets is a polar set; second, a surface of

(3) For the definition and basic properties of polar sets see [1]. The fact that the closed set $\mathscr{E}$ is polar allows us to conclude that the continuous function s, being subharmonic on V, must be subharmonic on U. This result extends lemma 1 in [3] which dealt with the special case when F is the gradient of a harmonic function.

dimension not higher than $n-2$ is a polar set. Given these results, it suffices to show that $\mathscr{E}$ is a countable union of surfaces of dimension not higher than $n-2$. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be an n-tuple of non-negative integers and

$$(G_1^{(\alpha)}, \ldots, G_k^{(\alpha)}) = G_\alpha = \frac{\partial^{\alpha_1+\ldots+\alpha_n} F}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} = \frac{\partial^{\alpha_1+\ldots+\alpha_n}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} (F_1, \ldots, F_k)$$

the corresponding partial derivative of F. Then G_α must also be a solution of (1.1). We claim, moreover, that the Jacobian matrix

$$D_x G_\alpha = \left(\frac{\partial G_i^{(\alpha)}}{\partial x_j}(x)\right), \quad 1 \leqslant i \leqslant k, 1 \leqslant j \leqslant n,$$

has rank at least 2 whenever it is not zero. In fact, the vectors $\partial G_\alpha/\partial x_1, \ldots, \partial G_\alpha/\partial x_n$ form a basis for the range of $D_x G_\alpha$. If this range were one-dimensional there must exist i, $1 \leqslant i \leqslant k$, and $\lambda = (\lambda_1, \ldots, \lambda_n)$ such that

$$\frac{\partial G_\alpha}{\partial x_j} = \lambda_j \frac{\partial G_\alpha}{\partial x_i} \quad \text{for } j = 1, \ldots, n.$$

But G_α being a solution of (1.1) this contradicts the ellipticity condition (1.2). On the other hand, since F is real analytic $\mathscr{E}$ is contained in the union, over all such n-tuples α, of the sets $\mathscr{E}_\alpha = \{x \in U: G_\alpha(x) = 0$ and $D_x G_\alpha \neq 0\}$. Since the rank of $D_x G_\alpha$ is at least two, the implicit function theorem implies that for each point $x \in \mathscr{E}_\alpha$ we can find a neighborhood $N(x)$ of x such that $\mathscr{E}_\alpha \cap N(x)$ is a surface of dimension not higher than $n-2$.

We now turn to the proof of part (c). In this case the function $s = A|F^0|^2|F^\lambda|^{p-2} - |F^\lambda|^p$ is of class C^∞ (in fact, it is real analytic) on $V = U - \mathscr{E}$ since the term $|F^0|^2$ is real analytic on all of U. Thus it suffices to show that $(\Delta s)(x) \geqslant 0$ for $x \in V$. We have, at each point of V,

$$\begin{aligned}
|F^\lambda|^{2-p}\Delta s &= |F^\lambda|^{2-p}\Delta\left(A|F^0|^2|F^\lambda|^{p-2} - |F^\lambda|^p\right) \\
&= 2A\sum_{j=1}^n\left|\frac{\partial F^0}{\partial x_j}\right|^2 + A(p-2)|F^\lambda|^{-2}|F^0|^2 \times \\
&\quad \times\left\{(p-4)\sum_{j=1}^n\left(\frac{F^\lambda}{|F^\lambda|}\cdot\frac{\partial F^\lambda}{\partial x_j}\right)^2 + \sum_{j=1}^n\left|\frac{\partial F^\lambda}{\partial x_j}\right|^2\right\} + \\
&\quad + 4A(p-2)|F^\lambda|^{-2}\sum_{j=1}^n\left(\frac{\partial F^\lambda}{\partial x_j}\cdot F^\lambda\right)\left(\frac{\partial F^0}{\partial x_j}\cdot F^0\right) - |F^\lambda|^{2-p}\Delta|F^\lambda|^p \\
&\geqslant 2A\sum_{j=1}^n\left|\frac{\partial F^0}{\partial x_j}\right|^2 + 4A(p-2)|F^\lambda|^{-2} \times \\
&\quad \times\left\{\sum_{j=1}^n\left(\frac{\partial F^\lambda}{\partial x_j}\cdot F^\lambda\right)\left(\frac{\partial F^0}{\partial x_j}\cdot F^0\right) - \sum_{j=1}^n\left(F^\lambda\cdot\frac{\partial F^\lambda}{\partial x_j}\right)^2\right\} - |F^\lambda|^{2-p}\Delta|F^\lambda|^p
\end{aligned}$$

$$\begin{aligned}
&= 2A\sum_{j=1}^n\left|\frac{\partial F^0}{\partial x_j}\right|^2 - 4A(p-2)|F^\lambda|^{-2}\lambda^2\sum_{j=1}^n F_k\frac{\partial F_k}{\partial x_j}\left(\frac{\partial F^\lambda}{\partial x_j}\cdot F^\lambda\right) - \\
&\quad - |F^\lambda|^{2-p}\Delta|F^\lambda|^p \\
&\geqslant 2A\sum_{j=1}^n\left|\frac{\partial F^0}{\partial x_j}\right|^2 - 4A(p-2)\lambda^2\sum_{j=1}^n\left|\frac{\partial F}{\partial x_j}\right|^2 - |F^\lambda|^{2-p}\Delta|F^\lambda|^p \\
&= A\left\{\sum_{j=1}^n\left|\frac{\partial F^0}{\partial x_j}\right|^2 - 4(p-2)\lambda^2\sum_{j=1}^n\left|\frac{\partial F}{\partial x_j}\right|^2\right\} + \\
&\quad + \left\{A\sum_{j=1}^n\left|\frac{\partial F^0}{\partial x_j}\right|^2 - |F^\lambda|^{2-p}\Delta|F^\lambda|^p\right\}.
\end{aligned}$$

We showed while proving part (b) that the term within the first bracket is non-negative provided $4(p-2)\lambda^2 \leqslant 1-\alpha$. The term in the second bracket equals

$$\begin{aligned}
A\sum_{j=1}^n\left|\frac{\partial F^0}{\partial x_j}\right|^2 - p\left\{\sum_{j=1}^n(p-2)\left(\frac{F^\lambda}{|F^\lambda|}\cdot\frac{\partial F^\lambda}{\partial x_j}\right)^2 + \left|\frac{\partial F^\lambda}{\partial x_j}\right|^2\right\} \\
\geqslant A\sum_{j=1}^n\left|\frac{\partial F^0}{\partial x_j}\right|^2 - p(p-1)\sum_{j=1}^n\left|\frac{\partial F}{\partial x_j}\right|^2
\end{aligned}$$

which, again by the proof of part (b), is non-negative if $A \geqslant p(1-\alpha)^{-1} \times (p-1)$.

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