This implies that the operator B is an epimorphism. Then basing ourselves on equations (5) and (5′), we infer that the system is controllable, q.e.d.

The problem of controllability of systems with equal numbers of strings and knots in the case where $\lambda_{i,j}$ are not commensurable is still open.

So far we have considered only systems in which two knots are connected at most by one string. From the engineering point of view cases where two knots are connected by two or more parallel strings are also important. Such systems are not controllable as follows from

THEOREM 3. *If two parallel strings are controlled by two common ends* (see Fig. 4), *then there is an initial state which cannot be quieted.*

Fig. 4

Proof. Of course, if there is a control quieting the system, then it satisfies equations (5) and (5′) for two ordered pairs (1,2) and (2,1). The rest of the proof is the same as the proof of Proposition 1.

The same results are obtained in another way by Butkowski [2].

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Decompositions of operator-valued representations of function algebras

by

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Suppose we are given the complex Hilbert space with the inner product (f, g) $(f, g \in H)$ and the norm $|f| = \sqrt{(f, f)}$. Let $L(H)$ stand for the algebra of all linear bounded operators in H. $|V|$ is the norm and V^* is the adjoint of $V \in L(H)$. I denotes the identity operator in H.

Let X be a compact Hausdorff space. $C(X)$ is the algebra of all complex-valued continuous functions on X, with the sup norm $\|u\| = \sup_X |u(x)|$. In what follows A stands for a sub-algebra of $C(X)$, which is uniformly closed, separates the points of X and contains constants. The homomorphic mapping T of A into $L(H)$ is called a *representation* of A. We may assume without any loss of generality that

$$T(1) = I. \tag{*}$$

We shall consider merely norm continuous representations that is such that T as a linear operator from the space A into $L(H)$ is bounded, i.e.

$$|T(u)| \leqslant M\|u\|, \quad \text{all } u \in A, \tag{**}$$

for some finite M. The representation is called *contractive* if $M = 1$.

If f and g are in H, then $u \to (T(u)f, g)$ is a linear functional on A, bounded by $M\,|f|\,|g|$. Using the Hahn-Banach theorem and the Riesz representation theorem we infer that there is a regular Borel measure $p(f, g)$ on X such that

$$(T(u)f, g) = \int u\,dp(f, g) \quad \text{for } u \in A \tag{***}$$

and

$$\|p(f, g)\| \leqslant M\,|f|\,|g|. \tag{**}$$

An arbitrary (Borel, regular) measure $p(f, g)$ which satisfies (***) and (**) is called an *elementary measure* for $f, g \in H$ of the representation T satisfying (**). In the case where $\dim H = 1$, that is if H is simply the complex plane, every homomorphic mapping of A is contractive. We show in the example below that if $\dim H \geqslant 2$, then, for a suitable A, there are representations with the norm so great as we wish.

Example. Suppose that $\dim H \geqslant 2$. Let A be the unit disc algebra of functions analytic in the open unit disc and continuous in the closed one. It is shown in [9], p. 48, Prop. 11.3, that for every $\varrho > 0$ there is a $V \in L(H)$ such that $V \in \mathscr{C}_\varrho$ and $|V| = \varrho$. Since $V \in \mathscr{C}_\varrho$, the generalized von Neumann inequality (see [9], p. 49, 11.12)

$$|u(V)| \leqslant \max_{|z|\leqslant 1} |\varrho u(z) + (1-\varrho) u(0)|$$

holds true for every polynomial $u \in A$. It follows that

$$|u(V)| \leqslant (\varrho + |1-\varrho|)\|u\|.$$

Now, if $u \in A$, then there is a sequence u_n such that $u_n \to u$ uniformly on the disc, which by the last inequality implies that the formula $T(u) = \lim u_n(V)$ determines a well defined representation of A with the norm not greater than $\varrho + |1-\varrho|$. Since for the function $u(z) = z$ we have $T(u) = V$ and $|V| = \varrho$, every M satisfying (**) for our representation is not less than ϱ, as was to be proved.

1. The closed set $\sigma \subset X$ is called a *peak set* of A, if there is a $u \in A$ such that $u(x) = 1$ on σ and $|u(x)| < 1$ for $x \in X - \sigma$. We say then that u *peaks on* σ. Let $P(A)$ be the totality of all peak sets of A. If $\sigma_1, \sigma_2 \in P(A)$ and $u_i \in A$ peaks on σ_i $(i = 1, 2)$, then $u_1 u_2$ peaks on $\sigma_1 \cap \sigma_2$. On the other hand, as simply proved by Bears (see [4], p. 435), the union $\sigma_1 \cup \sigma_2$ of $\sigma_i \in P(A)$ is again a peak set. It follows that $P(A)$ is a set lattice with intersections and forming unions (both in finite number) as the lattice operations. In what follows $P(A)$ stands always for this lattice.

The peak functions have been used in operator theory by C. Foiaş in 1959.

Let T be a representation of A and let $u \in A$ peak on $\sigma \in P(A)$. We take an elementary measure $p(f, g)$ and by (***) just get

$$\big(T(u^n)f, g\big) = \int u^n dp(f, g).$$

Since u peaks on σ, $u^n \to \chi_\sigma$ (the characteristic function of σ) pointwise, which implies that

$$\lim \big(T(u^n)f, g\big) = \int \chi_\sigma dp(f, g).$$

It is a simple matter to check that the limit on the left-hand side depends only on f, g and σ, but not on the choice of u and the choice of elementary measure $p(f, g)$. It follows now that there is a unique $P(\sigma) \in L(H)$ determined by σ and such that $\lim\big(T(u^n)f, g\big) = \big(P(\sigma)f, g\big)$ for arbitrary $f, g \in H$ and any u peaking on σ. Since for such u and arbitrary $v \in A$ we have

$$\big(T(v)T(u^n)f, g\big) = \int vu^n dp(f, g)$$

and

$$\lim\big(T(u^n)T(v)f, g\big) = \big(P(\sigma)T(v)f, g\big) = \lim\big(T(v)T(u^n)f, g\big) = \lim\big(T(u^n)f, T(v)^* g\big),$$

$P(\sigma)$ commutes with $T(A)$ and

$$\big(P(\sigma)T(v)f, g\big) = \int \chi_\sigma v dp(f, g) \tag{1.1}$$

for all $v \in A$ and arbitrary elementary measure $p(f, g)$. If $\sigma_1, \sigma_2 \in P(A)$, then $P(\sigma_1)$ commutes with $T(v^n)$, where v peaks on σ_2. It follows that $P(\sigma_1)$ and $P(\sigma_2)$ commute. We will show that every $P(\sigma)$ is a projection, that is $P(\sigma)^2 = P(\sigma)$. Indeed, if u peaks on σ, then, for arbitrary k,

$$\big(P(\sigma)f, g\big) = \lim\big(T(u^{n+k})f, g\big) = \lim\big(T(u^n)T(u^k)f, g\big) = \big(P(\sigma)T(u^k)f, g\big) = \big(T(u^k)P(\sigma)f, g\big).$$

If we let $k \to \infty$, then we get $\big(P(\sigma)f, g\big) = \big(P(\sigma)^2 f, g\big)$ for arbitrary $f, g \in H$, q.e.d. It follows now that the set $\hat{P}(A) = \{P(\sigma)\colon \sigma \in P(A)\}$ is a commutative family of projections. Hence, $\hat{P}(A)$ is a lattice if the lattice operations are defined by

$$P(\sigma_1) \vee P(\sigma_2) = P(\sigma_1)P(\sigma_2),$$

$$P(\sigma_1) \vee P(\sigma_2) = P(\sigma_1) + P(\sigma_2) - P(\sigma_1)P(\sigma_2).$$

PROPOSITION 1.1 *The mapping $\sigma \to P(\sigma)$ is a homomorphic mapping of $P(A)$ onto $\hat{P}(A)$.*

Proof. Suppose $\sigma_1, \sigma_2 \in P(A)$ and let $u \in A$ peak on σ_2. We have for $f, g \in H$

$$\big(P(\sigma_1 \cap \sigma_2)f, g\big) = \int \chi_{\sigma_1 \cap \sigma_2} dp(f, g).$$

On the other hand, by (1.1),

$$\big(P(\sigma_1)P(\sigma_2)f, g\big) = \lim\big(P(\sigma_1)T(u^n)f, g\big) = \lim \int \chi_{\sigma_1} u^n dp(f, g) = \int \chi_{\sigma_1 \cap \sigma_2} dp(f, g).$$

We may conclude that $P(\sigma_1 \cap \sigma_2) = P(\sigma_1)P(\sigma_2)$. Consequently, using (1.1) we get that

$$\Big(\big(P(\sigma_1) + P(\sigma_2) - P(\sigma_1)P(\sigma_2)\big)f, g\Big) = \int (\chi_{\sigma_1} + \chi_{\sigma_2} - \chi_{\sigma_1 \cap \sigma_2}) dp(f, g) = \int \chi_{\sigma_1 \cup \sigma_2} dp(f, g) = \big(P(\sigma_1 \cup \sigma_2)f, g\big)$$

which completes the proof.

A few remarks are now in order. First, it is obvious that $\hat{P}(A)$ is in the weak operator closure of $T(S)$, where S is the unit sphere of A.

However, a little more is true, namely $\hat{P}(A)$ belongs to the strong operator closure of $T(S)$. Indeed, let u peak on σ and let $f_i, g_i \in H$ for $i = 1, \ldots, k_0$. We write $\hat{H} = H \oplus H \oplus \ldots \oplus H$ (k_0 times). Then for $\hat{f} = (f_1, \ldots, f_{k_0})$, $\hat{g} = (g_1, \ldots, g_{k_0}) \in \hat{H}$

$$\big(\hat{T}(u^n)\hat{f}, \hat{g}\big) \to \sum_{i|1}^{k_0} \big(P(\sigma)f_i, g_i\big),$$

which means that $\hat{T}(u^n)\hat{f} \to \hat{P}(\sigma)\hat{f} = \big(P(\sigma)f_1, \ldots, P(\sigma)f_k\big)$ weakly in $\hat{H}$. By the classical theorem of S. Mazur, for suitable $\lambda_k^{(n)} \geqslant 0$ $(k = 1, \ldots, n)$ $\sum_{k|1}^{n} \lambda_k^{(n)} = 1$ we have

$$\hat{g}_n = \sum_{k|1}^{n} \lambda_k^{(n)} \hat{T}(u^k)\hat{f} \to \hat{P}(\sigma)\hat{f}$$

strongly in $\hat{H}$. But the i-th component of $\hat{g}_n$ is $T\big(\sum_{k|1}^{n} \lambda_k^{(n)} u^k\big) f_i$ and the convex combination of $u, u^2, \ldots, u^n$ is in S and peaks on σ. This completes the proof of our assertion. As a by-product we get therefore that for every $f \in H$ and every $\sigma \in P(A)$ there is a sequence of functions $u_n \in A$ peaking on σ such that $T(u_n)f \to P(\sigma)f$ strongly.

In general, since

$$\big|\big(P(\sigma)f, g\big)\big| \leqslant |\smallint \chi_\sigma dp(f, g)| \leqslant M|f|\,|g|,$$

we always have $|P(\sigma)| \leqslant M$. If T is contractive, then $|P(\sigma)| \leqslant 1$, which together with the idempotence property of $P(\sigma)$ yields that $P(\sigma)$ is an orthogonal projection.

In the next step we shall extend the homomorphism $\sigma \to P(\sigma)$ onto a wider class of subsets of X. We define $R(A)$ as the class of all subsets of X which are intersections of peak sets of A. If $\sigma \in R(A)$, then $\sigma = \bigcap \sigma_a$ where $\sigma_a \in P(A)$ and we will assume that the product on the right-hand side includes all peak sets including σ.

LEMMA 1.1 *Suppose that $\sigma \in R(A)$ and let p be a regular Borel measure on X. Then there is a sequence $\sigma_n \in P(A)$ such that $\sigma \subset \sigma_n$ and*

$$\lim |p|(\sigma_n \cap \sigma') = |p|(\sigma \cap \sigma')$$

for every Borel set σ'.

Proof. We define $q = |p|$ and $\beta = X - \sigma$. Since q is regular, there exists an increasing sequence of closed sets $\gamma_n \subset \beta$ such that $q(\gamma_n) \to q(\beta)$. On the other hand, $\sigma = \bigcap \sigma_a$, where σ_a ranges over the totality of all peak sets including σ. We will prove that there is a σ_a such that $\gamma_n \cap \sigma_a = \emptyset$ for a given γ_n. Indeed, if it were not true, then the family of closed sets $\beta_a = \gamma_n \cap \sigma_a$ should have the finite intersection property (the finite intersections $\sigma_{a_1} \cap \sigma_{a_2} \ldots \sigma_{a_k}$ are peak sets including σ). The compactness of X yields then that $\bigcap (\gamma_n \cap \sigma_a) = \gamma_n \cap \sigma \neq \emptyset$ which is impossible because $\gamma_n \subset X - \sigma$. It follows now that there exists a sequence $\sigma_n \in P(A)$, $\sigma \subset \sigma_n$, such that $\gamma_n \cap \sigma_n = \emptyset$. Now $q(X - \gamma_n) \to q(\sigma)$. Since $\sigma \subset \sigma_n \subset X - \gamma_n$, we get that

$$q(\sigma \cap \sigma') = q(\sigma \cap \sigma_n \cap \sigma') \leqslant q(\sigma_n \cap \sigma')$$
$$\leqslant q\big((X - \gamma_n) \cap \sigma'\big) \to q(\sigma \cap \sigma')$$

as was to be proved.

We are now able to prove that the following (see [4]) property holds true:

(1.2) *If $\sigma \in R(A)$ and $p \perp A$, then $p_\sigma \perp A$*[1].

Proof. Since, for $v \in A$, $\smallint v u^n dp = 0$ for u peaking on σ_a, we have $\int_{\sigma_a} v dp = 0$. Using Lemma 1.1, we get that

$$\smallint v dp_\sigma = \int_{\sigma_a} v dp = \lim \int_{\sigma_{a_n}} v dp = 0$$

for a suitable sequence $\sigma_{a_n} \in P(A)$, q.e.d.

As shown by Glicksberg in [4] (see also [3]) the sets in $R(A)$ are the only closed sets σ such that $p_\sigma \perp A$ for every $p \perp A$. It follows that (see [3], [4]) a union of any two sets in $R(A)$ belongs again to $R(A)$. Consequently, $R(A)$ is a set lattice under forming intersections and unions (in finite number) of sets. In what follows $R(A)$ stands for this lattice.

We come back to representations. Let $T\colon A \to L(H)$ be a representation and let $\{p(f, g)\}$ $(f, g \in H)$ be the system of its elementary measures. We take $p = p(f, g)$ in Lemma 1.1 and the sequence $\sigma_n \in P(A)$ corresponding to $\sigma \in R(A)$. Then

$$\big(P(\sigma_n)f, g\big) = \smallint \chi_{\sigma_n} dp(f, g) \to \smallint \chi_\sigma dp(f, g).$$

The limit $\lim(P(\sigma_n)f, g)$ does not depend on the choice of the sequence σ_n as well on the choice of elementary measures. Indeed, if $p'(f, g)$ is some other elementary measure for f and g and σ_n' is the corresponding sequence for σ, then by (1.2), since $p(f, g) - p'(f, g) \perp A$, we have necessarily $p_\sigma(f, g) - p_\sigma'(f, g) \perp A$, which proves that

$$\lim\big(P(\sigma_n)f, g\big) = \smallint \chi_\sigma dp(f, g) = \smallint \chi_\sigma dp'(f, g) = \lim\big(P(\sigma_n')f, g\big)$$

as was to be proved. Write now

$$\xi(\sigma\colon f, g) = \smallint \chi_\sigma dp(f, g) \quad \text{for } \sigma \in R(A).$$

[1] $p \perp A$ means that the measure p is orthogonal to A, that is $\smallint u dp = 0$ for $u \in A$; the measure p_σ is defined by $p_\sigma(\gamma) = p(\sigma \cap \gamma)$.

$\xi(\sigma: f, g)$ is a well defined functional. Since clearly $p(f+g, h)-p(f, h)-p(g, h) \perp A$, by (1.2) we get that $p_\sigma(f+g, h)-p_\sigma(f, h)-p_\sigma(g, h) \perp A$ which implies that

$$\xi(\sigma: f+g, h) = \xi(\sigma: f, h)+\xi(\sigma: g, h).$$

Using similar arguments one easily verifies that $\xi(\sigma: f, g)$ is a bilinear form in f and g for a fixed σ. Since $|\xi(\sigma: f, g)| \leqslant M|f|\,|g|$, there is a unique $R(\sigma) \epsilon L(H)$ such that $\xi(\sigma: f, g) = \big(R(\sigma)f, g\big)$ for all $f, g \epsilon H$. It is clear that $\big(R(\sigma)f, g\big) = \int \chi_\sigma dp\,(fg)$ for every elementary measure $p(f, g)$. It follows that $R(\sigma) = P(\sigma)$ if $\sigma \epsilon P(A)$.

Consider the finite sequences $f_i, g_i \epsilon H$ $(i = 1, \ldots, k)$. Applying Lemma 1.1 to the measure $p = \Sigma\,|p(f_i, g_i)|$ one easily gets that the following property holds true:

(1.3) For every $\sigma \epsilon R(A)$ there is a sequence $\sigma_n \epsilon P(A)$ such that $\lim\big(P(\sigma_n)f_i, g_i\big) = \big(R(\sigma)f_i, g_i\big)$ for $i = 1, \ldots, k$, i.e. $R(\sigma)$ belongs to the weak operator closure of $\hat{P}(A)$.

Since $\hat{P}(A)$ is included in the commutant of $T(A)$, (1.3) yields that the same property shares the set $\hat{R}(A) = \{R(\sigma): \sigma \epsilon R(A)\}$. Moreover,

$$\big(T(u)R(\sigma)f, g\big) = \int \chi_\sigma u dp\,(f, g)$$

for $u \epsilon A$ and arbitrary elementary measure $p(f, g)$. Indeed, by (1.2),

(1.4) $\big(T(u)R(\sigma)f, g\big) = \big(R(\sigma)T(u)f, g\big) = \lim\big(P(\sigma_n)T(u)f, g\big) = \int \chi_\sigma u dp\,(f, g)$, where $\sigma_n \epsilon P(A)$ is chosen in such way that $|p(f, g)|(\sigma_n-\sigma) \to 0$.

It follows from (1.4) that, for $\sigma, \beta \epsilon R(A)$,

$$\big(T(u)R(\sigma \cap \beta)f, g\big) = \int u\chi_{\beta\cap\sigma} dp\,(f, g) = \int u\, dp\big(R(\beta \cap \sigma)f, g\big)$$

for $u \epsilon A$. It results that

$$p_{\sigma\cap\beta}(f, g)-p\big(R(\sigma \cap \beta)f, g\big) \perp A,$$

which by (1.2) implies that $p_{\sigma\cap\beta}(f, g)-p_\sigma\big(R(\sigma \cap \beta)f, g\big) \perp 1$ and, consequently,

$$\big(R(\sigma \cap \beta)f, g\big) = \int \chi_{\sigma\cap\beta} dp\,(f, g) = \int \chi_\sigma dp\big(R(\sigma \cap \beta)f, g\big) = \big(R(\sigma)R(\sigma \cap \beta)f, g\big).$$

It follows now that

(1.5) $$R(\sigma \cap \beta) = R(\sigma)R(\sigma \cap \beta).$$

Now,

$$\big(T(u)R(\sigma)R(\beta)f, g\big) = \int u\chi_\sigma dp\big(R(\beta)f, g\big) = \int u\chi_\beta dp\big(f, R(\sigma)^* g\big),$$

which by (1.2) proves that $p_\sigma\big(R(\beta)f, g\big)-p_{\sigma\cap\beta}\big(f, R(\sigma)^* g\big) \perp 1$, i.e.

$$\big(R(\sigma)R(\beta)f, g\big) = \big(R(\sigma \cap \beta)f, R(\sigma)^* g\big).$$

Hence $R(\sigma)R(\beta) = R(\sigma)R(\sigma \cap \beta)$, which together with (1.5) proves that

(1.6) $$R(\sigma \cap \beta) = R(\sigma)R(\beta).$$

It follows now that operators in $\hat{R}(A)$ are pairwise commuting projections. We may conclude that $\hat{R}(A)$ becomes a lattice under the operations $\vee$ and $\wedge$. By (1.6)

$$\big(R(\sigma \cup \beta)f, g\big) = \int \chi_{\sigma\cup\beta} dp\,(f, g) = \Big(\big(R(\sigma)+R(\beta)-R(\sigma)R(\beta)\big)f, g\Big)$$

for arbitrary $f, g \epsilon H$. We deduce therefore that

(1.7) $$R(\sigma \cup \beta) = R(\sigma)+R(\beta)-R(\sigma)R(\beta).$$

Summing up we get the following theorem:

THEOREM 1.1. *There is a homomorphism of the lattice $R(A)$ onto $\hat{R}(A)$ such that $R(\sigma) = P(\sigma)$ for $\sigma \epsilon P(A)$. The projections $R(\sigma)$ commute with $T(A)$ and belong to the weak operator closure of $T(S)$.*

If T is contractive, then obviously $R(\sigma)$ are orthogonal projections and by (1.3) we may infer that $R(\sigma) = \inf P(\sigma_a)$, where σ_a ranges over the totality of all peak sets including σ.

2. It follows from Theorem 1.1 that every $\sigma \epsilon R(A)$ induces the direct decomposition of H of the form $H = H_\sigma + H_{X-\sigma}$, where $H_\sigma = R(\sigma)H$ and $H_{X-\sigma} = \big(I-R(\sigma)\big)H$. The restrictions T_σ and $T_{X-\sigma}$ of T to H_σ and $H_{X-\sigma}$ respectively are representations of A because $R(\sigma)$ commutes with $T(A)$. Moreover,

$$\big|\big(T(u)f, g\big)\big| = \Big|\int_\gamma u dp\,(f, g)\Big| \leqslant M|f|\,|g|$$

if $f \epsilon H_\gamma (\gamma = \sigma$ or $\gamma = X-\sigma)$, which proves that both T_σ and $T_{X-\sigma}$ have the same bound equal to M. Now the following definition is in order:

The representation T is *of type* $Z(\beta)$ (β is an arbitrary Borel subset of X) if it has a system of elementary measures vanishing outside of β.

We conclude from (1.4) that if $p(f, g)$ are elementary measures of T and $\sigma \epsilon R(A)$, then $p_\sigma(f, g)$ are elementary measures for $T_\sigma, f, g \epsilon H_\sigma$ and T_σ is in class $Z(\sigma)$. We formulate the following summarizing theorem:

THEOREM 2.1. *Suppose that $T\colon A \to L(H)$ is a representation with the bound M. Let $\sigma \epsilon R(A)$. Then T is a direct sum $T = T_\sigma + T_{X-\sigma}$ of representations with the bound M and such that $T_\sigma \epsilon Z(\sigma)$ and $T_{X-\sigma} \epsilon Z(X-\sigma)$.*

Notice that if $X = \bigcup_{i|1}^{s} \sigma_i$, where $\sigma_i \in R(A)$ are disjoint, then T is a direct sum

(2.1) $$T = T_{\sigma_1} + T_{\sigma_2} + \ldots + T_{\sigma_s},$$

where $T_{\sigma_i} \in Z(\sigma_i)$ and simply $T_{\sigma_i} = T_{|H_{\sigma_i}}$. If for instance $X =$ the maximal ideals space of A and $X = \bigcup_{i|1}^{s} \sigma_i$, where σ_i are closed and open in X, then by the theorem of Shilov $\chi_{\sigma_i} \in A$ peaks on σ_i. Consequently, $R(\sigma_i) = P(\sigma_i)$, $i = 1, \ldots, s$, and (2.1) holds true in this case. If $X = \bigcup_{i|1}^{\infty} \sigma_i$, where $\sigma_i \in R(A)$, then $T = \sum_{i|1}^{\infty} T_{\sigma_i}$ in the weak operator topology.

It is a simple matter to verify that if $T \in Z(\beta)$, then T reduces in fact to the representation of $A_{|\beta} = \{u_{|\beta}: u \in A\}$ and $|T(u)| \leqslant M\|u\|_\beta$, where $\|u\|_\beta = \sup_\beta |u|$.

If T is contractive, then $H_\sigma = R(\sigma)H$ reduces T and the decomposition of Theorem 2.1 becomes an orthogonal one. To be more precise, the following theorem holds true:

THEOREM 2.2 *Suppose that T: $A \to L(H)$ is a contractive representation of A. Let $\sigma \in R(A)$. Then T is the unique orthogonal sum $T = T_\sigma \oplus T_{X-\sigma}$ of contractive representations such that $T_\sigma \in Z(\sigma)$ and $T_{X-\sigma} \in Z(X-\sigma)$.*

It is obvious that only the uniqueness part of the assertion of Theorem 2.2 requires the proof and this one is almost the same as given in Theorem 2.3 below.

Suppose now that $X = \bigcup_{i|1}^{s} \sigma_i$, where $s \leqslant +\infty$ and $\sigma_i \in R(A)$ are pairwise disjoint sets. If the representation T is contractive, then by Theorem 1.1 the subspaces $G_{\sigma_i} = R(\sigma_i)H$ are pairwise orthogonal. Moreover, $I = \bigoplus_{i|1}^{s} R(\sigma_i)$ in this case. To prove this suppose $f \perp R(\sigma_i)H$ for all i. Then

$$(f, f) = \int_X dp(f, f) = \sum_{i|1}^{s} \int \chi_{\sigma_i} dp(f, f) = \sum_{i|1}^{s} \big(R(\sigma_i)f, f\big) = 0,$$

q.e.d. To this end we get the following theorem:

THEOREM 2.3. *Suppose that T: $A \to L(H)$ is a contractive representation. Assume that $X = \bigcup_{i|1}^{s} \sigma_i$ $(s \leqslant +\infty)$, where $\sigma_i \in R(A)$ are pairwise disjoint. Then T may be written uniquelly as the orthogonal sum of contractive representations $T = \bigoplus_{i|1}^{s} T_i$ such that $T_i \in Z(\sigma_i)$. In fact $T_i = T_{\sigma_i}$.*

Proof. All we have to show is that the conditions determine the decomposition in a unique way. Suppose just that $T = \oplus T_i'$, where $T_i' \in Z(\sigma_i)$ with H_i' being the representation space of T_i. Let $f = \Sigma f_i$, $g = \Sigma g_i$, $f_i, g_i \in H_i'$. Then $|f|^2 = \Sigma |f_i|^2$, $|g|^2 = \Sigma |g_i|^2$. Let p_i be elementary measures for T_i' vanishing outside of σ_i, such that $\|p_i(f_i, g_i)\| \leqslant |f_i|\,|g_i|$. Then the measure $p(f, g) = \sum_i p_i(f_i, g_i)$ is a well defined elementary measure for T, f and g. Indeed, $\|p_i(f_i, g_i)\| \leqslant \frac{1}{2}(|f|^2 + |g_i|^2)$ which ensures the convergence of the series defining $p(f, g)$ and yields, on the other hand, that

$$\int u dp(f, g) = \Sigma\big(T_i'(u)f_i, g_i\big) = \big(T(u)f, g\big)$$

for all $u \in A$. It follows that $\big(T_k'(u)f_k, g_k\big) = \int u\chi_{\sigma_k} dp(f, g)$ which proves that $(f_k, g) = (f_k, g_k) = \big(R(\sigma_k)f, g\big)$. Since g is arbitrary, we may conclude that $f_k = R(\sigma_k)f$. We have proved that $H_k' \subset R(\sigma_k)H$ for every k. Since $H = \oplus H_k'$, we get $H_k' = R(\sigma_k)H$ as was to be proved.

Let us now consider a special case, namely assume that the single point set $\{x\}$ $(x \in X)$ is in $R(A)$, that is x belongs to the Choquet boundary of A. We have for arbitrary $g \in H$

$$\big((T(u) - u(x)I)f, g\big) = \big(R(\{x\})(T(u) - u(x)I)f, g\big)$$
$$= \int \big(u - u(x)\big)\chi_{\{x\}} dp(f, g) = 0$$

if $f = R(\{x\})f$. Hence $T(u)f = u(x)f$ for such f. Conversely, if $T(u)f = u(x)f$ for every $u \in A$, then $f = R(\{x\})f$. Indeed, for arbitrary $g \in H$ we then have for every u peaking on σ which includes x, $\big(T(u^n)f, g\big) = (f, g)$ which proves that $P(\sigma)f = f$. Since $R(\{x\})$ is the weak operator limit of such $P(\sigma)$, we necessarily have $R(\{x\})f = f$, q.e.d. We have proved that

(2.2) $$R(\{x\})H = \{f: T(u)f = u(x)f \text{ for all } u \in A\}.$$

If T is contractive, then $H_{\{x\}} = R(\{x\})H$ reduces T and the adjoint of the part $T_{\{x\}}(u)$ equals to the part in $H_{\{x\}}$ of the adjoint $T(u)^*$. It follows that $T_{\{x\}}(u)^*$ is the operator of multiplication by $\overline{u(x)}$ which implies that

(2.3) $$\{f: T(u)f = u(x)f \text{ for } u \in A\} = \{f: T(u)^*f = \overline{u(x)}f \text{ for } u \in A\}.$$

Summing up we get the following theorem:

THEOREM 2.4. *Suppose $x \in X$ and $\{x\} \in R(A)$. Then* (2.2) *holds true. If T is contractive, then it satisfies* (2.3).

This theorem generalizes the results of Sz.-Nagy and Foiaş [8] and of Lebow [5], Th. 3, p. 74, which concerned representations associated with von Neumann spectral sets of operators. In contrary to the results of those authors, our theorem does not require that x is a peak point as well that the representation has an X-dilation(2) which is the case for representations induced by suitable spectral sets. Extending the results of [5] and of [8] we get for contractive T:

(a) If $\{x\} \in R(A)$, then, for every $u \in A$, $u(x)$ is not in the residual spectrum of $T(u)$.

(b) If $\{x\}, \{y\} \in R(A)$ and $x \neq y$, then $R(\{x\})H \perp R(\{y\})H$.

3. This is the result of Bishop [1] completed by Glicksberg in [4] that if A is the subalgebra of $C(X)$, then

$$X = \bigcup \sigma_a, \tag{3.0}$$

where σ_a are pairwise disjoint closed sets such that the following properties hold true:

(3.1) Every σ_a is a maximal set of antisymmetry of A.

(3.2) If $u \in C(X)$ and $u_{|\sigma_a} \in A_{|\sigma_a}$ for every a, then $u \in A$.

(3.3) $A_{|\sigma_a}$ is closed in $C(\sigma_a)$ for every a.

We shall call (3.0) the *Bishop decomposition* of X (relative to A). Glicksberg proved in [4] that

(3.4) Every σ_a belongs to $R(A)$.

Throughout the present section we assume that T is a contractive representation of A. If (3.0) is the Bishop decomposition of X, then (3.4) together with our previous results implies that, for every a, $H_a = R(\sigma_a)H$ reduces T, the part T_a of T in H_a belongs to $Z(\sigma_a)$ and is in fact a representation of the antisymmetric algebra $A_{|\sigma_a}$. By Theorem 1.1, $H_a \perp H_\beta$ if $a \neq \beta$ because $\sigma_a \cap \sigma_\beta = 0$. Let us now define $H'' = \oplus H_a$, $H' = H \ominus H''$. $T' =$ the part of T in H', $T'' =$ the part of T in H''. To this end we have the following theorem:

THEOREM 3.1. *Let* (3.0) *be the Bishop decomposition of X relative to A and let $T\colon A \to L(H)$ be a contractive representation of A. Then T is a unique orthogonal sum*

$$T = T' \oplus T'' = T' \oplus (\oplus T_a), \tag{3.5}$$

where $T_a \in Z(\sigma_a)$ are representations of A (in fact T_a is isomorphic to a representation of $A_{|\sigma_a}$) and T' is a representation of A such that every its elementary measure vanishes on σ_a for each a.

Proof. The only thing we have to prove is the uniqueness of the decomposition. Suppose just that

$$T = \hat{T} \oplus (\oplus \hat{T}_a), \tag{3.6}$$

(2) See [2] and [6] for definition.

where $\hat{T}_a \in Z(\sigma_a)$ and $\hat{T}$ has the property that its elementary measures vanish on σ_a for every a. Let $H = \hat{H} \oplus (\oplus \hat{H}_a)$ be the decomposition corresponding to (3.6). $\hat{T}(\hat{T}_a)$ is the part of T in $\hat{H}(\hat{H}_a)$. Suppose now that $f \in \hat{H}_a$. Since $\hat{T}_a \in Z(\sigma_a)$, $(f, f) = \int dp(f, f)$, where $p(f, f)$ is an elementary measure vanishing outside of σ_a, i.e. $p(f, f) = p_{\sigma_a}(f, f)$. It follows that $(f, f) = (R(\sigma_a)f, f)$ and, consequently, $f = R(\sigma_a)f$. We just proved that $\hat{H}_a \subset H_a$. Now, if $f \in H_a$ and $f \perp \hat{H}_a$, then $f \in (\bigoplus_{\beta \neq a} \hat{H}_\beta) \oplus \hat{H}$. It follows that $f = \sum_{n|0}^{\infty} f_n$, where $f_0 \in \hat{H}$ and $f_n \in \hat{H}_{\beta n}$ for $\beta_n \neq a$. We conclude that $(f, f) = \sum_{n|0}^{\infty} (f_n, f_n)$, that is, by non-negativity of measures involved

$$p(\gamma\colon f, f) = \sum_{n|0}^{\infty} p(\gamma\colon f_n, f_n)$$

for every Borel set γ. If $\gamma = \sigma_a$, then, since $f \in H_a$, the left-hand side is equal to (f, f) and the right-hand side vanishes because $\hat{T}_{\beta n} \in Z(\sigma_{\beta n})$ and $\sigma_{\beta n} \cap \sigma_a = \emptyset$ and $p(f_0, f_0)$ vanishes on σ_a. This proves that $\hat{H}_a = H_a$, which completes the proof.

If decomposition (3.0) contains at most a countable number of σ_a, then by Theorem 2.3 the part T' vanishes. Notice that it may happen that the whole representation T reduces to T'. Indeed, let X be the unit circle, $A = C(X)$ and $H = L^2(m)$, where m is the Lebesgue measure on X. Then every σ_a reduces to a single point set which implies that for the representation defined by $T(u)f = uf$ every T_a is trivial. The example below shows that the part T of (3.5) need not be in general an X-representation.

Example. Let Γ be the unit circle on the complex plane. We define $X = \Gamma \times \Gamma$ and A as the tensor product $C(\Gamma) \oplus A(\Gamma)$, where $A(\Gamma)$ is the algebra of functions continuous on Γ and having analytic extensions to the open unit disc. A may be identified with the uniform closure in $C(\Gamma \times \Gamma)$ of functions of the form

$$f(y, z) = \sum_{i|1}^{s} g_i(y) h_i(z) \qquad ((y, z) \in \Gamma \times \Gamma), \tag{3.7}$$

where $g_i \in C(\Gamma)$ and $h_i \in A(\Gamma)$. It follows from the theorem of Mochizuki [7] that the maximal sets of antisymmetry of A are of the form $\sigma_a = \{y_a\} \times \times \Gamma$, where y_a ranges over the totality of all points of Γ. We define H as the $L^2(m \times m)$ closure of functions of the form (3.7). Notice that the function $h(y, z) = \bar{z}$ is orthogonal to H that is

$$\bar{z} \perp H. \tag{3.8}$$

It follows from the definition of H that it is invariant under multiplication by functions of A. Hence, the formula $T(u)f = uf$, $u \in A$ and $f \in H$, determines a well defined contractive representation of A. It is a simple matter to check that $p(\gamma\colon f, g) = \int_\gamma f\bar{g}\, dm \times m$ are elementary measures of this representations. They evidently vanish on σ_a for every a which proves that $T = T'$. But T is not an X-representation. This means that T is not a restriction to A of a $*$-representation of $C(\Gamma \times \Gamma)$. Indeed, if it were true, then every $T(u)$ should be a normal operator. Since $T(u_1)$ is obviously an isometry for $u_1 \equiv z$, the normality would imply that $T(u_1)$ is unitary. Consequently, $u_0 = 1 \in T(u_1)H$, i.e. $1 = zf$ for some $f \in H$. It follows that $\bar{z} = f \in H$ which is impossible by (3.8), the desired contradiction.

We say that a representation is *irreducible* if $\{0\}$ and H are the only subspaces reducing simultaneously all $T(u)$. The representation is *reducible* if it is not irreducible. It follows that every irreducible T must coincide with exactly one component of (3.5). The following theorem gives a sufficient condition for some representations to be of type $Z(\sigma_a)$:

THEOREM 3.2. *Suppose that p is a non-negative regular Borel measure on X. Let H be the closed linear span of A in $L^2(p)$ and let $T\colon A \to L(H)$ be the contractive representation defined by $T(u)f = uf$ $(u \in A, f \in H)$. Then, if T is irreducible, the closed carrier of p is a set of antisymmetry of A and, consequently, $T = T_a$ for some a of decomposition* (3.5).

Proof. Suppose that $u_0 \in A$ is real on the closed carrier γ of p. Since $\big(T(u_0)f, f\big) = \int u_0 |f|^2 dp$ for $f \in H$, it follows that $T(u_0)$ is a selfadjoint operator commuting with $T(A)$. But T is irreducible. Consequently, $T(u_0) = \lambda I$ for some real constant λ. Hence, for $f = 1$ we have

$$0 = |\big(T(u_0) - \lambda I\big)f|^2 = \int |u_0 - \lambda|^2 dp,$$

which proves that $u_0 = \lambda$ on γ, q.e.d.

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