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Some uncomplemented subspaces of $C(X)$ of the type $C(Y)$

by

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Introduction. In 1962, Amir [1] proved that $C[0,1]$ contains an uncomplemented subspace which is isometric to $C[0,1]$. In 1965, Arens [4] constructed a countable, closed subset X of $[0,1]$ and a decomposition of X such that the subspace of functions which are constant on each set of the decomposition is uncomplemented in $C(X)$. The crucial step in Arens' construction depends upon the following theorem:

If Y is a compact set in a metric space X and if the boundary of Y contains n points, then each linear projection of $C(X)$ onto the subspace of functions constant on Y is of norm at least $3-2/n$.

Theorem 1.3. substantially generalizes this result and replaces the hypothesis of metrizability with normality and T_1. In Corollary 1.4, sufficient conditions on a decomposition D are given in order that $C(X/D)$ will be uncomplemented in $C(X)$.

In Chapter 2, the main result is Theorem 2.9 which states that if X is a T_4-space and if all successive derived sets $X^{(1)}, X^{(2)}, \ldots$ are nonempty, then there is a decomposition D such that $C(X/D)$ is uncomplemented in $C(X)$. The following characterization is obtained if X is a compact metric space: Some finite derived set of X is empty if and only if for each Hausdorff decomposition D of X, $C(X/D)$ is complemented in $C(X)$. This answers a question raised by A. Pełczyński (cf. [17], p. 74).

The notation and terminology used herein follow Kelley's *General Topology*, except for the following: a *decomposition* D of a topological space X is a disjoint collection of *closed* subsets of X such that $X = \bigcup \{A : A \in D\}$. The notation $C(X)$ is used to denote the space of *bounded* continuous scalar-valued functions in a topological space X with $\|f\| = \sup_{x\in X}|f(x)|$. A continuous function from a topological space X onto a topological

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space Y is called an *epimorphism*. Two Banach spaces X and Y are *isomorphic* (*isometric*) if there exists a one-to-one linear epimorphism T of X onto Y (with $\|Tx\| = \|x\|$). If T is also multiplicative, we say that X and Y are *algebraically isomorphic* (*algebraically isometric*). A *projection* is a continuous linear operator from a Banach space X into X which is idempotent. A subspace Y of X is *complemented* in X if there exists a projection from X onto Y.

1. Lower bounds for norms of projections. The notation X/D denotes the decomposition D of a topological space X with the quotient topology. If D and M are two decompositions of X, then M *implies* D if and only if for each A in M there is a B in D such that $A \subset B$. If M implies D and if we define $h(B) = \{A \in M\colon A \subset B\}$ for each $B \in D$, then $\{h(B)\colon B \in D\}$ is a decomposition of the quotient space X/M. This decomposition is denoted by D/M and is called the *quotient decomposition*.

Suppose that G is a family of disjoint subsets of a topological space X. If $B \subset X$, we say that B is *saturated* (with respect to G) if and only if for each $A \in G$ either $A \subset B$ or $A \cap B = \emptyset$. A set A in G is called *plural* if it contains at least two members. An element A of G is a *limit set* if each neighborhood of A intersects a plural set in $G \sim \{A\}$. Observe that a set in G need not be plural in order to be a limit set. The notation $\bigcup G$ denotes $\{x\colon x \in A$ for some A in $G\}$. The family $G^{(1)}$. is the decomposition of $\bigcup G$ consisting of the plural limit sets of G and remaining singleton sets. For an ordinal number $\alpha > 1$, the decomposition $G^{(\alpha)}$ of $\bigcup G$ is defined inductively. If $\alpha = \beta+1$, then $G^{(\alpha)} = (G^{(\beta)})^{(1)}$; if α is a limit ordinal, then $G^{(\alpha)}$ is the decomposition of $\bigcup G$ consisting of the plural sets in $\bigcap G^{(\lambda)}$ and singleton sets. It is convenient to let $G^{(0)} = G$. If G is a decomposition of X, $G^{(\alpha)}$ is called the α^{th} *derived decomposition* of X. These concepts are consistent with those introduced by R. Arens in [4].

A major difficulty with upper semicontinuous decompositions is that if a plural set is replaced by its singleton subsets, the resulting decompositions need not be upper semicontinuous. For example, for each t in $\{0, 1, 1/2, 1/3, \ldots\}$ let X_t denote the vertical segment $\{(t, y)\colon 0 \leqslant y \leqslant 1\}$ in the plane. Let $X = \bigcup X_t$. Let D be the decomposition of X whose elements are the sets X_t. Then D is upper semicontinuous, but the decomposition obtained by replacing X_0 with its singleton subsets is not upper semicontinuous.

In order to avoid this difficulty, the notion of a *contracting decomposition* is introduced. Suppose that X is a topological space, D is a decomposition of X, and $A \in D$. We say that D is *contracting at* A if and only if the decomposition of X whose plural sets are the sets in $D \sim \{A\}$ is upper semicontinuous. Therefore, D is contracting at A if and only if D is upper semicontinuous, and for each x in A and each neighborhood U of x there is a $(D \sim \{A\})$-saturated neighborhood V of x such that $V \subset U$. It is easy to see that D is contracting at A if and only if D is upper semicontinuous and if $\{A_\gamma\}_{\gamma\in\Gamma}$ is a family of sets of $D \sim \{A\}$ with a_γ and b_γ in A_γ, then either (i) neither of the nets $\{a_\gamma\}$ and $\{b_\gamma\}$ converge to a point of A or (ii) both converge to the same point. If D is contracting at each of its sets, we say that D is *contracting*. Thus, D is contracting if and only if for each family $\{A_\gamma\}_{\gamma\in\Gamma}$ of D-sets with a_γ and b_γ in A_γ and $A_\gamma \neq A$, either neither of the nets $\{a_\gamma\}$ and $\{b_\gamma\}$ converge or both converge to the same point. These definitions extend the definition of R. L. Moore (cf. [15], p. 285).

A basic property of contracting decompositions is stated in the following lemma:

LEMMA 1.1. *Let D be an upper semicontinuous decomposition of a topological space X, let $D_1 \subset D$, $D_2 \subset D$, $D_1 \cap D_2 = \emptyset$, and suppose that D is contracting at each set in $D_1 \cup D_2$. Then the decomposition of X whose plural sets are the plural in $D \sim D_1$ is upper semicontinuous and contracting at each set in D_2.*

If q is a continuous map from a topological space X into a topological space Y, then the *induced mapping* q^0 defined by $q^0 f = f \circ q$ is a continuous, multiplicative, linear operator from $C(Y)$ into $C(X)$ of norm one (cf. [11], Chap. 10, and [19], p. 331). In particular, if Y is the decomposition space X/D and q is the quotient map, then q^0 is an algebraic isometry of $C(X/D)$ onto the subspace of $C(X)$ consisting of the functions which are constant on each set in D. This fact is an easy consequence of Theorem 9 of [13], p. 95, and is part of the folklore of quotient space theory. It is generally convenient to identify the space $C(X/D)$ with the subspace of functions in $C(X)$ which are constant on each set in D without specific reference to the isomorphism q^0.

Our next lemma removes the restriction in Theorem 3.1 of [4] that the decomposition set must contain only one plural set and weakens the restriction that the space must be metrizable. Another generalization of this theorem is given in Theorem 1.3.

LEMMA 1.2. *Let X be a T_4-space and let D be an upper semicontinuous decomposition of X. Suppose there exists a plural set Y in D such that D is contracting at Y and the boundary ∂Y of Y contains at least n points. If P is a projection of $C(X)$ onto $C(X/D)$, then $\|P\| \geqslant 3-2/n$.*

Moreover, if $\varepsilon > 0$, if U is a neighborhood of Y, and if $y_1, y_2, \ldots, y_n$ are distinct points in ∂Y, then there exists an i and a neighborhood V of y_i such that for each t in $V \sim Y$ there exists f in $C(X)$ with $f|(X-U) = 0$, $\|f\| = f(t) = 1$, and $Pf(t) > 3-2/n-\varepsilon$.

Proof. If $n = 0$, then the conclusion is clearly true. Hence, we assume $n \geqslant 1$. Let M be the decomposition of X consisting of the plural sets in $D \sim \{Y\}$ and let p be the quotient map of X onto X/M. Then M is upper semicontinuous (because D is contracting at Y) and X/M is a T_4-space (see [14], p. 185, or [13], p. 134, Problem M).

Suppose $B = \{y_1, y_2, \ldots, y_n\}$ is a set of n distinct points in ∂Y and U is a neighborhood of Y. By possibly passing to a smaller neighborhood, we may assume that U is both open and D-saturated. Then $p[B]$ consists of n distinct points of $p[Y]$. Note that $p[Y]$ is the only plural set in the decomposition D/M of X/M. Since $C(X/D) \subset C(X/M) \subset C(X)$, the restriction P' of P to $C(X/M)$ is a projection of $C(X/M)$ onto $C(X/D)$. But X/D is homeomorphic to $(X/M)/(D/M)$ (cf. [5], p. 40, or [9], p. 72); hence, it follows from our identifications that $P' = (p^0)^{-1} P p^0$ and that P' is a projection of $C(X/M)$ onto the functions in $C(X/M)$ which are constant on $p[Y]$.

Let R be the restriction operator of $C(X/M)$ onto $C(pB)$. Suppose $\varepsilon > 0$ and $\delta = \varepsilon/3$. There exists g of norm 1 in $C(X/M)$ and y in B such that

$$|(P'g-g)(py)| > \|R(P'-I)\| - \delta,$$

where I denotes the identity operator on $C(X/M)$. By the continuity of g, there is an open neighborhood W of $p(y)$ in X/M such that $|(P'g-g)(x)|$, $> |(P'g-g)(py)| - \delta$ for each x in W. It we define $V = p^{-1}[W] \cap U|$ then V is an open M-saturated X-neighborhood of y in X. Let t be an element of $V \sim Y$. Then $p(t)$ belongs to the open subset $p[U] \sim p[Y]$ of X/M. We may assume that $(P'g-g)(pt) \geqslant 0$. By the Urysohn-Tietze Extension Theorem, there exists f' in $C(X/M)$ such that f' and g are equal on $p[Y]$, f' vanishes off $p[U]$, and $\|f'\| = f'(pt) = 1$. Since $f'-g$ is constant on $p[Y]$, we have $P'(f'-g) = f'-g$ and $(P'-I)f' = (P'-I)g$. Let $f = p^0 f'$. Since $p(t)$ belongs to W,

$$\begin{aligned} Pf(t) &= p'f'(pt) = f'(pt) + (P'f'-f')(pt) \\ &= 1 + (P'g-g)(pt) > 1 + |(P'g-g)(py)| - \delta > 1 + \|R(P'-I)\| - 2\delta. \end{aligned}$$

Let Φ be the functional on $C(X/M)$ such that $\Phi(f)$ is the constant value of $P'f$ on $p[Y]$. Let $\{U_1, U_2, \ldots, U_n\}$ be a family of disjoint open sets in X/M such that $p(y_i)$ belongs to U_i for each i. There exists q in $C(X/M)$ such that $\|q\| = 1$ and $\Phi(q) > \|\Phi\| - \delta$. For each i, there exists by the Urysohn-Tietze Extension Theorem an f_i in $C(X/M)$ such that $f_i = q$ on the complement of U_i, $\|f_i\| = 1$, and $f_i(py_i) = -1$. Suppose that $\Phi(f_i) \leqslant \|\Phi\|\,(1-2/n) - \delta$ for each i. Put $h = \sum (q - f_i)$. Since $q - f_i = 0$ on the complement of U_i, we have $\|h\| = \max\|q - f_i\| \leqslant 2$. Then we obtain the contradiction,

$$\begin{aligned} 2\|\Phi\| &\geqslant \|h\|\,\|\Phi\| \geqslant \Phi(h) \\ &= \sum \Phi(q-f_i) > \sum [\|\Phi\| - \delta - \|\Phi\| + 2/n\|\Phi\| + \delta] = 2\|\Phi\|. \end{aligned}$$

Thus for some j, $\Phi(f_j) > \|\Phi\|(1-2/n) - \delta$. Then

$$\begin{aligned} \|R(P'-I)\| \geqslant \|R(P'-I)f_j\| &\geqslant (P'f_j - f_j)(py_j) \\ &= \Phi(f_j) + 1 > 1 + \|\Phi\|(1-2/n) - \delta. \end{aligned}$$

Since $\|\Phi\| = \|RP'\| \geqslant 1$, we obtain that

$$Pf(t) > 1 + \|R(P'-I)\| - 2\delta > 2 + \|\Phi\|(1-2/n) - 3\delta \geqslant 3 - 2/n - \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, and $\|f\| = 1$, it follows that $\|P\| \geqslant 3 - 2/n$.

The next theorem shows that the existence of repeated limits of plural sets is fundamental in raising the norm of projections of $C(X)$ onto decomposition subspaces of $C(X)$. A convenient feature of this theorem is that it can be applied to a decomposition that has no zero derived decompositions. One natural choice for the sets $S_1, S_2, \ldots, S_n$ is to let S be a family of plural sets of the decomposition such that $S^{(n-1)}$ is non-zero and to let $S_i = S^{(i-1)}$ for each i. Another natural choice is to let S be a family of plural sets such that $S^{(n-1)}$ contains a plural non-limit set and to define S_i to be the family of non-limit sets in $S^{(i-1)}$ for each i. The decomposition is required to be contracting at each set in each S_i so that each neighborhood of each *limit point of plural sets* in these sets not only intersects a plural set but also contains a plural set. (If q denotes the quotient map of the decomposition, a point x in X is a *limit point of plural sets* if for each neighborhood U of x, $U \sim q(x)$ intersects a plural set.) An example is given following the theorem to show that the *contracting* condition can not be dropped. A more general theorem for the case in which X is compact has been obtained independently by S. Ditor in his dissertation [7].

It is convenient to introduce the following definition prior to stating the theorem:

Definition. Let $m_1, m_2, \ldots, m_n$ be positive integers. A decomposition D of a topological space X has *property* $L_n(m_1, m_2, \ldots, m_n)$ if and only if there exists non-empty collections $S_1, S_2, \ldots, S_n$ of plural D-sets such that

(a) D is contracting at each set in $\bigcup_{i=1}^{n} S_i$;

(b) the boundary of each set in S_1 contains at least m_1 points;

(c) the boundary of each set A in S_{i+1} contains at least m_{i+1} limit points of the sets in $S_i \sim \{A\}$.

In particular, if D is a contracting decomposition such that $D^{(n)}$ is non-zero and each plural set A in $D^{(i)}$ contains at least k limit points of the plural sets in $D^{(i-1)} \sim \{A\}$ for $i \leqslant n$, then D has property $L_n(k, k, \ldots, k)$. If the boundary of each non-limit set in D contains at least k points, then D also has property $L_{n+1}(k, k, \ldots, k)$.

THEOREM 1.3. *If D is a decomposition of a T_4-space X with property $L_n(m_1, m_2, \ldots, m_n)$ and P is a projection of $C(X)$ onto $C(X/D)$, then*

$$\|P\| \geqslant 2n+1-\sum_{i=1}^{n} 2/m_i.$$

Proof. For $n=1$ this inequality is a consequence of Lemma 1.2. Let P be a projection of $C(X)$ onto $C(X/D)$. We shall consider the following property:

(*) If $\delta > 0$, $A \in S_n$, and G is a neighborhood of A, there exists t in $G \sim A$ and f in $C(X)$ such that $\|f\| = 1$, $f|X \sim G = 0$, and

$$Pf(t) > 2n+1-\Big(\sum_{i=1}^{n} 2/m_i\Big)-\delta.$$

If $n=1$, (*) follows from Lemma 1.2.

Next, we suppose that the theorem and property (*) hold for some positive integer n. Let $S_1, S_2, \ldots, S_n, S_{n+1}$ be non-empty collections of plural D-sets that satisfy (a), (b), and (c). Let $\delta > 0$. Choose $\varepsilon > 0$ so that $\varepsilon < 2\delta$. Let Y be a plural set in S_{n+1} and let M be the decomposition of X consisting of the plural sets in $D \sim \{Y\}$. Since D is contracting at Y, it follows that M is upper semicontinuous and X/M is a T_4-space. Let $y_1, y_2, \ldots, y_{m_{n+1}}$ be distinct points in Y which are limit points of the plural sets in $S_n \sim \{Y\}$. Let q denote the quotient map of M. Then $q(y_1), q(y_2), \ldots, q(y_{m_{n+1}})$ are distinct boundary points of $q[Y]$ in X/M. Since D/M has only the one plural set $q[Y]$, it is contracting.

Because P is a projection of $C(X)$ onto $C(X/D)$ and $C(X/D) \subset C(X/M) \subset C(X)$, the restriction P' of P to $C(X/M)$ is a projection of $C(X/M)$ onto $C(X/D)$ (i.e., the functions in $C(X/M)$ which are constant on $q[Y]$). Technically, $P' = (q^0)^{-1}Pq^0$. Let G be a D-saturated neighborhood of Y. By Lemma 1.2 there exists an index i and an open X/M-neighborhood U of $q(y_i)$ such that for each x in $U \sim q[Y]$ there exists an f_x in $C(X/M)$ such that f_x vanishes off of $q[G]$, $\|f_x\| = 1$, and

$$P'f_x(x) > 3-2/m_{n+1}-\varepsilon.$$

We may assume that $U \subset G$. Since $q(y_i)$ is the limit of points in $q[S_n]$, there is an A in S_n such that $q[A]$ belongs to U. Since X/M is T_4, there is a closed neighborhood V of $q[A]$ contained in $U \sim q[Y]$. If $V^* = \bigcup V$, then V^* is an X-neighborhood of A. By property (*) there is a point t in $V^* \sim A$ and g in $C(X)$ such that $g|X \sim V^* = 0$, $\|g\| = 1$ and

$$Pg(t) > 2n+1-\sum_{i=1}^{n} 2/m_i-\varepsilon.$$

Let $K = \big((X/M) \sim U\big) \cup q(Y)$ and observe that K is a closed set in X/M which does not intersect V. Since X/M is T_4, there exists by the Urysohn-Tietze Extension Theorem h in $C(X/M)$ such that $\|h\| = 1$, $h|K = f_{qt}|K$ and $h|V = 0$. Therefore, $h-f_{qt}$ vanishes on $q(Y)$ and

$$\begin{aligned} P'h(qt) &= [P'f_{qt}+P'(h-f_{qt})](qt) = P'\big(f_{qt}(qt)\big)+(h-f_{qt})(qt) \\ &\geqslant (3-2/m_{n+1}-\varepsilon)-f_{qt}(qt) = 2-2/m_{n+1}-\varepsilon \end{aligned}$$

since $h(qt) = 0$. Observe that q^0h belongs to $C(X)$ and $\|q^0h\| = 1$. In fact, $\|q^0h+g\| = 1$ since $q^0h(x) = h(qx) = 0$ if x belongs to V^* and $g(x) = 0$ if x does not belong to V^*. Therefore,

$$\begin{aligned} P(q^0h+g)(t) &= Pq^0h(t)+Pg(t) = (q^0)^{-1}Pq^0h(qt)+Pg(t) \\ &= P'h(qt)+Pg(t) > (2-2/m_{n+1}-\varepsilon)+\Big(2n+1-\varepsilon-\sum_{i=1}^{n} 2/m_i\Big) \\ &= 2(n+1)+1-\Big(\sum_{i=1}^{n+1} 2/m_i\Big)-2\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have that

$$\|P\| \geqslant 2(n+1)+1-\sum_{i=1}^{n+1} 2/m_i.$$

It is easy to see that for $f = q^0h+g$, t and f satisfy (*) for $n+1$. This completes the proof.

COROLLARY 1.4. *If D is a decomposition of a T_4-space X with property $L_n(2, 2, \ldots, 2)$ for all n, then $C(X/D)$ is not complemented in $C(X)$.*

The following example is due to R. Arens (cf. [4], p. 475):

EXAMPLE 1.5. For each t in $[0, 1]$ let X_t denote the vertical segment $\{(t, y)\colon 0 \leqslant y \leqslant 1\}$ in the plane. Let $X = \bigcup X_t$ and D be the decomposition of X whose elements are the sets X_t. Then D is an upper semicontinuous decomposition of X. It we choose the families $S_1, S_2, \ldots, S_n$ of plural sets from D by letting $S_i = D$ for each i, then each hypothesis of Theorem 1.3 with the exception of the contracting condition holds. However, X/D is homeomorphic to the unit interval and there is a projection P of $C(X)$ onto $C(X/D)$ with $\|P\| = 1$.

Let X and Y be Hausdorff spaces and φ be an epimorphism of X onto Y. A continuous linear operator u from $C(X)$ onto $C(Y)$ is a *linear averaging operator* for φ if and only if $u\varphi^0$ is the identity on $C(Y)$. There is a simple relationship between linear averaging operators and projections (cf. [17], p. 16): φ has a linear averaging operator with norm $\leqslant \lambda$ if and only if there is a projection with norm $\leqslant \lambda$ of $C(X)$ onto its subspace $\varphi^0[C(Y)]$. We let D_φ denote the decomposition of X consisting of the sets $\varphi^{-1}(Y)$ for $y \in Y$. If φ is closed or, equivalently, D_φ is upper semicontinuous, then X/D_φ is algebraically isometric to $C(Y)$. In [17], Pełczyński introduced the concept $p(\varphi) = \inf\{\|u\|: u$ is a linear averaging operator for $\varphi\}$. Therefore, $p(\varphi) = +\infty$ if and only if φ does not admit a linear operator of averaging. By use of these notations, we can restate the two preceding results in terms of properties of continuous functions.

Corollary 1.6. *Let φ be an epimorphism of a T_4-space X onto a Hausdorff space Y. If D_φ has property $L_n(m_1, m_2, \ldots, m_n)$, then*

$$p(\varphi) \geqslant 2n+1 - \sum_{i=1}^{n} 2/m_i.$$

Corollary 1.7. *Let φ be an epimorphism of a T_4-space X onto a Hausdorff space Y. If D_φ has property $L_n(2, 2, \ldots, 2)$ for all n, then $p(\varphi) = +\infty$.*

Suppose X is a topological space and D is a decomposition of X. We say that D is a *metric decomposition* of X if X/D is metrizable. Also, D is *lower semicontinuous* if the quotient map is open (cf. [14], p. 185, and [13], p. 97). In [4], R. Arens established results similar to Lemma 1.8 and Theorem 1.9 under the additional hypotheses that the decomposition is lower semicontinuous (line 2.05) and metric. He has communicated privately that upper semicontinuity should be included in the hypothesis of both results. The proof given here is similar to that given by Arens.

Lemma 1.8. *Let X be a metric space and let D be an upper semicontinuous decomposition of X. If P_0 is a projection of $C(X)$ onto $C(X/D^{(1)})$, then there exists a projection P of $C(X)$ onto $C(X/D)$ with $\|P\| \leqslant \|P_0\|+2$.*

The proof of this lemma is the same as the proof of statement 2.53 of Lemma 2.5 in [4], except that the definition of Q should be changed to

$$(Qf)(x) = \begin{cases} (P_0 f)(x(S)) & \text{for } x \in (X_1 \sim X_0), \\ (P_0 f)(x) & \text{for } x \in X. \end{cases}$$

The additional hypotheses of Lemma 2.5 are not needed.

Theorem 1.9. *Let X be a metric space and let D be an upper semicontinuous decomposition of X with $D^{(n)} = 0$. Then there exists a projection P of $C(X)$ onto $C(X/D)$ with $\|P\| \leqslant 2n+1$.*

Proof. Since D is upper semicontinuous, the decompositions $D^{(1)}, D^{(2)}, \ldots, D^{(n)}$ are also upper semicontinuous. Also, $(D^{(n-1)})^{(1)} = 0$; hence, it follows from Theorem 2.2 of [4], p. 471, or Theorem 8 of [20], p. 596, that there is a projection P_1 of $C(X)$ onto $C(X/D^{(n-1)})$ with $\|P\| \leqslant 3$. Let m be a positive integer less than n and P_m be a projection of $C(X)$ onto $C(X/D)^{(n-m)}$), with $\|P_m\| \leqslant 2m+1$. Since $(D^{(n-m-1)})^{(1)} = D^{(n-m)}$, it follows from Lemma 1.8 that there exists a projection P_{m+1} of $C(X)$ onto $C(X/D^{(n-m-1)})$ with $\|P_{m+1}\| \leqslant \|P_m\|+2$. Therefore, $\|P_{m+1}\| \leqslant 2(m+1)+1$ and the theorem is established.

Corollary 1.10. *Let φ be a closed epimorphism of a metric space X onto a Hausdorff space Y. If $D_\varphi^{(n)} = 0$, then $p(\varphi) \leqslant 2n+1$.*

The following corollary is an immediate consequence of Corollaries 1.6 and 1.10:

Corollary 1.11. *Let φ be an epimorphism of a metric space X onto a Hausdorff space Y. If D_φ has property $L_{n-1}(k, k, \ldots, k)$ for each positive integer k and $D_\varphi^{(n)} = 0$, then $p(\varphi)$ is attained and $p(\varphi) = 2n+1$.*

In particular, it should be noted that a contracting decomposition D_φ with $D_\varphi^{(n)} = 0$ has property $L_{n-1}(k, k, \ldots, k)$ for each positive integer k if the following is true: $D_\varphi^{(n-1)} \neq 0$, the boundary of each plural set is infinite, and each non-limit plural set in $D_\varphi^{(i)}$ contains infinitely many limit points of the plural sets in $D_\varphi^{(i-1)}$.

As pointed out by R. Arens (cf. [4], p. 475) and as illustrated by Example 1.5, the condition that $D^{(n)} = 0$ is far from being necessary for the existence of a projection of $C(X)$ onto $C(X/D)$. However, combining the results of Corollary 1.4 and Theorem 1.9, we obtain for a certain type of decomposition that the condition that $D^{(n)} = 0$ is necessary and sufficient for the existence of a projection. Example 1.5 also illustrates that the *contracting* hypothesis cannot be replaced with *upper semicontinuous and lower semicontinuous* (*i.e. continuous*).

Corollary 1.12. *Let X be a metric space. Suppose D is a contracting decomposition of X such that each plural set A in $D^{(k)}$ contains at least two limit points of the plural sets in $D^{(k-1)} \sim \{A\}$ for each positive integer k. Then $C(X/D)$ is complemented in $C(X)$ if and only if there exists a positive integer n such that $D^{(n)} = 0$.*

2. Some uncomplemented subspaces of $C(X)$.

In this chapter we use the preceding results to determine a sufficient topological condition on X so that $C(X)$ will have an uncomplemented subspace (Theorem 2.9). This result establishes a close relationship between uncomplemented subspaces of $C(X)$ and the successive derived sets of X (cf. [14], p. 261, or [20], p. 64). We first restrict our attention to the derived sets of the subset $S(X)$ consisting of the points of X which have a countable neighborhood base. If λ is an ordinal number, then $[S(X)]^{(\lambda)}$ is denoted by $S^\lambda(X)$ and a point in $S^\lambda(X)$ is called an *S^λ-point*.

The quotient space X/D inherits many of the properties of X. This is especially the case if D is upper semicontinuous and each of its plural sets is compact (e.g., see related discussion and problems in [5], [6], [8], [13], and [18]). In each result prior to Lemma 2.7, we also obtain $\mathrm{Card}(X) = \mathrm{Card}(X/D)$ and for each ordinal number λ, $X^{(\lambda)} \neq \emptyset$ implies $(X/D)^{(\lambda)} \neq \emptyset$ and $S^{\lambda}(X) \neq \emptyset$ implies $S^{\lambda}(X/D) \neq \emptyset$; moreover, if X is a compact metric space, then $C(X)$ and $C(X/D)$ are isomorphic by Corollary 8.7 of [17], p. 42. In this chapter, the hypothesis of each theorem and corollary is satisfied by the resulting space X/D whenever it is satisfied by X (except it may not be possible to use the same t for X/D in Theorem 2.1 and Corollary 2.4).

Let D and M be decompositions of X. We say that M is a *plural refinement* of D if each plural set in D belongs to M.

THEOREM 2.1. *Suppose that X is a T_4-space, n is a positive integer, and G is an open set in X with $S^n(G) \neq \emptyset$. For each positive integer k and for each integer t with $1 \leqslant t \leqslant \mathrm{Card}\, S^n(G)$, there exists a contracting decomposition D of X with the following properties:*

(1) *Each plural set is contained in G.*

(2) *Each plural set consists of either t or k distinct points.*

(3) *If M is an upper semicontinuous plural refinement of D which is contracting at each plural set of D and P is a projection of $C(X)$ onto $C(X/M)$, then*

$$\|P\| \geqslant 1+2n-\frac{2(n-1)}{k}-\frac{2}{t}.$$

Before proving this theorem, we state three lemmas. The purpose of the first and second lemmas is to simplify the task of showing that a decomposition is either contracting or upper semicontinuous. Most of the proof of the theorem is contained in the proof of the third lemma.

LEMMA 2.2. *Let X be a topological space and both D and M be decompositions of X such that*

(1) *M is contracting (upper semicontinuous).*

(2) *D implies M.*

(3) *For each A in M, the decomposition of X consisting of the plural sets of $\{B \in D\colon B \subset A\}$ is contracting (upper semicontinuous).*

(4) *Each limit set of M belongs to D.*

Then D is also contracting (respectively, upper semicontinuous).

LEMMA 2.3. *Let X be a topological space, D_1 and D_2 decompositions of X, and D_2 a plural refinement of D_1. If D_1 is contracting (upper semicontinuous) and the decomposition of X consisting of the plural sets in $D_2 \sim D_1$ is contracting (respectively, upper semicontinuous), then D_2 is contracting (respectively, upper semicontinuous).*

LEMMA 2.4. *Suppose X is a T_4-space, n is a positive integer and G is an open subset of X such that $\mathrm{Card}\, S^n(G) > 0$. For each integer t with $1 \leqslant t \leqslant \mathrm{Card}\, S^n(G)$ and for each positive integer k, there is a contracting decomposition D of X such that*

(1) *$D^{(n)}$ consists of singleton sets and a set F with t distinct points.*

(2) *Each plural set in $D \sim \{F\}$ consists of k points.*

(3) *Each point in each plural set in $D^{(i)}$ is a limit point of the plural sets in $D^{(i-1)}$.*

(4) *Each plural set in D is a subset of G.*

Proof. We inductively select non-empty families $C_1, C_2, \ldots, C_{n+1}$ and $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_{n+1}$ of subsets of G such that if $1 \leqslant m \leqslant n+1$, then:

(a) Each set in C_m consists of k points from $S^{n-m+1}(G)$ if $m > 1$.

(b) Each point of each set in C_{m-1} is a limit of the sets in C_m.

(c) $\mathcal{U}_m$ is a family of disjoint, closed subsets such that for each A in C_m, there is a neighborhood U_A of A in $\mathcal{U}_m$ which does not contain and other set in C_m.

(d) If $U \in \mathcal{U}_m$, then U does not intersect any set in C_j for $1 \leqslant j < m$.

(e) The decomposition D_m of X consisting of the plural sets in $(\bigcup_{i=1}^{m-1} C_i) \cup \mathcal{U}_m$ is contracting and each set in $\mathcal{U}_m$ is a non-limit set of D_m.

We select C_1 and $\mathcal{U}_1$ first. Let $\{x_i\}_{i=1}^{t}$ be t distinct points of $S^n(G)$ and define $F = \{x_i\}_{i=1}^{t}$. Let $C_1 = \{F\}$ and $\mathcal{U}_1 = \{G\}$. It is easy to check that conditions (a) through (e) are satisfied for $m = 1$.

Next, suppose $C_1, C_2, \ldots, C_m$ and $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_m$ have been selected and $m \leqslant n$. Let A be a set in C_m. We may suppose that $A = \{a_1, a_2, \ldots, a_z\}$, where each a_i belongs to $S^{n-m+1}(G)$, $z = t$ if $m = 1$, and $z = k$ if $m > 1$. There exists a neighborhood U_A of A in $\mathcal{U}_m$ which does not intersect any other set in C_m. We may select a family $\{U_i\}_{i=1}^{z}$ of closed disjoint sets such that for each i, U_i is a neighborhood of a_i and is contained in U_A. Let $\{V_{ij}\}_{j=1}^{\infty}$ be a closed neighborhood base for a_i such that $V_{ij} \subset U_i$ for each j. Let $\{a_{ij}\}_{j=1}^{\infty}$ be a set of distinct S^{n-m}-points such that $a_{ij} \neq a_i$ and a_{ij} is an element of the interior of V_{ij} for each i and each j. We may select a family $\{W_{ij}\}_{j=1}^{\infty}$ of disjoint closed sets such that W_{ij} is a neighborhood of a_{ij} and $V_{ij} \sim \{a_i\}$ is a neighborhood of W_{ij}.

Next, we define

$$A_{ij} = \{a_{i(jk+r)}\,|\,1 \leqslant r \leqslant k\} \quad \text{for } 1 \leqslant i \leqslant z \text{ and } j = 0, 1, 2, \ldots,$$

$$U_{ij} = \bigcup_{r=1}^{k} W_{i(jk+r)} \quad \text{for } 1 \leqslant i \leqslant z \text{ and } j = 0, 1, 2, \ldots,$$

$$C_A = \{A_{ij}\,|\,1 \leqslant i \leqslant z \text{ and } j = 0, 1, 2, \ldots\},$$

$$\mathcal{U}_A = \{U_{ij}\,|\,1 \leqslant i \leqslant z \text{ and } j = 0, 1, 2, \ldots\}.$$

Observe that each set in C_A contains k points from $S^{n-m}(G)$. Also, since a_{ij} converges to a_i in A, each point in A is a limit point of the sets in C_A. Therefore, if we define $C_{m+1} = \bigcup \{C_A: A \in C_m\}$, the hypotheses (a) and (b) are satisfied for $m+1$. If we also define $\mathscr{U}_{m+1} = \bigcup \{\mathscr{U}_A: A \in C_m\}$, it also follows that (c) and (d) are valid for $m+1$.

For each A in C_m, we let D_A be the decomposition of X whose plural sets are the plural in $\mathscr{U}_A$. It follows that each set in $\mathscr{U}_A$ is a non-limit set in D_A. However, if $K \in \mathscr{U}_{m+1}$, then $K \in \mathscr{U}_A$ for some A in C_m and U_A is a neighborhood of K. Since the only plural sets of D_{m+1} that intersect U_A are the plural sets in D_A, K is a non-limit set of D_{m+1}. Therefore, each set in $\mathscr{U}_{m+1}$ is a nonlimit set of D_{m+1}.

The only limit sets in D_A are the singleton sets $\{a_i\}$ for $a_i \in A$. Using this fact, it follows by a direct argument that D_A is contracting. Let M_A denote the decomposition of X whose plural sets are the sets in $\mathscr{U}_A \cup \{A\}$. Since the decomposition of X consisting of the one plural set A is contracting, it follows from Lemma 2.3 that M_A is contracting. Observe that M_A is precisely the decomposition of X whose plural sets are the plural sets of D_{m+1} which are contained in U_A. Since (e) implies that each limit set of D_m belongs to D_{m+1}, we have satisfied the hypothesis of Lemma 2.2 for $M = D_m$ and $D = D_{m+1}$. Thus D_{m+1} is contracting and (e) is established for $m+1$.

By induction, it follows that $C_1, C_2, \ldots, C_{n+1}$ and $\mathscr{U}_1, \mathscr{U}_2, \ldots, \mathscr{U}_{n+1}$ can be selected subject to the conditions (a)-(e). Let D be the decomposition of X consisting of the plural sets in $\bigcup_{j=1}^{n+1} C_j$. It follows from Lemma 2.2 that since D_{n+1} is contracting, D is also contracting (i.e., let $D_{n+1} = M$ in this lemma).

By induction, we can show that for $0 \leqslant m \leqslant n$, $D^{(m)}$ satisfies the following properties:

(a′) $\bigcup_{j=1}^{n-m+1} C_j$ is the set of plural sets.

(b′) C_{n-m+1} is the set of non-limit plural sets.

From (a′) we obtain that the set of plural sets of $D^{(n)}$ is C_1. However, $C_1 = \{F\}$ by our construction, and F consists of exactly t points. This completes the proof of the lemma.

Proof of Theorem 2.1. By Lemma 2.4, there is a contracting decomposition R of X which satisfies conditions (1) through (4) of this lemma. Let $D = R^{(1)}$ and suppose M is an upper semicontinuous plural refinement of D which is contracting at each set in D. Let S_i be the family of non-limit plural sets in $D^{(i-1)} \sim D^{(i)}$ for $1 \leqslant i \leqslant n$. It follows by Theorem 1.3 that if P is a projection of $C(X)$ onto $C(X/M)$, then

$$\|P\| \geqslant 1+2n-2/t-\sum_{i=1}^{n-1} 2/k.$$

This completes the proof.

A special case of Theorem 2.1 is stated in the following corollary:

COROLLARY 2.5. *Suppose X is a first-countable T_4-space with* Card $X^{(n)} \geqslant t$. *For each $\varepsilon > 0$, there is a contracting decomposition D of X with each plural set finite such that if P is a projection of $C(X)$ onto $C(X/D)$, then $\|P\| \geqslant 2n+1-2/t-\varepsilon$.*

We shall see in Remark 2.10 that the requirement that $S^\omega(X)$ be non-empty in Theorem 2.6 can be replaced with the requirement that $X^{(n)}$ be non-empty for each positive integer n if X is first countable. Since the decomposition D selected by this theorem is such that $S^\omega(X) \neq \emptyset$, it follows that if X is a T_4-space with $S^\omega(X) \neq \emptyset$, then $C(X)$ contains infinitely many uncomplemented subspaces of the type $C(Y)$.

THEOREM 2.6. *Suppose X is a T_4-space such that $S^\omega(X) \neq \emptyset$. There is a contracting decomposition D such that $C(X/D)$ is not complemented in $C(X)$. If k is an integer and $k \geqslant 2$, D can be selected so that each plural set of D contains exactly k elements.*

Proof. Let x be an S^ω-point of X and $\{U_n\}_{n=1}^\infty$ an open neighborhood base for x. By induction, we may select a sequence $\{x_n\}_{n=1}^\infty$ of distinct points such that for each n, x_n is an S^n-point, $x_n \neq x$, and U_n is a neighborhood of x_n. Next, we inductively select subsets $V_1, V_2, \ldots, V_n, \ldots$ such that for each n

(a) V_n is a closed neighborhood of x_n.

(b) $\{V_i\}_{i=1}^n$ is a family of disjoint closed sets.

(c) $V_n \subset U_n \sim \{x\}$.

Let R be the decomposition of X consisting of the plural sets $\{V_n\}_{n=1}^\infty$. This decomposition is contracting. Let k be an integer greater than 1. By Theorem 2.1 there exists for each n a contracting decomposition M_n of X with each plural set contained in V_n such that if M is a contracting, plural refinement of M_n and P is a continuous linear projection of $C(X)$ onto $C(X/M)$, then $\|P\| \geqslant n$. In fact, M_n can be selected so that each plural set contains exactly k elements. Let D be the decomposition of X consisting of the plural sets in M_n for each n and singleton sets. It follows by Lemma 2.2 that D is contracting.

Suppose P is a continuous linear projection of $C(X)$ onto $C(X/D)$. Since D is a contracting plural refinement of M_n for each n, $\|P\| \geqslant n$ for each n. It follows from this contradiction that $C(X/D)$ is not complemented in $C(X)$.

Many interesting spaces, such as βN and $\beta N \sim N$, do not have non-trivial convergent sequences. It is known (cf. [22], p. 483) that no extremely disconnected space contains a non-trivial convergent sequence. (See [11] for additional information on βN, $\beta N \sim N$, and extremely disconnected spaces.) Therefore, it is desirable to remove the dependence of the preceding theorems upon convergent sequences. The following lemma is our first step in this direction.

LEMMA 2.7. *Suppose X is a T_3-space, G is an open set in X, and n is a non-negative integer. If $x \in G^{(n)}$, then there is an upper semicontinuous decomposition D of X with each plural set of D contained in G such that if q is the quotient map, then $q(x)$ is an S^n-point of $q(G)$.*

Proof. First, we establish the lemma for $n = 0$. We may select open neighborhoods $U_1, U_2, \ldots, U_k, \ldots$ of x such that $\mathrm{Cl}(U_1) \subset G$ and, for each k, $\mathrm{Cl}(U_{k+1}) \subset \mathrm{Cl}(U_k)$. Define

$$K = \bigcap_{k=1}^{\infty} \mathrm{Cl}(U_k).$$

Then K is a closed subset of G and $\{U_k\}_{k=1}^{\infty}$ is an open neighborhood base for K. Let D be the decomposition of X with the one plural set K and q the induced quotient map. Then D is upper semicontinuous and $q(x)$ has a countable base. This establishes the lemma for $n = 0$.

Next, we suppose the lemma has been established for some $m < n$ and show that it is valid for $m+1$. We inductively select $x_1, x_2, \ldots, x_k, \ldots$ from X and neighborhoods $U_1, U_2, \ldots, U_k, \ldots$ of x such that for each k,

(a) $x_k \in G^{(m)}$.

(b) $x_k \in U_k$ and U_k is an open subset of G.

(c) $\mathrm{Cl}(U_k) \subset U_{k-1} \sim \{x_{k-1}\}$ if $k > 1$.

Let $V_k = U_k \sim \mathrm{Cl}(U_{k+1})$. Then $\{V_k\}_{k=1}^{\infty}$ is a family of disjoint open sets such that $x_k \in V_k$. Let W_k be a closed neighborhood of x_k contained in V_k. Define $F = \bigcap_{k=1}^{\infty} \mathrm{Cl}(U_k)$ and let M be the decomposition of X consisting of the plural sets $\{W_k\}_{k=1}^{\infty} \cup \{F\}$. Then M is upper semicontinuous.

By inductive hypothesis, there is an upper semicontinuous decomposition D_k of X with each plural set of D_k contained in W_k such that if q_k is the induced quotient map, then $q_k(x_k)$ is an S^m-point of X/D_k. Let D be the decomposition of X whose plural sets are the sets in $\{F\} \cup \cup (\bigcup_{k=1}^{\infty} D_k)$. By Lemma 2.2, it follows that D is upper semicontinuous. If q is the induced quotient map, then $q(x_k)$ is an S^m-point for each k. Since $q(F)$ has a countable base in X/D and is the limit of a sequence of S^m-points, it is an S^{m+1}-point. This completes the proof.

The next theorem replaces the requirement that $S^n(X) \neq \emptyset$ in Theorem 2.1 with the requirement that $X^{(n)} \neq \emptyset$. As a result, it is also applicable to spaces in which sequences of distinct points do not converge. Although the decomposition D of X obtained from this theorem is upper semicontinuous, the plural sets are not necessarily finite. However, if X

is compact, the plural sets must be compact since they are closed. It is an immediate consequence of the Stone-Čech compactification theory that $C(X)$ is isometric to $C(\beta X)$; hence, we may require that X be compact without loss of generality. The construction of D in this theorem ensures that X/D has an S^n-point.

THEOREM 2.8. *Let X be a T_4-space and n be a positive integer such that $X^{(n)} \neq \emptyset$. For each integer t with $1 \leqslant t \leqslant \mathrm{Card}\, X^{(n)}$ and for each $\varepsilon > 0$ there is a decomposition D of X such that each projection of $C(X)$ onto $C(X/D)$ has norm at least $2n+1-2/t-\varepsilon$.*

Proof. Let $x_1, x_2, \ldots, x_t$ be distinct points in $X^{(n)}$. Let $\{U_i\}_{i=1}^{t}$ be a family of disjoint open sets such that $x_i \in U_i$ for each i. By Lemma 2.7, there exists an upper semicontinuous decomposition H_i of X with each plural set contained in U_i such that if q_i is the induced quotient map, then $q_i(x_i)$ is an S^i-point of X/H_i. Let H be the decomposition of X consisting of the plural sets in each H_i. Then H is upper semicontinuous. If q is the associated quotient map, then $q(x_1), q(x_2), \ldots, q(x_t)$ are distinct S^n-points of X/H.

Let k be a positive integer such that $2(n-1)/k < \varepsilon$. By Theorem 2.1, there is a contracting decomposition K of X/H such that if P is a projection of $C(X/H)$ onto $C((X/H)/K)$, then

$$\|P\| \geqslant 1+2n-\frac{2(n-1)}{k}-\frac{2}{t}.$$

Let $D = \{\bigcup A:\ A \in K\}$. Then $K = D/H$ and by [5], p. 128, Problem 5, D is upper semicontinuous. Suppose P is a continuous linear projection of $C(X)$ onto $C(X/D)$. Since $C\big((X/H)/(D/H)\big) \subset C(X/H) \subset C(X)$ and $(X/H)/(D/H)$ is homeomorphic to X/D according to [8], p. 72, or [5], p. 40, the restriction P' of P to $C(X/H)$ is a projection of $C(X/H)$ onto $C\big((X/H)/K\big)$. Therefore, $\|P'\| \geqslant 1+2n-2/t-\varepsilon$. And since $\|P\| \geqslant \|P'\|$, the proof is complete.

The next theorem removes the dependence of Theorem 2.6 upon convergent sequences. The decomposition D is selected so that $S^n(X/D) \neq \emptyset$ for each positive integer n. Therefore, it follows from this theorem that if X is a T_4-space and $X^{(n)} \neq \emptyset$ for each positive integer n, then there are infinitely many uncomplemented subspaces of $C(X)$ of the type $C(Y)$.

THEOREM 2.9. *Let X be a T_4-space such that $X^{(n)} \neq \emptyset$ for all n. Then there is an upper semicontinuous decomposition D of X such that $C(X/D)$ is not complemented in $C(X)$.*

Proof. We inductively select points $x_1, x_2, \ldots, x_n, \ldots$ from X and subsets $V_1, V_2, \ldots, V_n, \ldots$ of X such that

(a) x_n belongs to $X^{(n)}$;

(b) $\{V_i\}_{i=1}^n$ is a family of closed disjoint sets and V_n is a neighborhood of x_n;

(c) if $G_n = \bigcup_{i=1}^n V_i$ contains an element of $X^{(n)}$, then $X \sim G_n$ contains an element of $X^{(n)}$.

The remainder of the proof consists of the consideration of two cases.

Case I. Suppose that $S = \{x_i\}_{i=1}^\infty$ is closed. Let

$$F = \mathrm{Cl}(\bigcup_{i=1}^\infty V_i) \sim \mathrm{Int}(\bigcup_{i=1}^\infty V_i),$$

where Int denotes the interior operator. Then F is a closed set disjoint from S. By normality, there exists a closed neighborhood V of S which does not intersect F. Let $W_i = V_i \cap V$ for each i. Then $\bigcup_{i\in J} W_i$ is closed for each subset J of positive integers. If M denotes the decomposition of X consisting of the plural sets in $\{W_i\}_{i=1}^\infty$, then it follows from the preceding remark that M is upper semicontinuous.

Since W_i is a neighborhood of x_i and $x_i \in X^{(i)}$, it follows by Lemma 2.7 that there is an upper semicontinuous decomposition K_i of X such that W_i is a neighborhood of each plural set in K_i and if k_i is the induced quotient map, then $k_i(x_i)$ is an S^i-point of X/K_i. Let K be the decomposition of X consisting of each plural set in K_i for each i. As M is upper semicontinuous and each K_i is upper semicontinuous, it follows from Lemma 2.2 that K is upper semicontinuous. Let k be the quotient map of X onto K. Then $k(x_i)$ is an S^i-point of X/K. By Theorem 2.1, there is a contracting decomposition H_i of X/K with each plural set of H_i contained in $k(W_i)$ such that if H is an upper semicontinuous plural refinement of H_i which is contracting at each set in H_i and P is a continuous linear projection of $C(X/K)$ onto $C\big((X/K)/H_i\big)$, then $\|P\| \geq i$.

Let L be the decomposition of X/K whose plural sets are the sets $k(W_i)$ for $i = 1, 2, \ldots$ Since M does not have any singular or plural limit sets, L does not either. Therefore, L is a contracting decomposition. Let H be the decomposition of X/K consisting of each plural set in H_i for each i. Since L is contracting and H_i is contracting for each i, it follows from Lemma 2.2 that H is also contracting.

Let $D = \{\bigcup A\colon A \in H\}$. Then D is a decomposition of X and $H = D/K$. Since H and K are both upper semicontinuous, it follows that D is upper semicontinuous ([5], p. 128, Problem 5). Suppose that P is a projection of $C(X)$ onto $C(X/D)$. Since $C(X) \subset C(X/K) \subset C\big((X/K)/(D/K)\big)$ and $(X/K)/(D/K)$ is homeomorphic to X/D by [5], p. 40, or [9], p. 72, we have that the restriction P' of P to $C(X/K)$ is a projection of $C(X/K)$ onto $C\big((X/K)/H\big)$. But H is a contracting plural refinement of each H_i: hence $\|P\| \geq i$ for each i. This contradiction establishes that $C(X/D)$ is not complemented in $C(X)$.

Case II. Suppose that $S = \{x_i\}_{i=1}^\infty$ is *not* closed. Let $F = \mathrm{Cl}(S) \sim S$. By normality, there is a closed neighborhood W_i of x_i contained in $\mathrm{Int}(V_i)$. Let

$$F_j = \mathrm{Cl}\big((\bigcup_{i=j}^\infty W_i) \cup F\big)$$

for each positive integer j. Then F_j is a closed set contained in the closed set $\bigcap_{i=1}^{j-1} \big(X \sim \mathrm{Int}(V_i)\big)$ and $W_i \cap F_j = \emptyset$ for $i < j$.

Next, we inductively select $U_1, U_2, \ldots, U_n, \ldots$ so that for each n,

(a) U_n is an open neighborhood of F_{n+1},

(b) $\mathrm{Cl}(U_n) \subset U_{n-1}$,

(c) $\mathrm{Cl}(U_n)$ and $\bigcup_{i=1}^n W_i$ are disjoint.

We define $F^* = \bigcap_{i=1}^\infty \mathrm{Cl}(U_i)$ and observe that $\{U_i\}_{i=1}^\infty$ is a countable base for F^*. If M denotes the decomposition of X consisting of the plural sets in $\{F^*\} \cup \{W_i\}_{i=1}^\infty$, then M is upper semicontinuous.

As W_n is a neighborhood of $x_n \in X^{(n)}$, there is an upper semicontinuous decomposition K_n of X with each plural set contained in W_n such that if k_n is the associated quotient map, then $k_n(x_n)$ is an S^n-point of X/K_n. Let K denote the decomposition of X consisting of the plural sets in K_n for each n and the set F^*. According to Lemma 2.2, K is upper semicontinuous. Let k be the quotient map of X onto X/K. Then $k(W_n)$ is a neighborhood of $k(x_n)$ and $k(x_n)$ is an S^n-point of X/K. Also, $k(F^*)$ has a countable base and is the limit of the sequence $\{k(x_i)\}_{i=1}^\infty$. Thus, $q(F^*)$ is an S^ω-point of X/K.

By Theorem 2.6, there is an upper semicontinuous decomposition H of X/K such that $C\big((X/K)/H\big)$ is not complemented in $C(X)$. Let $D = \{\bigcup A\colon A \in H\}$. Then $H = D/K$ and by [5], p. 128, Problem 5, D is upper semicontinuous since both H and K are upper semicontinuous. Also, $(X/K)/(D/K)$ is homeomorphic to X/D; hence, $C(X/D)$ is not complemented in $C(X)$. This completes the proof.

Remark 2.10. If X is a T_4-space with a closed sequence $\{x_i\}_{i=1}^\infty$ of distinct points such that $x_n \in S^n(X)$ for each n, then in Case I of the preceding argument the decompositions K_i and K are unnecessary. In this case, for each integer $n \geq 2$, D can be selected so that it is contracting and each plural set consists of n points.

Amir [2], [3] and Pełczyński [17], p. 55, have established the equivalence of conditions (1) through (4) of Theorem 2.11. Recall that according to Theorem 2.9, the condition that $X^{(n)} \neq \emptyset$ for all n is sufficient to ensure that a T_4-space X has an upper semicontinuous decomposition D such that $C(X/D)$ is uncomplemented in $C(X)$. It follows from conditions (1) and (5) of Theorem 2.11 that if X is a compact metric space, then this condition is also necessary. Equivalence (10) gives an affirmative answer to question 30 (b) of Pełczyński in [17], p. 74. He communicated privately that the statement of Problem 30 should be changed to read: "Are the conditions (9.13.1)-(9.13.3) equivalent to the negation of the following conditions?"

For an ordinal number ζ, we let $[\zeta]$ denote the set $\{\eta: \eta$ is an ordinal number and $\eta \leqslant \zeta\}$ with the order topology. Let $\lambda \geqslant 1$. A separable Banach space X is a $\mathfrak{P}'_s$ space if for each separable Banach space Y and for each linear isometry $u: X \to Y$, there is a projection P of Y onto $u(X)$ with $\|P\| \leqslant s$.

THEOREM 2.11. *Suppose X is an infinite compact metric space. Then the following conditions are equivalent*:

(1) *Some derived set of X of finite order is empty*;

(2) *$C(X)$ is isomorphic to the space c*;

(3) *X is homeomorphic to $[\zeta]$ for some ordinal $\zeta < \omega^\omega$*;

(4) *$C(X)$ is a $\mathfrak{P}'_s$ space for some $s \geqslant 1$*;

(5) *for each upper semicontinuous decomposition D of X, $C(X/D)$ is complemented in $C(X)$*;

(6) *for each Hausdorff space Y and each epimorphism f of X onto Y, $f^0[C(Y)]$ is complemented in $C(X)$*;

(7) *for each compact metric space Y and each epimorphism f of X onto Y, $f^0[C(Y)]$ is complemented in $C(X)$*;

(8) *for each compact Hausdorff space Y and for each epimorphism f of X onto Y, there exists a linear averaging operator for f.*

Proof. The equivalence of conditions (1) through (4) is established by [2], [3], and [17], p. 55. We prove the following implications: (1) → (5) → (6) and (7) → (8) → (1). The implication (6) → (7) is trivial.

(1) → (5). Suppose that D is an upper semicontinuous decomposition of X. Since $X^{(n)} = \emptyset$ for some positive integer n, we obtain that $D^{(n)} = 0$. Therefore, by Theorem 1.9, $C(X/D)$ is complemented in $C(X)$.

(5) → (6). Suppose that f is an epimorphism of X onto a Hausdorff space Y. Since X is a compact, f is closed and Y has the quotient topology. Therefore, if $D = \{f^{-1}(y): y \in Y\}$, then Y is homeomorphic to X/D. Since f is closed, D is upper semicontinuous; hence, by (5), $f^0[C(Y)]$ is complemented in $C(X)$.

(7) → (8). Suppose that f is a continuous function from X onto a compact Hausdorff space Y. Since X is a compact metric space and f is a closed continuous map, it follows from Theorem 1 of [16] that Y is a metric space. Therefore, by (7) there is a projection of $C(X)$ onto $f^*[C(Y)]$. It follows that f has a linear averaging operator.

(8) → (1). Suppose that $X^{(n)} \neq \emptyset$ for each n. By Theorem 2.9, there is an upper semicontinuous decomposition D of X such that $C(X/D)$ is not complemented in $C(X)$. Let q be the quotient map of X onto X/D. Then X/D is compact and T_4; hence, by (8), q has a linear averaging operator. This implies there is a projection of $C(X)$ onto $q^0[C(X/D)]$, which is a contradiction. Therefore, $X^{(n)} = \emptyset$ for some positive integer n.

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