STUDIA MATHEMATICA, T. XXXVI, (1970)

On controllability of systems of strings

by

S. ROLEWICZ (Warszawa)

Let a system of strings with n knots $p_1, \ldots, p_n$ be given (see Fig. 1). By $\mathfrak{A}$ we denote the set of non-ordered pairs (i, j) such that there is a string $L_{i,j}$ connecting the knots p_i and p_j. The length of $L_{i,j}$ we denote by $S_{i,j}$, the tension by $T_{i,j}$ and the linear density by $\varrho_{i,j}$.

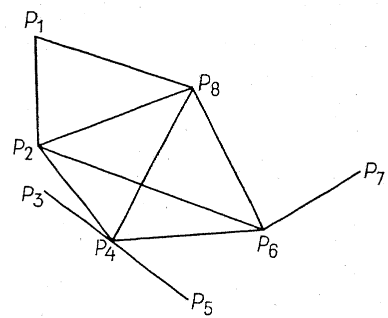

Fig. 1

We consider perpendicular vibrations of the strings. By $Q^{i,j}(x, t)$ we denote the perpendicular deviation of the string $L_{i,j}$ at the point x at the moment t. Of course, $Q^{i,j}(x, t)$ satisfies the wave equation

$$\frac{\partial^2}{\partial t^2} Q^{i,j}(x, t) = a_{i,j}^2 \frac{\partial^2}{\partial x^2} Q^{i,j}(x, t), \tag{1}$$

where

$$a_{i,j} = \sqrt{\frac{T_{i,j}}{\varrho_{i,j}}}.$$

We assume that the system is controlled by perpendicular deviations $(u_1(t), \ldots, u_n(t))$ at the knots $p_1, \ldots, p_n$, i.e.

$$\lim_{\substack{x \to p_i \\ x \in L_{i,j}}} Q^{i,j}(x, t) = u_i(t). \tag{2}$$

The problem considered is the following. We have an initial state at the moment $t = 0$, i.e.

$$(3)\qquad Q^{i,j}(x,0) = Q_0^{i,j}(x), \qquad \frac{\partial}{\partial t} Q^{i,j}(x,t)|_{t=0} = \dot{Q}_0^{i,j}(x),$$

where $Q_0^{i,j}(x)$ and $\dot{Q}_0^{i,j}(x)$ are given square integrable functions defined on $L_{i,j}$. We are looking for a time T and a square integrable control $u(t) = (u_1(t), \ldots, u_n(t))$ quieting the system at the moment T, i.e. such that for the functions $Q^{i,j}(x,t)$ satisfying (1), (2), (3)([1]) we have

$$(4)\qquad Q^{i,j}(x,T) = 0 = \frac{\partial}{\partial t} Q^{i,j}(x,t)|_{t=T}.$$

If for arbitrary square integrable functions $Q_0^{i,j}(x)$ and $\dot{Q}_0^{i,j}(x)$ such a time T and such a control $u(t)$ exist, we say that the system is *controllable*.

A. G. Butkowski in his book [1] have considered a system consisting of one string. He proved that the system is controllable, even if we put a control equal to 0 at one of the ends. (In fact he showed much more since he gave an effective method of describing $u(t)$).

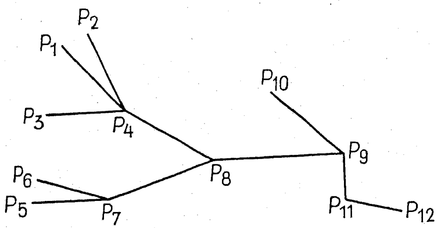

Fig. 2

From Butkowski's results it follows that if in the system the number of strings is less than the number of knots, then the system is controllable. In fact, in this case the system cannot contain a homeomorphic image of a circle (see Fig. 2). Then we can choose an arbitrary knot, for example p_1, and we put $u_1(t) = 0$. Now we take a string $L_{1,i}$, $(1,i)\in\mathfrak{A}$. By Butkowski's theorem there is a control $u_i(t)$ quieting the string $L_{1,i}$ at the time T_i. Then we put $u_i(t) = 0$ for $t > T_i$. Now we take a string $L_{i,j}$, $(i,j)\in\mathfrak{A}$. Then there is a control $u_j(t)$ quieting $L_{i,j}$ at the moment T_j. Continuing this process we can quiet sequentially the whole system.

([1]) By solutions of (1), (2), (3) we understand generalized solutions in the sense of Sobolev.

The main result of this paper is a proof that if the number of a string is greater than the number of knots, then the system is not controllable.

As follows from Butkowski's considerations, the controls $u_i(t)$ which quiet the system at the moment T ought to satisfy the following equations:

$$(5)\qquad \int_0^T [u_i(t)\cos\pi\lambda_{i,j}kt + u_j(t)(-1)^k\cos\pi\lambda_{i,j}kt]\,dt = \frac{A_{i,j}^k}{2\lambda_{i,j}},$$

$$(5')\qquad \int_0^T [u_i(t)\sin\pi\lambda_{i,j}kt + u_j(t)(-1)^k\sin\pi\lambda_{i,j}kt]\,dt = \frac{B_{i,j}^k}{2\lambda_{i,j}}$$

$$(k = 1, 2, \ldots, (i,j)\in\mathfrak{A}),$$

where

$$\lambda_{i,j} = \frac{a_{i,j}}{S_{i,j}},$$

$$A_{i,j}^k = \frac{2}{S_{i,j}} \int_0^{S_{i,j}} Q_0^{i,j}(x)\sin\frac{\pi k}{S_{i,j}}x\,dx$$

and

$$B_{i,j}^k = \frac{2}{\pi k a_{i,j}} \int_0^{S_{i,j}} \dot{Q}_0^{i,j}(x)\sin\frac{\pi k}{S_{i,j}}x\,dx.$$

Let us consider the space $L_n^2[0,T]$ of n-dimensional square integrable vector functions defined on the interval $[0,T]$. Let $f_{i,j}^k$ be the following functionals defined on $L_n^2[0,T]$:

$$f_{i,j}^k(u) = \int_0^T [u_i(t)\cos\pi\lambda_{i,j}kt + u_j(t)(-1)^k\cos\pi\lambda_{i,j}kt]\,dt$$

$$(k = 1, 2, \ldots, (i,j)\in\mathfrak{A}).$$

Let us order the functionals $f_{i,j}^k$ in a sequence $\{F_m\}$, $m = 1, 2, \ldots$, and let A be an operator mapping $L_n^2[0,T]$ into l^2 defined as follows:

$$A(u) = \{F_m(u)\}.$$

If the system is controllable, then equations (5) imply that the operator A is an epimorphism. But

PROPOSITION 1. *If the number of strings is greater than the number of knots, then the operator A cannot be an epimorphism.*

Proof. By Kronecker's theorem for any number $\delta > 0$ there is a real r and integers $n_{i,j}^r$ such that

$$|n_{i,j}^r\lambda_{i,j} - r| < \delta, \qquad (i,j)\in\mathfrak{A}.$$

Since $\mathfrak{A}$ has more than n elements, this implies that for any $\varepsilon > 0$ there is an index $i(\varepsilon)$ and indices $i_1(\varepsilon), \ldots, i_n(\varepsilon), i(\varepsilon) \neq i_1(\varepsilon), \ldots, i_n(\varepsilon)$ such that

$$\inf \|F_{i(\varepsilon)} - \sum_{k=1}^{n} a_k F_{i_k(\varepsilon)}\| < \varepsilon, \tag{6}$$

where $a_1, \ldots, a_n$ are scalars.

Suppose that there is an element $u \in L_n^2[0, T]$ such that $F_{i(\varepsilon)}(u) = a$ and $F_{i_k(\varepsilon)}(u) = 0$ $(k = 1, 2, \ldots, n)$. Then

$$|a| = |F_{i(\varepsilon)}(u)| = \inf_{a_1, \ldots, a_n} |F_{i(\varepsilon)}(u) - \sum_{k=1}^{n} {}_k F_{i_k(\varepsilon)}(u)| < \varepsilon \|u\|. \tag{7}$$

Hence

$$\|u\| > \frac{|a|}{\varepsilon}. \tag{8}$$

Let us put $\varepsilon_n = 1/2^n$. Without loss of generality we can assume that $i(\varepsilon_n) \neq i(\varepsilon_{n'})$ for $n \neq n'$. Let $a = \{a_n\}$, where

$$a_n = \begin{cases} 1/n^2 & \text{for } n = i(\varepsilon_n), \\ 0 & \text{elsewhere.} \end{cases} \tag{9}$$

Of course $a \in l^2$, but, on the other hand, inequality (8) implies that there exists no $u \in L_n^2[0, T]$ such that $A(u) = a$, because if such a u existed, then $\|u\| > 2^n/n^2$ $(n = 1, 2, \ldots)$. Therefore A is not an epimorphism, q.e.d.

As an obvious consequence we obtain

THEOREM 1. *If the number of strings is greater than the number of knots, then the system is not controllable.*

Using the same technique we can obtain even a stronger result. Namely, we can consider the operator A also as a continuous operator mapping the space of integrable vector functions $L_n^1[0, T]$ into the space c_0 of all sequences tending to 0. Repeating the same considerations, we are able to prove that if $a \in c_0$ is given by formula (9), then there is no element $u \in L_n^1[0, T]$ satisfying the equation $A(u) = a$.

Let us remark that the sequence a_n is a sequence of coefficients of continuous functions. Therefore the following theorem holds:

THEOREM 1'. *If the number of strings is greater than the number of knots, then there is an initial state given by continuous functions $Q_0^{i,j}(x)$ and $Q_0^{i,j}(x)$ such that no integrable control can quiet the system.*

We have considered the problem of controllability for systems in which the number of strings is different from the number of knots. Now we shall consider systems in which the number of knots is equal to the number of strings, i.e. systems which contain one homeomorphic image of the circle (see Fig. 3). The main problem is the controllability of the "circle". In fact, if the "circle" is controllable, then from similar considerations as for systems containing more knots than strings we are able to prove the controllability of the whole system.

THEOREM 2. *If the number of strings is equal to the number of knots and all $\lambda_{i,j}$ are commensurable, then the system is controllable.*

Proof. Let us denote by $g_{i,j}^k$ functionals defined on $L_n^2[0, T]$

$$g_{i,j}^k(u) = \int_0^T [u_i(t) \sin \pi\lambda_{i,j} kt + u_j(t)(-1)^k \sin \pi\lambda_{i,j} kt]\, dt.$$

Let us order all functionals $f_{i,j}^k, g_{i,j}^k$ into a sequence $\{G_m\}$ and let us consider an operator B mapping $L_n^2[0, T]$ into l^2 defined as follows: $B(u) = \{G_m(u)\}$.

We shall show that for certain T the operator B is an epimorphism.

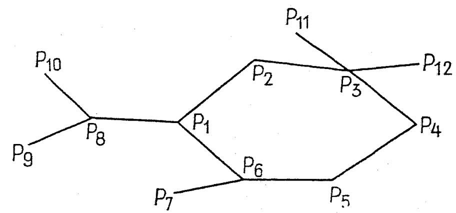

Fig. 3

Let r be a common divisor of all $\lambda_{i,j}$ and let $T = 2/r$. Let X_l be a space spanned by elements

$$(\cos \pi lrt, 0, \ldots, 0), \quad (0, \cos \pi lrt, 0, \ldots, 0), \quad (0, \ldots, 0, \cos \pi lrt),$$
$$(\sin \pi lrt, 0, \ldots, 0), \quad (0, \sin \pi lrt, 0, \ldots, 0), \quad (0, \ldots, 0, \sin \pi lrt)$$

and let P_l be an orthogonal projection of the space $L_n^2[0, T]$ onto X_l. It is obvious that X_l and $X_{l'}$ are orthogonal if $l \neq l'$, i.e. $P_l P_{l'} = 0$ if $l \neq l'$.

Let us fix l and let $f_1, \ldots, f_p, g_1, \ldots, g_p$ be functionals $f_{i,j}^k, g_{i,j}^k$ which belongs to X_l (of course $p \leqslant n$). By simple calculations we find that there are positive constant c and C independent of l such that

$$c \sum_{i=1}^{p} (\alpha_i^2 + \beta_i^2) \leqslant \|u\| \leqslant C \sum_{i=1}^{p} (\alpha_i^2 + \beta_i^2) \tag{10}$$

for

$$u = \sum_{i=1}^{p} (\alpha_i f_i + \beta_i g_i) \in \chi_l.$$

This implies that the operator B is an epimorphism. Then basing ourselves on equations (5) and (5′), we infer that the system is controllable, q.e.d.

The problem of controllability of systems with equal numbers of strings and knots in the case where $\lambda_{i,j}$ are not commensurable is still open.

So far we have considered only systems in which two knots are connected at most by one string. From the engineering point of view cases where two knots are connected by two or more parallel strings are also important. Such systems are not controllable as follows from

THEOREM 3. *If two parallel strings are controlled by two common ends* (see Fig. 4), *then there is an initial state which cannot be quieted.*

Fig. 4

Proof. Of course, if there is a control quieting the system, then it satisfies equations (5) and (5′) for two ordered pairs (1,2) and (2,1). The rest of the proof is the same as the proof of Proposition 1.

The same results are obtained in another way by Butkowski [2].

References

[1] А. Г. Бутковский, *Теория оптимального управления с распределенными параметрами*, Москва 1965.

[2] — *Финитное управление распределенными системами*, ДАН СССР (in print).

INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
INSTITUTE OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES

Reçu par la Rédaction le 15. 5. 1969

STUDIA MATHEMATICA, T. XXXVI, (1970)

Decompositions of operator-valued representations of function algebras

by

W. MLAK (Kraków)

Suppose we are given the complex Hilbert space with the inner product (f, g) $(f, g \in H)$ and the norm $|f| = \sqrt{(f, f)}$. Let $L(H)$ stand for the algebra of all linear bounded operators in H. $|V|$ is the norm and V^* is the adjoint of $V \in L(H)$. I denotes the identity operator in H.

Let X be a compact Hausdorff space. $C(X)$ is the algebra of all complex-valued continuous functions on X, with the sup norm $\|u\| = \sup_X |u(x)|$. In what follows A stands for a sub-algebra of $C(X)$, which is uniformly closed, separates the points of X and contains constants. The homomorphic mapping T of A into $L(H)$ is called a *representation* of A. We may assume without any loss of generality that

$$T(1) = I. \tag{*}$$

We shall consider merely norm continuous representations that is such that T as a linear operator from the space A into $L(H)$ is bounded, i.e.

$$|T(u)| \leqslant M\|u\|, \quad \text{all } u \in A, \tag{**}$$

for some finite M. The representation is called *contractive* if $M = 1$.

If f and g are in H, then $u \to (T(u)f, g)$ is a linear functional on A, bounded by $M\,|f|\,|g|$. Using the Hahn-Banach theorem and the Riesz representation theorem we infer that there is a regular Borel measure $p(f, g)$ on X such that

$$(T(u)f, g) = \int u\,dp(f, g) \quad \text{for } u \in A \tag{***}$$

and

$$\|p(f, g)\| \leqslant M\,|f|\,|g|. \tag{**}$$

An arbitrary (Borel, regular) measure $p(f, g)$ which satisfies (***) and (**) is called an *elementary measure* for $f, g \in H$ of the representation T satisfying (**). In the case where $\dim H = 1$, that is if H is simply the complex plane, every homomorphic mapping of A is contractive. We