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A general separation theorem for mappings, saddle-points, duality and conjugate functions

by

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This paper concerns the following groups of problems:

(a) Separation of non-linear functions and general theory of inequalities.

(b) Existence of Lagrangian saddle-points in the theory of convex programming in linear spaces.

(c) Duality for mathematical programming in linear and linear topological spaces.

(d) The theory of conjugate functions in linear and linear topological spaces.

Ad (a). In 1952 Mazur and Orlicz ([8], p. 147) proved a very important and general theorem on inequalities. Several important applications to various problems are also given there (e.g. extensions of linear functionals, separation of convex sets, scalar inequalities). In 1962 Mil'man [9] proved a very interesting and essential generalization of the Mazur-Orlicz theorem. It covers also the monotone extension theorem ([2], p. 20) and applies also to infinite systems of scalar inequalities in linear spaces. In 1968 [1], Appendix, we presented another generalization of the Mazur-Orlicz theorem on inequalities. However, our generalization is not contained in the Mil'man theorem. The reason for is that the Mazur-Orlicz argument as well as that of Mil'man involves necessity for a sublinear functional to be separated, whereas in our argument this sublinear functional can be replaced by a convex one.

Ad (b). Hurwicz ([5], p. 91) and Hurwicz-Uzawa [6] have proved a general theorem on the existence of Lagrangian saddle-points for general concave programming in linear topological spaces. The case considered there is very general, so that the objective function is a concave mapping into a linear topological space with an ordering relation defined by a convex cone. Thus, the case of an objective functional is obviously covered.

Ad (c). A short time ago Gol'štein ([4], p. 16) proved the most general duality theorem for convex programming. Later on Joffe and Tihomirov ([7], 77) proved a generalization of Gol'štein's theorem (without the convexity assumption) by using the Fenchel-Moreau theory of conjugate functions for a pair of dual spaces. However, in both cases the objective function is a functional. Thus, the programming problem considered in this duality theorem has not yet achieved its full generality as in the Lagrangian saddle-point theorem [6], where the objective function is of a more general character than a functional.

Ad (d). The theory of conjugate functions has been developed by Fenchel in the finite-dimensional spaces and by Moreau in the case of a pair of dual spaces. References are in the paper of Joffe-Tihomirov [7]. This paper contains an interesting approach to various extremal problems which is based on the theory of conjugate functions mentioned above.

At first sight it is not obvious that there is a possibility of finding a uniform technique designed to handle all these groups of problems from a common point of view. Nevertheless, it is the purpose of this paper to present an unified approach to all the above-mentioned questions. Moreover, we shall show that using this technique one can obtain even significantly more general results concerning the problems in (a), (c) and (d). Now, let us give a brief outline of the results contained in this paper.

In section 3 a general separation theorem for mappings is proved. As a special case of this theorem we obtain Mil'man's [9] theorem on the separation of non-linear functionals as well as our generalization [1] of the Mazur-Orlicz [8] theorem on inequalities.

Section 4 contains a separation theorem for convex mappings which is actually a particular case of the general separation theorem in section 3.

As a special case of the separation theorem for convex mappings we obtain in section 5 the Hurewicz-Uzawa [6] result concerning the existence of Lagrangian saddle-points for the general concave programming problem in linear topological spaces with an ordering relation defined by a convex cone.

Slightly modifying the same technique, we obtain in section 6 a general duality theorem for the convex programming problem, which in a special case gives the duality theorem of Gol'štein ([4], p. 16). Let us remark that the duality theorem presented in section 6 seems to be the first one for the general convex programming problem in which the objective functional is replaced by a convex mapping with values in a space with an ordering relation defined by a convex cone. In all known duality theorems the objective function is only a functional.

Section 7 contains an extension to linear and linear topological spaces of the duality property of convex functions. Moreover, it is shown that this duality property is a special case of a more general property of the Lagrange function. This property is rather of algebraic character in the sense that no topology is needed to prove it. Besides, there is also a proof of a general duality theorem for a general programming problem in a linear topological space. As a special case we obtain the Joffe-Tihomirov ([7], p. 77) duality theorem, which is proved for a pair of dual spaces. Another duality theorem with the existence of Lagrangian multipliers is proved for programming in linear spaces and in linear topological spaces endowed with ordering convex cones having certain properties.

Section 8 concerns mainly convex-concave functions. However, some properties of conjugate functions are also investigated. Thus, the necessary and sufficient conditions for a minimum of a convex function on a linear space and on a linear topological space are given in terms of the subdifferential of the conjugate functions. Further, an extension to linear and linear topological spaces of the Moreau theorem concerning the commutativity of the $\inf_x \sup_u$ of a function $f(x, u)$ is proved. Besides, the theorem of Joffe-Tihomirov which gives the necessary and sufficient conditions for the existence of a saddle-point of a convex-concave function is extended to linear and linear topological spaces. Moreover, two new criteria for the existence of a saddle-point of a convex-concave function are presented. These criteria are discussed for linear spaces, linear topological spaces and locally convex linear topological spaces. Let E, Y, Z be real linear spaces. Suppose that there exist convex $K_Y \subset Y, K_Z \subset Z$. The cone K_Y has the property: $y \in K_Y$ and $-y \in K_Y$ imply $y = 0$ (zero-element). It is not excluded that the cone K_Z can be reduced to the zero-element. In the linear spaces Y, Z an ordering relation is defined by the convex cones K_Y, K_Z, respectively. Thus, $y_1 \geqslant y_2$ (or $y_2 \leqslant y_1$) means $y_1 - y_2 \in K_Y$ and $z_1 \geqslant z_2$ (or $z_2 \leqslant z_1$) means $z_1 - z_2 \in K_Z$. We shall denote by y^*, z^* linear (i.e. additive and homogeneous) functionals on Y, Z, respectively.

1. Separation problem. Given three subsets X_0, X_1, X_2 of E and three pairs of mappings

$$a_i\colon X_i \to Y, \quad b_i\colon X_i \to Z, \quad i = 0, 1, 2.$$

The problem is to find the necessary and sufficient conditions for the existence of a non-trivial pair of linear functionals y_0^* and z_0^* such that

$$(1) \qquad \begin{cases} y_0^*[a_2(x)] + z_0^*[b_2(x)] \leqslant 0 & \text{for all } x \text{ of } X_2, \\ y_0^*[a_0(x)] + z_0^*[b_0(x)] = 0 & \text{for all } x \text{ of } X_0, \\ y_0^*[a_1(x)] + z_0^*[b_1(x)] \geqslant 0 & \text{for all } x \text{ of } X_1. \end{cases}$$

2. Assumptions. Let us assume that the following relations are satisfied:

$$
(2)\quad \begin{cases}
a_1(0) \leqslant 0 & \text{and } b_1(0) \leqslant 0,\ 0 \in X_1,\\
0 \leqslant a_2(0) & \text{and } 0 \leqslant b_2(0),\ 0 \in X_2,\\
a_1(x) \leqslant a_0(x) & \text{and } b_1(x) \leqslant b_0(x) \text{ for } x \text{ in } X_1 \cap X_0,\\
a_1(x) \leqslant a_2(x) & \text{and } b_1(x) \leqslant b_2(x) \text{ for } x \text{ in } X_1 \cap X_2,\\
a_0(x) \leqslant a_2(x) & \text{and } b_0(x) \leqslant b_2(x) \text{ for } x \text{ in } X_0 \cap X_2.
\end{cases}
$$

We assume the central symmetry of the set X_0 with respect to the zero-element of E and suppose that the mappings $a_0(x)$ und $b_0(x)$ are odd, i.e.

$$(3)\quad a_0(-x) = -a_0(x) \text{ and } b_0(-x) = -b_0(x) \quad \text{for } x \text{ in } X_0.$$

Denote by $\tilde{X}_2$ the subset of elements of the set $(-X_1 \cap X_2)\setminus X_0$ that satisfy the following relations:

$$(4)\quad a_2(x)+a_1(-x) \leqslant 0 \quad \text{and} \quad b_2(x)+b_1(-x) \leqslant 0.$$

Denote by $\tilde{X}_1$ the subset of elements such that

$$(5)\quad \tilde{X}_1 \subseteq (-X_1 \cap X_2)\setminus X_0 \setminus \tilde{X}_2$$

with the following property:

$$(6)\quad a_2(x)+a_1(-x) \geqslant 0 \text{ and } b_2(x)+b_1(-x) \geqslant 0 \quad \text{for } x \text{ in } \tilde{X}_1.$$

Now let us define the sets

$$(7)\quad X_1' = \tilde{X}_1' \cup \tilde{X}_0 \quad \text{and} \quad X_2' = \tilde{X}_2' \cup \tilde{X}_0,$$

where $\tilde{X}_0 = (-X_1 \cap X_2)\setminus X_0 \setminus \tilde{X}_2 \setminus \tilde{X}_1$,

$$\tilde{X}_1' = X_1\setminus(X_0 \cup -\tilde{X}_2) \quad \text{and} \quad \tilde{X}_2' = X_2\setminus(X_0 \cup \tilde{X}_1).$$

Thus, the set $\tilde{X}_1'$ is obtained from the set $X_1\setminus X_0$ by removing those elements of the set $X_1 \cap -X_2$ that satisfy relation (4) in which x is replaced by $-x$. It is also obvious that the set $\tilde{X}_2'$ is obtained from the set $X_2\setminus X_0$ by removing those elements of the set $-X_1 \cap X_2$ that satisfy relation (4) or (6), in virtue of (5).

In the product space $Y\times Z$ let us define the set $W \subset Y\times Z$ of elements (u, v), where $u \in Y$, $v \in Z$ and

$$
(8)\quad \begin{aligned}
u &= \sum_j t_j'' a_2(x_j'') - \sum_j t_j^0 a_0(\mathring{x}_j) - \sum_j t_j' a_1(x_j') + y_K,\\
v &= \sum_j t_j'' b_2(x_j'') - \sum_j t_j^0 b_0(\mathring{x}_j) - \sum_j t_j' b_1(x_j') + z_K
\end{aligned}
$$

for all x_j'', $\mathring{x}_j$, x_1', z_K and y_K running over X_2', X_0, X_1', K_Z and $K_Y\setminus\{0\}$, respectively, and for all finite systems of non-negative t_j'', $\mathring{t}_j$ and t_j' such that

$$(9)\quad \sum_j t_j'' \leqslant 1, \quad \sum_j t_j^0 \leqslant 1 \quad \text{and} \quad \sum_j t_j' \leqslant 1.$$

3. The general separation theorem for mappings. The following theorem is basic for all further considerations:

THEOREM 1. *Suppose that the set W has an internal point* (see Appendix). *Then for the existence of linear functionals $y_0^* \leqslant 0$, $z_0^* \leqslant 0$ satisfying relations* (1) *and $(y_0^*, z_0^*) \neq (0, 0)$ it is sufficient that for any arbitrary finite system of non-negative numbers $\{t_j^0, t_j', t_j''\}$ satisfying relation* (9), *elements*

$$\{x_j^0\} \subseteq X_0, \quad \{x_j'\} \subseteq X_1', \quad \{x_j''\} \subseteq X_2'$$

and for arbitrary z_K in K_Z

$$
(10)\quad \begin{cases}
\textit{the equality}\\
\qquad \sum_j t_j^0 b_0(x_j^0) + \sum_j t_j' b_1(x_j') - \sum_j t_j'' b_2(x_j'') = z_K\\
\textit{implies the relation}\\
\qquad \sum_j t_j^0 a_0(x_j^0) + \sum_j t_j' a_1(x_j') - \sum_j t_j'' a_2(x_j'') \notin K_Y\setminus\{0\}.
\end{cases}
$$

Proof. It is easily seen that W is a convex set and it follows from implication (10) that the point $(0, 0)$ is not in W. Since W has an internal point by assumption, it follows from the basic separation theorem (see Appendix) that there exist linear functionals y_0^* on Y, z_0^* on Z and a real number c such that $(y_0^*, z_0^*) \neq (0, 0)$,

$$(11)\quad y_0^*(u)+z_0^*(v) \geqslant c \quad \text{and} \quad y_0^*(0)+z_0^*(0) \geqslant c$$

for all u and v such that $(u, v) \in W$. Hence, we obtain $c \leqslant 0$ and, consequently, one can put in (11) $c = 0$. In particular, putting in (8) $t_j^0 = t_j' = t_j'' = 0$ and $z_k = 0$, we obtain $y_0^* \leqslant 0$. By the same argument it follows from (11) that $z_0^* \leqslant 0$, since, for $t > 0$, $ty_K \in K_Y\setminus\{0\}$ whenever $y_K \in K_Y\setminus\{0\}$. In particular, putting in (8) $t_j^0 = t_j' = 0$ and $z_K = 0$ we obtain from (11)

$$(12)\quad y_0^*[a_2(x)]+z_0^*[b_2(x)] \geqslant 0 \quad \text{for all } x \text{ in } X_2'.$$

Analogously, putting in (8) $t_j^0 = t_j'' = 0$ and $z_K = 0$ we obtain from (11)

$$(13)\quad y_0^*[a_1(x)]+z_0^*[b_1(x)] \geqslant 0 \quad \text{for all } x \text{ in } X_1'.$$

Similarly, putting in (8) $t_j' = t_j'' = 0$ and $z_K = 0$ we obtain from (11)

$$y_0^*[a_0(x)]+z_0^*[b_0(x)] \leqslant 0 \quad \text{for all } x \text{ in } X_0.$$

Hence, replacing x by $-x$ in the last inequality we infer from assumption (3) that

(14) $$y_0^*[a_0(x)]+z_0^*[b_0(x)] = 0 \quad \text{for all } x \text{ in } X_0.$$

We shall show that relations (12), (13) hold when X_2', X_1' are replaced by X_2, X_1, respectively. In virtue of definition (7), if x is in X_1 but not in X_1', then x is not in $\tilde{X}_1'$ and we can distinguish two cases: either (a) x is in $X_1 \cap X_0$ or (b) x is in $X_1 \setminus X_0$ and x is in $-\tilde{X}_2$. In case (a) we have, by (14),

$$0 = y_0^*[a_0(x)]+z_0^*[b_0(x)] \leqslant y_0^*[a_1(x)]+z_0^*[b_1(x)],$$

since $y_0^* \leqslant 0$, $z_0^* \leqslant 0$ and $x \in X_1 \cap X_0$ implies $a_1(x) \leqslant a_0(x)$ and $b_1(x) \leqslant b_0(x)$, by relations (2). In case (b) $-x$ is in $\tilde{X}_2 \subseteq \tilde{X}_2'$ and $-x \in X_2'$, by (7). Hence, it follows in virtue of (12) that

(15) $$0 \geqslant y_0^*[a_2(-x)]+z_0^*[b_2(-x)] \geqslant -y_0^*[a_1(x)]-z_0^*[b_1(x)],$$

since $y_0^* \leqslant 0$, $z_0^* \leqslant 0$, and replacing in relations (4) x by $-x$, we have

$$a_2(-x) \leqslant -a_1(x) \quad \text{and} \quad b_2(-x) \leqslant -b_1(x).$$

Thus, relation (13) holds for all x in X_1, in virtue of relations (15).

Suppose now that x is in X_2. In virtue of definition (7), if x is not in X_2' one can distinguish two cases: either (i) x is in $X_0 \cap X_2$ or (ii) x is in $X_2 \setminus X_0$ and x is in $\tilde{X}_1$. In case (i) we have, in virtue of (14)

$$0 = y_0^*[a_0(x)]+z_0^*[b_0(x)] \geqslant y_0^*[a_1(x)]+z_0^*[b_1(x)],$$

since $y_0^* \leqslant 0$, $z_0^* \leqslant 0$ and for x in $X_0 \cap X_2$ we have, in virtue of relations (2)

$$a_0(x) \leqslant a_2(x) \quad \text{and} \quad b_0(x) \leqslant b_2(x).$$

In case (ii) x is in $\tilde{X}_1$. It follows from definitions (5), (7) that $-\tilde{X}_1 \subseteq \tilde{X}_1'$, since $\tilde{X}_2 \cap \tilde{X}_1 = \emptyset$, the empty set. Hence $-x$ is in X_1' and we infer from relation (13), where x is replaced by $-x$, that

(16) $$0 \leqslant y_0^*[a_1(-x)]+z_0^*[b_1(-x)] \leqslant -y_0^*[a_2(x)]-z_0^*[b_2(x)],$$

since $y_0^* \leqslant 0$, $z_0^* \leqslant 0$ and in virtue of (6) we get

$$a_1(-x) \geqslant -a_2(x) \text{ and } b_1(-x) \geqslant -b_2(x) \quad \text{for } x \text{ in } \tilde{X}_1.$$

Thus we infer from relation (16) that relation (12) is true for all x in X_2. This completes the proof of the theorem.

Remark 1. Let us suppose that the following conditions are satisfied: b_0, b_1 and b_2 denote the identity mapping of the linear space $E = Z$; the cone K_Z consists of the zero-element only; the set $\mathring{K}_Y$ of internal

points of the cone K_Y is not empty. Under these hypotheses condition (10) of Theorem 1 is necessary and sufficient provided that $K_Y \setminus \{0\}$ is replaced by $\mathring{K}_Y$ and that the smallest wedge (see Appendix) containing the set $-X_1 \cup X_2 \cup X_0$ is a linear subspace of E.

Indeed, suppose that for a finite system of non-negative numbers $\{t_j^0, t_j', t_j''\}$ and elements $\{x_j^0\} \subseteq X_0$, $\{x_j'\} \subseteq X_1'$ and $\{x_j''\} \subseteq X_2'$ we have

$$\sum_j t_j^0 x_j^0 + \sum_j t_j' x_j' = \sum_j t_j'' x_j''.$$

Hence, we obtain

$$\sum_j t_j^0 z_0^*(x_j^0) + \sum_j t_j' z_0^*(x_j') = \sum_j t_j'' z_0^*(x_j'')$$

and it follows from (1) that

(17) $$-\sum_j t_j^0 y_0^*[a_0(x_j^0)] - \sum_j t_j' y_0^*[a_1(x_j')] \leqslant -\sum_j t_j'' y_0^*[a_2(x_j'')].$$

Suppose that

$$\sum_j t_j^0 a_0(x_j^0) + \sum_j t_j' a_1(x_j') - \sum_j t_j'' a_2(x_j'') \in \mathring{K}_Y.$$

Since $y_0^* \leqslant 0$, it follows from (17) that y_0^* vanishes at an internal point. Thus, $y_0^* \equiv 0$. It follows from relations (1) that the linear functional z_0^* assumes non-positive values on the smallest wedge containing the set $-X_1 \cup X_2 \cup X_0$. Since this wedge is a linear subspace of E by assumption, it follows that the linear functional z_0^* vanishes on X_i $(i = 0, 1, 2)$, in contradiction to our assumption. In order to prove the sufficiency of condition (10) modified above, one must replace the sets $K_Y \setminus \{0\}$ and K_Z, involved in the construction of the convex set W, by the sets $\mathring{K}_Y$ and $\{0\}$, respectively.

Let us observe that under the hypotheses in Remark 1 it follows that the linear functional y_0^* is not identically zero.

Remark 2. It follows from Remark 1 that in this particular case where the linear space Y is the space of all real numbers we obtain Mil'man's [9] theorem on separators of non-linear functionals as a corollary to the theorem contained in Remark 1. In this case the linear functional y_0^* yields a negative constant number and the set $\mathring{K}_Y$ is the set of all positive numbers. It is easily seen that condition (6) yields in this case $a_2(x)+a_1(-x) > 0$ for x in $\tilde{X}_1$ and follows from the first inequality (4), i.e. $a_2(x)+a_1(-x) \leqslant 0$ for x in $\tilde{X}_2$. We have also $b_2(x)+b_1(-x) = x-x = 0$ and $\tilde{X}_0$ is empty.

Remark 3. Denote by V the convex set of elements

$$v = \sum_j t_j'' b_2(x_j'') - \sum_j t_j^0 b_0(x_j^0) - \sum_j t_j' b_1(x_j') \quad \text{for } x_j'', x_j^0, x_j'$$

running over X_2', X_0, X_1', respectively, and $t_j'' \geqslant 0, t_j^0 \geqslant 0, t_j' \geqslant 0$ with restriction (9). If for every (non-trivial) linear functional $z^* \geqslant 0$ there exists in V an element v depending on z^* such that

$$(18) \qquad z^*(v) < 0, \quad v \in V,$$

then the linear functional y_0^* in Theorem 1 is non-trivial, i.e. $y_0^* \neq 0$.

Indeed, if $y_0^* = 0$ in Theorem 1, then it follows from (1) that $z_0^*(v) \leqslant 0$ for all v in V. But in virtue of (18) we have for $z^* = -z_0^*$ that $-z_0^*(v_0) < 0$ for an element v_0 in V. Thus, assuming $y_0^* = 0$, we obtain a contradiction.

Remark 4. If Y and Z are linear topological spaces, then it is natural that the linear functionals y_0^* and z_0^* are supposed to be continuous. For this purpose it is sufficient to postulate the existence of an interior point in the convex set W instead of an internal point. In particular, it is sufficient to postulate that the cones K_Y and K_Z have non-empty interiors. In the case where these cones have empty interiors it is sufficient to assume that

(a) W has an internal point and

(b) Y and Z are complete metric linear spaces with closed convex cones K_Y and K_Z such that $K_Y - K_Y$ and $K_Z - K_Z$ have non-empty interiors.

Assumption (b) guarantees the continuity of y_0^* and z_0^*. This statement follows from a theorem of Klee (see [6], p. 104).

Remark 5. In the case considered in Remark 1 it is assumed that b_0, b_1, b_2 denote the identity mapping of $E = Z$ and $K_Z = \{0\}$. Besides, the smallest wedge containing the set $-X_1 \cup X_2 \cup X_0$ is the linear space E. Suppose in addition that Y and E are linear topological spaces. Thus, if W has an interior point or, in particular, the cone K_Y has an interior point, then the linear functionals y_0^* and z_0^* are continuous.

4. The separation theorem for convex mappings. We shall now discuss the case where one of the mappings a_2, b_2 or both of them are convex.

THEOREM 2. *Suppose that in addition to the assumptions of Theorem* 1 *the following conditions are satisfied: the subset X_2 of E is convex and the mappings*

$$a_2\colon X_2 \to Y \quad \text{and} \quad b_2\colon X_2 \to Z$$

are convex. Then Theorem 1 *is true if assumption* (10) *is replaced by the following one: for arbitrary non-negative numbers $\{t_j^0, t_j', t_j''\}$ such that $\sum_j t_j'' \leqslant 1$*

(19) *the equality*

$$\sum_j t_j^0 b_0(x_j^0) + \sum_j t_j' b_1(x_j') - b_2\Big(\sum_j t_j'' x_j''\Big) = z_K$$

implies the relation

$$\sum_j t_j^0 a_0(x_j^0) + \sum_j t_j' a_1(x_j') - a_2\Big(\sum_j t_j'' x_j''\Big) \notin K_Y \setminus \{0\}.$$

Proof. We shall show that implication (10) follows from (19). Suppose that

$$(20) \qquad \sum_j t_j^0 b_0(x_j^0) + \sum_j t_j' b_1(x_j') - \sum_j t_j'' b_2(x_j'') = z_K.$$

One can assume that $\sum_j t_j'' = 1$. For if this is not the case, then one can multiply the above equation by the number $(\sum_j t_j'')^{-1}$. We have, by (20),

$$\sum_j t_j^0 b_0(x_j^0) + \sum_j t_j' b_1(x_j') - b_2\Big(\sum_j t_j'' x_j''\Big)$$
$$= \sum_j t_j^0 b_0(x_j^0) + \sum_j t_j' b_1(x_j') - \sum_j t_j'' b_2(x_j'') +$$
$$+ \Big[\sum_j t_j'' b_2(x_j'') - b_2\Big(\sum_j t_j'' x_j''\Big)\Big] = z_K + z_1 \in K_Z,$$

since $z_1 \geqslant 0$ in virtue of the convexity of b_2. Hence, it follows that the second relation of implication (19) is satisfied. On the other hand, we have

$$\sum_j t_j^0 a_0(x_j^0) + \sum_j t_j' a_1(x_j') - \sum_j t_j'' a_2(x_j'')$$
$$= \sum_j t_j^0 a_0(x_j^0) + \sum_j t_j' a_1(x_j') - a_2\Big(\sum_j t_j'' x_j''\Big)$$
$$- \Big[\sum_j t_j'' a_2(x_j'') - a_2\Big(\sum_j t_j'' x_j''\Big)\Big] \notin K_Y \setminus \{0\},$$

in virtue of (19), since the expression enclosed in brackets is $\geqslant 0$, by the convexity of a_2. Thus, the second relation of implication (10) holds.

Remark 6. It is easy to see that if only one of the mappings a_2, b_2 is convex, then in condition (10) only the corresponding expression has to be changed. Let, say, a_2 be convex; then the sum $\sum_j t_j'' a_2(x_j'')$ in (10) must be replaced by the expression $a_2(\sum_j t_j'' x_j'')$. This follows from the proof of Theorem 2.

Let us observe that all the remarks made above are valid also in this case.

We shall now consider a particular case of Theorem 2 where b_1 and b_2 denote the indentity mapping of the linear space $E = Z$; the cone K_Z

consists of the zero-element only; the subset X_0 of E is empty and the set $\mathring{K}_Y$ of internal points of the cone K_Y is not empty.

THEOREM 3. *Given: two subsets X_1 and X_2 of the linear space E and two mappings a_1: $X_1 \to Y$; a_2: $X_2 \to Y$, where X_2 is a convex set and a_2 is a convex mapping. Suppose that 0 is an internal point of the smallest convex set containing the set $-X_1 \cup X_2$ and that the assumptions of section 2 are fulfilled. If condition*

$$(21)\qquad x = \sum_j t_j' x_j' \in X_2 \ \textit{implies}\ \sum_j t_j' a_1(x_j') \leqslant a_2(x)$$

is satisfied, then there exist linear functionals z_0^* on $E = Z$ and $y_0^* \leqslant 0$ on Y satisfying the inequalities of relation (1).

Proof. We have $\sum_j t_j' x_j' = \sum t_j'' x_j''$, which implies $\sum_j t_j' a_1(x_j') \leqslant \sum_j t_j'' a_2(x_j'')$, by (21), assuming without loss of generality that $\sum_j t_j'' = 1$. Thus, condition (10) is satisfied.

Remark 7. If $X_1 \subset X_2$, then relation (21) is replaced by the following $x = \sum_j t_j' x_j'$, $t_j' \geqslant 0$, $\sum_j t_j' \leqslant 1$ and $x_j' \in X_1$ imply $\sum_j t_j' a_1(x_j') \leqslant a_2(x)$. Then the elements v in (8) are of the form

$$v = \sum_j t_j'' x_j'' - \sum_j t_j' x_j',$$

where $x_j'' \in X_2$, $x_j' \in X_1$, $t_j' \geqslant 0$, $t_j'' \geqslant 0$, $\sum_j t_j' \leqslant 1$, $\sum_j t_j'' \leqslant 1$.

In the particular case where the linear space Y is the real line, we obtain the generalization of the Mazur-Orlicz theorem presented in the Appendix to [1]. Let us note that this generalization does not follow from the corresponding theorem of Mil'man [8], where the functional a_2 is supposed to be sublinear, i.e. subadditive and positive-homogeneous. It is clear that this requirement is stronger than the convexity of a_2 assumed in [1].

5. The Lagrangian saddle-points. In this section we are concerned with the problem of maximization under constraints in linear spaces. Let Y and Z be two linear spaces with ordering relations defined on Y and Z respectively by the convex cones K_Y and K_Z. A point y_0 of Y is said to be *maximal* over the set $Y_0 \subseteq Y$, if $y_0 \in Y$ and, for each $y \in Y_0$, $y \geqslant y_0$ implies $y \leqslant y_0$.

Let X be a convex subset of a linear space E. Given two concave functions f: $X \to Y$ and g: $X \to Z$, we shall say that x_0 *maximizes* $f(x)$ subject to $x \in X$ and $g(x) \geqslant 0$ if $f(x_0)$ is maximal over the set $[f(x)\colon x \in X, g(x) \geqslant 0]$. We are interested in the conditions under which the Lagrangian expression

$$(22)\qquad \varphi(x, z^*, y^*) = y^*[f(x)] + z^*[g(x)],$$

where y^* and z^* are respectively linear functionals over Y and Z, has a (non-negative) saddle-point with $x = x_0$, i.e., the following conditions are satisfied: there exist linear functionals y_0^* and z_0^* such that

$$y_0^* \geqslant 0 \quad \text{and} \quad z_0^* \geqslant 0,$$

$$(23)\qquad \Phi(x, z_0^*, y_0^*) \leqslant \Phi(x_0, z_0^*, y_0^*) \leqslant \Phi(x_0, z^*, y_0^*)$$

for all x in X and $z^* \geqslant 0$.

Hurwicz ([5], Theorem v. 3.1.) has shown that the Lagrangian Φ has such a saddle-point if the convex cones K_Y and K_Z have non-empty interiors. Hurwicz and Uzawa [6] have proved a stronger theorem which shows that the Lagrangian saddle-point might exist in some situations where the positive orthants (i.e. the order-defining cones) have no interior. These results are covered by the following theorem, which is a particular case of Theorem 2:

THEOREM 4. *Suppose that x_0 maximizes $f(x)$ subject to $x \in X$, $g(x) \geqslant 0$ and that the following condition is satisfied: the set W of elements (u, v), where*

$$u = -f(x) + y_K, \quad v = -g(x) + z_K \quad \textit{for } x, z_K$$

and y_K running over X, K_Z and $K_Y \setminus \{0\}$ respectively, has an internal point. Then there exist linear functionals $y_0^* \geqslant 0$ and $z_0^* \geqslant 0$ such that the Lagrangian expression $\Phi(x, z^*, y^*)$ defined by (22) *has a saddle-point at (x_0, z_0^*) for all x of X and $z^* \geqslant 0$; i.e., Φ satisfies inequalities* (23). *If Y and Z are linear topological spaces and W has an interior point instead of an internal one, then y_0^* and z_0^* are continuous.*

(i) *If, in addition, for any non-null non-negative linear functional z^*, there exists an element x of X (depending on z^*) such that $z^*[g(x)] > 0$, then $y_0^* \neq 0$.*

(ii) *Suppose that y_0^* and z_0^* are continuous. In this case, in condition* (i) *"any" is replaced by "any continuous".*

Proof. Consider the set W_1 of elements (u, v), where $u = -f(x) + f(x_0) + y_K$, $v = -g(x) + z_K$ for x, z_K and y_K running over X, K_Z and $K_Y \setminus \{0\}$ respectively. W is convex and does not contain the element $(0, 0)$, since $f(x_0)$ is maximal. By assumption, it follows that W_1 has an internal point. In virtue of the basic separation theorem, there exist linear functionals $y_0^* \geqslant 0$ on Y and $z_0^* \geqslant 0$ on Z such that

$$y_0^*[f(x) - f(x_0)] + z_0^*[g(x)] \leqslant 0 \quad \text{for } x \text{ in } X.$$

Hence, we have $y_0^*[f(x)] + z_0^*[g(x)] \leqslant y_0^*[f(x_0)]$ for x in X. Thus, we obtain $z_0^*[g(x_0)] \leqslant 0$. Since $g(x_0) \geqslant 0$ and $z_0^* \geqslant 0$, it follows that $z_0^*[g(x_0)] = 0$. Thus, we obtain

$$y_0^*[f(x)] + z_0^*[g(x)] \leqslant y_0^*[f(x_0)] + z_0^*[g(x_0)] \leqslant y_0^*[f(x_0)] + z^*[g(x_0)]$$

for all x in X and $z^* \geqslant 0$, which proves (23).

We have actually repeated the same argument as in the proof of Theorem 2, where X_1 and X_0 are empty, $a_1 = a_0 = b_1 = 0$, $a_2(x) = -f(x)+ +f(x_0)$, and $b_2(x) = -g(x)$ and $X_2 = X$. Assertion (i) follows from Remark 3 and assertion (ii) follows from the same by a similar argument.

6. The general duality theorem. There are two important groups of theorems in the theory of mathematical programming. One of them concerns the existence of Lagrangian saddle-points and a general theorem of this kind is contained in section 5. The second group of theorems is connected with the duality problem. A general duality theorem has been recently proved by Gol'štein ([4], p. 16). Another approach to this problem is contained in the paper by Joffe and Tihomirov [7]. The argument used there is based on the theory of conjugate functions developed by Fenchel(¹) in the finite-dimensional space and by Moreau(¹), Rockafellar in the general case. However, the mathematical programming problem considered in the duality theory is not as general as that discussed in section 5. In other words, all duality theorems pertain to the case where the objective function of the corresponding mathematical programming problem is a functional. Thus, the question arises of extending the duality problem to the case where the objective function is an operator, i.e. of covering also the case considered in section 5. It is shown in this section that such an extension is possible by using a technique which is similar to that exploited in our general separation theorem for mappings.

Let Y and Z be two Banach spaces with ordering relations defined on Y and Z by the convex cones K_Y and K_Z. Suppose that K_Y has a non-empty interior.

Let X be a convex subset of a linear space E. Given: two functions, $f\colon X\to Y$ and $g\colon X\to Z$, where f is convex and g is concave. Consider the set $\tilde{A}$ of all generalized sequences $\{x_a\}$, $a\in A$, A being a directed set (see Appendix) such that $x_a\in X$ and $g(x_a) = z'_a+z''_a$, where $z'_a\geqslant 0$ and $z''_a\to 0$. We shall suppose that $\tilde{A}$ is not empty. The sequence of elements x_a is called a *feasible sequence*. For the set $\tilde{A}$ of all feasible sequences let us introduce the notion of a weak minimum solution as follows.

Definition. The point y_0 of Y is called the *weak minimum solution* of the generalized mathematical programming problem if for any non-null non-negative continuous linear functional y^* the following relation is satisfied:

$$\inf_{\{x_a\}\in\tilde{A}} \liminf_{a} y^*[f(x_a)] = y^*(y_0), \qquad y^*\geqslant 0. \tag{24}$$

(¹) For references see [7].

In order to formulate the general duality theorem let us introduce the following notation. Put

$$\varphi(y^*, z^*) = \inf_{x\in X}\{y^*[f(x)]-z^*[g(x)]\} \tag{25}$$

for $y^*\geqslant 0$, $\|y^*\| = 1$ and $z^*\geqslant 0$.

THEOREM 5. *If y_0 is the weak minimum solution to the generalized mathematical programming problem, then*

$$\inf_{\substack{y^*\geqslant 0\\ \|y^*\|=1}} y^*(y_0) \leqslant \sup_{\substack{y^*\geqslant 0\\ \|y^*\|=1}} \sup_{z^*\geqslant 0} \varphi(y^*, z^*) \leqslant \sup_{\substack{y^*\geqslant 0\\ \|y^*\|=1}} y^*(y_0). \tag{26}$$

Proof. Since the interior of the convex cone K_Y is not empty, by assumption, let $\mathring{y}_K$, $\|\mathring{y}_K\| = 1$, be an interior point in K_Y. For the positive numbers δ and $\varepsilon(\delta)$ let us consider the set W_δ of elements (u, v), where

$$u = -y_0+f(x)+\delta\mathring{y}_K+y_K \quad\text{ and }\quad v = -g(x)+z_K+\varepsilon(\delta)z,$$

and where x, y_K, z_K and z are running over the sets X, K_Y, K_Z and the set $[z\colon \|z\|\leqslant 1]$, respectively. It is easily seen that the set W_δ is convex. We shall show that for every positive δ there exists a positive $\varepsilon(\delta)$ such that $(0, 0)$ is not in W_δ. If this is not the case, then there are a positive $\hat{\delta}$, a sequence of positive numbers a and sequences of elements $x_a\in X$, $y_K^a\in K_Y$, $z_K^a\in K_Z$ and z_a such that $f(x_a) = y_0-\hat{\delta}\mathring{y}_K-y_K^a$, $g(x_a) = z_K^a+az_a$ with $a\to 0$ and $\|z_a\|\leqslant 1$. Hence, it follows that $\{x_a\}$ is a feasible sequence and for any non-null non-negative linear continuous functional y^* with $\|y^*\| = 1$ we have $y^*[f(x_a)]\leqslant y^*(y_0)-\hat{\delta}y^*(\mathring{y}_K)$. Thus,

$$\liminf_{a} y^*[f(x_a)]\leqslant y^*(y_0-\hat{\delta}\mathring{y}_K) < y^*(y_0),$$

since $y^*(\mathring{y}_K) > 0$. The last relation shows that y_0 is not a weak minimum solution in spite of our assumption. This contradiction proves that $(0, 0)$ is not in W_δ with arbitrary positive δ. Since W_δ has a non-empty interior, it follows from the separation theorem that there exist linear continuous functionals $-y_\delta^*$ on Y and $-z_\delta^*$ on Z such that

$$-y_\delta^*(u)-z_\delta^*(v)\leqslant 0 \quad\text{ for all } (u, v) \text{ in } W_\delta.$$

Hence, it follows that $y_\delta^*\geqslant 0$, $z_\delta^*\geqslant 0$ and $y_\delta^*\neq 0$, by the same argument as in our general separation theorem for mappings. Further, we obtain

$$y_\delta^*(-y_0)+y_\delta^*[f(x)]+\delta y_\delta^*(\mathring{y}_K)-z_\delta^*[g(x)]\geqslant 0 \tag{27}$$

for all x in X. One can assume that $\|y_\delta^*\| = 1$. Hence we obtain from (27) and (25)

$$y_\delta^*(y_0)-\delta\leqslant y_\delta^*(y_0)-\delta y_\delta^*(\mathring{y}_K)\leqslant \Psi(y_\delta^*, z_\delta^*), \tag{28}$$

where $\delta > 0$ is arbitrary and $y_\delta^*(\mathring{y}_K)\leqslant 1$.

On the other hand, we have, by (25),

$$\psi(y^*, z^*) \leqslant y^*[f(x_a)] - z^*[g(x_a)] = y^*[f(x_a)] - z^*(z'_a) - z^*(z''_a) \leqslant y^*[f(x_a)] - z^*(z''_a)$$

for any feasible sequence $\{x_a\} \subset X$, where $z'_a \geqslant 0$ and $z''_a \to 0$. Hence, we infer from the last relation that

$$\psi(y^*, z^*) \leqslant \liminf_a y^*[f(x_a)]$$

for any feasible sequence $\{x_a\}$.

Thus, it follows from (24) that $\psi(y^*, z^*) \leqslant y^*(y_0)$ and, consequently, we have

(29) $$\sup\{\psi(y^*, z^*) \mid y^* \geqslant 0, \|y^*\| = 1, z^* \geqslant 0\} \leqslant \sup\{y^*(y_0) \mid y^* \geqslant 0, \|y^*\| = 1\}.$$

Relations (28) and (29) imply relation (26), which proves the theorem.

In the particular case where Y is the real line we obtain the following duality theorem of Gol'štein ([4], p. 16) as a corollary to Theorem 5:

THEOREM 6. *If the set $\tilde{A}$ of feasible sequences $\{x_a\}$ is not empty, then*

(30) $$\sup_{z^* \geqslant 0} \psi(z^*) = \inf_{A} \liminf_{a} f(x_a),$$

where $\psi(z^*) = \inf_{x \in X}\{f(x) - z^*[g(x)]\}$, and $f(x)$ is a real-valued function on X; $\inf_A$ means inf over all feasible sequences $\{x_a\} \in \tilde{A}$.

It is obvious that relation (30) follows from (26), since for $y^* \geqslant 0$ and $\|y^*\| = 1$ we have $y^*[f(x)] = f(x)$ in relations (24), (25) and so on.

Let us note that the method of conjugate functions [7] is not applicable to Theorem 5.

7. The conjugate functions. In this section we shall show that the same technique of separation can be applied to the theory of conjugate functions of Fenchel-Moreau.

Let X be a linear space. Given: real-valued function $f(x)$, $x \in X$, defined on X. A function is called *trivial* if either $f(x) \equiv \infty$ or $f(x) = -\infty$ for some $x \in X$. Such functions are not considered in this section. Let Y be the linear space of all linear functionals defined on X. The notation $y(x) = \langle x, y\rangle$ means the value of the linear functional $y \in Y$ at the point $x \in X$. Put

(31) $$f^*(y) = \sup_{x \in X}\{\langle x, y\rangle - f(x)\}, \quad y \in Y.$$

$f^*(y)$ is called the *conjugate function*. From this definition we immediately obtain the following

LEMMA 1. $f(x) \geqslant f^{**}(x)$, $x \in X$.

Proof. In virtue of (31) we have

$$f(x) \geqslant \langle x, y\rangle - f^*(y), \quad x \in X, y \in Y.$$

Hence, we obtain in virtue of (31)

$$f(x) \geqslant \sup_{y \in Y}\{\langle x, y\rangle - f^*(y)\} = f^{**}(x).$$

THEOREM 8. *Let $f(x)$ be a convex real-valued finite function on X. Then*

(32) $$f^{**}(x) = f(x) \quad \text{for } x \in X,$$

where

$$f^{**}(x) = \sup_{y \in Y}\{\langle x, y\rangle - f^*(y)\} = \langle x, y_0\rangle - f^*(y_0)$$

for some y_0 of Y.

Proof. Consider the set W of elements (u, v), where

$$u = f(z) - f(x) + t \quad \text{and} \quad v = z - x$$

for fixed x of X, arbitrary z of X and arbitrary $t > 0$. It is clear that W is a convex set and $(0, 0)$ is not in W. Besides, the set of internal points of W is not empty. For instance, $(-f(x) + f(0) + 1, -x)$ is an internal point of W. Hence, it follows from the basic separation theorem that there exist a negative number c and a linear functional y'_0 on Y such that

$$c[f(z) - f(x)] + \langle z - x, y'_0\rangle \leqslant 0 \quad \text{for all } z \text{ in } X.$$

Let $y_0 = -c^{-1}y'_0$; then we obtain

$$-[f(z) - f(x)] + \langle z - x, y_0\rangle \leqslant 0 \quad \text{for all } z \text{ in } X.$$

Hence, it follows that

$$\langle z, y_0\rangle - f(z) \leqslant \langle x, y_0\rangle - f(x) \quad \text{for all } z \text{ in } X$$

and, consequently, we obtain

$$f^*(y_0) = \sup_{y \in Y}\{\langle z, y_0\rangle - f(z)\} \leqslant \langle x, y_0\rangle - f(x).$$

The last relation yields

$$f(x) \leqslant \langle x, y_0\rangle - f^*(y_0) \leqslant \sup_{y \in Y}\{\langle x, y\rangle - f^*(y)\}$$

and $f(x) \leqslant f^{**}(x)$. This relation and Lemma 1 imply (32).

Remark 9. a. If X is a linear topological space and f is continuous at x, then on replacing Y by the space X^* of all linear continuous functionals on X, Theorem 8 remains true for all points x of continuity of f. The proof is exactly the same as that of Theorem 8.

The continuity of y_0 follows from the inequality $\langle z-x, y_0\rangle \leqslant f(z)-f(x)$, since $f(z)$ is continuous at x and y_0 is linear.

b. Let X be a locally convex linear topological space and f be a lower semi-continuous function on X. Put $\operatorname{dom} f = [x \in X\colon f(x) < \infty]$. Then $f^{**}(x) = f(x)$, $x \in \operatorname{dom} f^{**}$.

Indeed, suppose that $f(x_0) > f^{**}(x_0)$. Then $\big(f^{**}(x_0), x_0\big)$ is not in the closed convex set W of elements (u, v), where $u = f(x)+t$ and $v = x$ for all $t \geqslant 0$ and all x of $\operatorname{dom} f$. In virtue of the strict separation theorem there exist a number c and y_0^* of X^* such that

$$\sup_{(t,x)}\{c[f(x)+t]+\langle x, y_0^*\rangle\} < cf^{**}(x_0)+\langle x_0, y_0^*\rangle$$

for $t \geqslant 0$ and $x \in \operatorname{dom} f$. Since $\operatorname{dom} f \subset \operatorname{dom} f^{**} \subset \overline{\operatorname{dom} f}$ (see [7], p. 58), it follows that $x_0 \in \overline{\operatorname{dom} f}$ and $c \neq 0$. The assumption $c > 0$ leads to a contradiction. Thus $c < 0$ and one can put $c = -1$. Thus, $\sup\{\ \}$ is attained at $t = 0$ and we obtain

$$f^*(y_0^*) = \sup_{x \in \operatorname{dom} f}\{\langle x, y_0^*\rangle - f(x)\} < \langle x_0, y_0^*\rangle - f^{**}(x_0),$$

i.e. $f^*(y_0^*)+f^{**}(x_0) < \langle x_0, y_0^*\rangle$, which is impossible by the definition of f^{**}. Thus, $f^{**}(x_0) = f(x_0)$, by Lemma 1.

In the case of a pair of dual spaces (X, Y) this theorem is proved by Moreau (see [7], p. 57).

Let us emphasize the existence of a linear functional y_x maximizing the expression $\langle x, y\rangle - f^*(y)$ in Theorem 8 as well as in Remark 9 a, in which y_x is continuous if f is continuous at x. This observation is important and an application will be given.

Let $f(x)$ be a real-valued function on the linear (topological) space X, and let us define

(33) $$\operatorname{cof}(x) = \inf\{\sum_i a_i f(x_i) \mid x = \sum_i a_i x_i;\ a_i \geqslant 0,\ \sum_i a_i = 1\}$$

for all finite representations of x;

(34) $$\bar{f}(x) = \inf \liminf_{a} f(x_a),$$

where inf is taken over all generalized sequences $\{x_a\}$ convergent to x.

The following lemma is an obvious consequence of definition (31):

LEMMA 2. *If $f_1 \geqslant f_2$, then $f_2^* \geqslant f_1^*$, where $f_1 \geqslant f_2$ means $f_1(x) \geqslant f_2(x)$ for all x in X. f^* is always convex.*

Let f be a real-valued function on a locally convex linear topological space.

COROLLARY 1. *The following relation holds:*

$$f^{**} = \overline{\operatorname{cof}}.$$

Proof. Since $\overline{\operatorname{cof}} \leqslant f$, it follows from Lemma 2 that $(\overline{\operatorname{cof}})^* \geqslant f^*$. Suppose that $a = (\overline{\operatorname{cof}})^*(y^*) > f^*(y^*) = b$ for some linear continuous functional y^* on X. Then we have

$$a = \sup_{x \in X}\{\langle x, y^*\rangle - \overline{\operatorname{cof}}(x)\} > \sup_{x \in X}\{\langle x, y^*\rangle - f(x)\} = b.$$

For $\varepsilon_0 > 0$ let $x_0 \in X$ be so chosen that $a = \langle x_0, y^*\rangle - \overline{\operatorname{cof}}(x_0)+\varepsilon_0 = b+(a-b)$. For $\varepsilon_1 > 0$, in virtue of definition (34), there is a generalized sequence $\{x_a\}$ convergent to x_0 such that $\overline{\operatorname{cof}}(x_0) = \liminf_a \operatorname{cof}(x_a) - \varepsilon_1$. Hence, it follows that, for the positive number ε_2, a can be chosen so as to satisfy the equation $\overline{\operatorname{cof}}(x_0) = \operatorname{cof}(x_a)-\varepsilon_2-\varepsilon_1$. In virtue of definition (33), for $\varepsilon_3 > 0$, there exist a finite set of real numbers a_i^a and elements $x_i^a \in X$ such that $a_i^a \geqslant 0$, $\sum_i a_i^a = 1$, $x_a = \sum_i a_i^a x_i^a$ and $\operatorname{cof}(x_a) = \sum_i a_i^a f(x_i^a) - \varepsilon_3$. Thus we obtain

$$a = \langle x_0 - x_a, y^*\rangle + \langle x_a, y^*\rangle - \sum_i a_i^a f(x_i^a) + \sum_{i=0}^{3} \varepsilon_i = b+(a-b).$$

Hence, it follows that choosing a and ε_i so as to satisfy the inequality

$$\langle x_0 - x_a, y^*\rangle + \sum_{i=0}^{3}\varepsilon_i < (a-b)/2,$$

we obtain

$$\langle x_a, y^*\rangle - \sum_i a_i^a f(x_i^a) > b + \frac{a-b}{2} \geqslant \langle x_i^a, y^*\rangle - f(x_i^a) + \frac{a-b}{2}.$$

Multiplying these inequalities by a_i^a and summing over i, we obtain

$$\langle x_a, y^*\rangle - \sum_i a_i^a f(x_i^a) > \langle x_a, y^*\rangle - \sum_i a_i^a f(x_i^a) + \frac{a-b}{2}.$$

Hence, it follows that $a < b$, in contradiction to our assumption. Thus,

$$(\overline{\operatorname{cof}})^* = f^* \quad \text{and} \quad f^{**} = (\overline{\operatorname{cof}})^{**} \leqslant (\operatorname{cof})^{**} \leqslant f^{**}.$$

But $\overline{\operatorname{cof}} = (\operatorname{cof})^{**}$ in virtue of Remark 9 b.

COROLLARY 2. *If the function $\bar{f}$ defined by (34) is convex, then $f^{**} = \bar{f}$.*

COROLLARY 1*. *If cof (x) is finite, then for every x of the linear space X there exists a linear functional y_x defined on X such that*

$$(*)\qquad \mathrm{cof}(x) = \langle x, y_x\rangle - f^*(y_x) = \sup_{y\in Y}\{\langle x, y\rangle - f^*(y)\} = f^{**}(x).$$

Proof. Using the same argument as in the proof of Corollary 1, we obtain $(\mathrm{cof})^* = f^*$. Hence, it follows in virtue of Theorem 8 that $\mathrm{cof} = (\mathrm{cof})^{**} = f^{**}$. Moreover, there exists a linear functional $y_x \in Y$ such that

$$\mathrm{cof}(x) = \langle x, y_x\rangle - (\mathrm{cof})^*(y_x)$$
$$= \sup_{y\in Y}\{\langle x, y\rangle - (\mathrm{cof}^*(y)\} = \sup_{y\in Y}\{\langle x, y\rangle - f^*(y)\} = f^{**}(x),$$

since $(\mathrm{cof})^* = f^*$, where Y denotes the linear space of linear functionals y on X.

Let X be a locally convex linear topological space with an ordering relation defined by the convex cone K. Given: a set U and a real-valued function f defined on U. Let g be a mapping $g\colon U \to X$. Put

$$(35)\qquad M(x) = \inf\{f(u)\,|\, g(u) \geqslant x,\ u\in U\},\qquad x\in X,$$

and $M(x) = \infty$ if there is no u in U which satisfies the inequality $g(u) \geqslant x$.

$$(36)\qquad \bar{M}(x) = \inf\{\liminf_a M(x_a)\,|\, x_a \to x\},$$

where inf is over all generalized sequences convergent toward x. Denote by X^* the space of all continuous linear functionals y^* on X. Put

$$(37)\qquad \psi(y^*) = \sup_{u\in U}\{\langle g(u), y^*\rangle - f(u)\},\qquad y^*\in X^*.$$

THEOREM 9. *If the function $\bar{M}(x)$ is convex, then*

$$(38)\qquad \bar{M}(x) = \sup_{y^*\in X^*}\{\langle x, y^*\rangle - \psi(y^*)\} = \sup_{y^*\geqslant 0}\{\langle x, y^*\rangle - \psi(y^*)\} = \psi^*(x).$$

Proof. The inequality $g(u)\geqslant x$ implies

$$\langle x, y^*\rangle - f(u) \leqslant \sup_{x:x\leqslant g(u)}\{\langle x, y^*\rangle - f(u)\}.$$

Hence, it follows that

$$\sup_{u:g(u)\geqslant x}\{\langle x, y^*\rangle - f(u)\} \leqslant \sup_{u\in U}\sup_{x:x\leqslant g(u)}\{\langle x, y^*\rangle - f(u)\}.$$

Thus, we have

$$(39)\qquad \sup_x\sup_{u:g(u)\geqslant x}\{\langle x, y^*\rangle - f(u)\} \leqslant \sup_{u\in U}\sup_{x:x\leqslant g(u)}\{\langle x, y^*\rangle - f(u)\}.$$

On the other hand, the inequality $x\leqslant g(u)$ implies

$$\langle x, y^*\rangle - f(u) \leqslant \sup_{u:g(u)\geqslant x}\{\langle x, y^*\rangle - f(u)\}.$$

Hence, it follows that

$$\sup_{x:x\leqslant g(u)}\{\langle x, y^*\rangle - f(u)\} \leqslant \sup_x\sup_{u:g(u)\geqslant x}\{\langle x, y^*\rangle - f(u)\}.$$

Thus, we obtain

$$(40)\qquad \sup_{u\in U}\sup_{x:x\leqslant g(u)}\{\langle x, y^*\rangle - f(u)\} \leqslant \sup_x\sup_{u:g(u)\geqslant x}\{\langle x, y^*\rangle - f(u)\}.$$

If y^* is non $\geqslant 0$, then there is an $\hat{x}$ in K such that $\langle \hat{x}, y^*\rangle < 0$. Thus, $g(u)\geqslant 0$ implies $g(u)\geqslant -t\hat{x}$ and $\langle -t\hat{x}, y^*\rangle > 0$ for arbitrary positive t. Hence, it follows that

$$\sup_{x:x\leqslant g(u)}\{\langle x, y^*\rangle - f(u)\} \geqslant \langle -t\hat{x}, y^*\rangle - f(u) \to \infty$$

if $t\to\infty$. Thus, we obtain

$$(41)\qquad \sup_{u\in U}\sup_{x:x\leqslant g(u)}\{\langle x, y^*\rangle - f(u)\} = \begin{cases} \infty & \text{if } y^* \text{ is non} \geqslant 0,\\ \sup_{u\in U}\{\langle g(u), y^*\rangle - f(u)\} & \text{if } y^*\geqslant 0.\end{cases}$$

In virtue of Corollary 2, we have

$$\bar{M}(x) = M^{**}(x),$$

$$M^*(y^*) = \sup_x\{\langle x, y^*\rangle - M(x)\}$$
$$= \sup_x\{\langle x, y^*\rangle - \inf_{u:g(u)\geqslant x} f(u)\} = \sup_x\sup_{u:g(u)\geqslant x}\{\langle x, y^*\rangle - f(u)\}$$
$$= \sup_{u\in U}\sup_{x:x\leqslant g(u)}\{\langle x, y^*\rangle - f(u)\} = \begin{cases}\infty & \text{if } y^* \text{ is non} \geqslant 0,\\ \psi(y^*) & \text{if } y^* \geqslant 0,\end{cases}$$

in virtue of relations (35)-(37) and (39)-(41). Thus, we have obtained $M^{**}(x) = \psi^*(x)$, which proves relation (38).

Let us note that for a pair of dual spaces (X, Y) the same Theorem 9 is contained in [7], p. 77. As an application of Theorem 9 it is easy to obtain a general duality theorem in locally convex linear topological spaces. The generalized sequence $\{u_a\}$, $a\in A$ (directed set) is called *feasible* if $g(u_a) = x'_a + x''_a$, where $x'_a \geqslant x$, $x''_a \to 0$, x being a fixed element in X. Put

$$(42)\qquad m(x) = \inf\{\liminf_a f(u_a)\,|\, g(u_a) = x'_a + x''_a,\ x'_a\geqslant x,\ x''_a\to 0\},$$

where inf is over all feasible sequences. $m(x) = \infty$ if there is no feasible sequence.

LEMMA 3. *$m(x) = \bar{M}(x)$, where m and $\bar{M}$ are defined by* (36) *and* (42), *respectively.*

Proof. For an arbitrary feasible sequence $\{u_a\}$ we have, by (42), $g(u_a) = x'_a + x''_a \geqslant x + x''_a$. Thus, $x + x''_a \to x$ and $M(x + x''_a) \leqslant f(u_a)$, by definition. Hence, it follows that

$$\inf \liminf_a M(x + x''_a) \leqslant m(x).$$

On the other hand, it follows from definition (36) that for $x_a \to x$, there exist a with $g(u_a) \geqslant x_a$ (if $M(x_a) < \infty$). Thus, $g(u_a) = (g(u_a) - x_a + x) + (x_a - x) = x'_a + x''_a$, where $x'_a \geqslant x$ and $x''_a \to 0$. Hence, there is a feasible sequence $\{u_a\}$ and, by (42), $m(x) \leqslant \liminf_a M(x_a)$. Since the sequence of $x_a \to x$ is arbitrarily chosen, from (36) we obtain $m(x) \leqslant \bar{M}(x)$.

Putting $x = 0$ in Theorem 9 and in Lemma 3, we obtain the following duality theorem in locally convex linear topological spaces:

$$m(0) = \bar{M}(0) = \sup_{y^* \geqslant 0} \inf_{u \in U} \{f(u) - \langle g(u), y^* \rangle\}.$$

For a pair of dual spaces (X, Y) this theorem is proved in [7], p. 77. As a corollary it contains the general duality theorem by Gol'štein ([4], p. 16), where U is a convex set, $f(u)$ is a convex real-valued function and $g(u)$ is a concave mapping.

In some general cases, a stronger duality theorem can be obtained as well as the existence of Lagrangian multipliers on the basis of Corollary 1*.

Let X be a linear space with an ordering relation defined by the convex cone K. Let $M(x)$ have the same meaning as in (35) and put

$$\psi(y) = \sup_{u \in U} \{\langle g(u), y \rangle - f(u)\}, \qquad y \in Y,$$

i.e. replacing in (37) X^* by Y, the linear space of all linear functionals y on X. Then we obtain

THEOREM 9*. *For every x of X, there exists a linear functional $y_x \geqslant 0$ such that*

$$\operatorname{co} M(x) = \langle x, y_x \rangle - \psi(y_x) = \sup_{y \geqslant 0} \{\langle x, y \rangle - \psi(y)\}$$
$$= \sup_{y \in Y} \{\langle x, y \rangle - \psi(y)\} = \psi^*(x).$$

If X is a linear topological space and (a) *K has a non-empty interior or* (b) *X is complete metric linear and $K - K$ has a non-empty interior, K being closed, then Y can be replaced by X^* and y_x by $y^*_x \in X^*$.*

Proof. In the same way as in the proof of Theorem 9 we obtain

$$M^*(y) = \begin{cases} \infty & \text{if } y \text{ is non} \geqslant 0, \\ \psi(y) & \text{if } y \geqslant 0. \end{cases}$$

Hence, it follows in virtue of Corollary 1* that there exists a linear functional y_x satisfying relations (*) for $f = M$. Thus, replacing M^* by ψ we obtain the required equalities. If X is a linear topological space and K has non-empty interior, then it follows from Lemma 7 of [3], p. 417, that y_x is continuous, since $y_x \geqslant 0$. If X is a complete metric linear space and $K - K$ has a non-empty interior, then by a theorem of Klee (see [6], p. 104, footnote 9) y_x is continuous, since $y_x \geqslant 0$. Thus in both cases (a) and (b) Y can be replaced by X^*, the linear space of all linear continuous functionals on X. It is known that all of the following spaces are complete metric linear and have closed space-spanning positive orthants K (i.e. $K - K = X$): (S), (s), (Lp), $p > 0$, (lp), $p > 0$. Thus, in all these spaces Y can be replaced by X^*. As an immediate consequence of Theorem 9* we obtain the following

Duality Theorem and Existence of Lagrangian Multipliers. If X is a linear space, then there exists a linear functional $y_0 \geqslant 0$ such that

$$(**) \qquad \operatorname{co} M(0) = \inf_{u \in U} \{f(u) - \langle g(u), y_0 \rangle\} = \sup_{y \geqslant 0} \inf_{u \in U} \{f(u) - \langle g(u), y \rangle\}.$$

If X is a linear topological space satisfying condition (a) or (b) in Theorem 9*, then $y_0 \geqslant 0$ and $y \geqslant 0$ in (**) denote continuous linear functionals.

If U is a convex set, $f(u)$ is a real-valued convex function defined on U and g: $U \to X$ is a concave mapping, then $\operatorname{co} M(0) = M(0)$, and we substitute in (**)

$$\operatorname{co} M(0) = M(0) = \inf\{f(u) \mid g(u) \geqslant 0,\, u \in U\}.$$

In the particular case where $U = X$ let us put instead of (37)

$$(f \circ g)(y^*) = \sup_{x \in X} \{\langle g(x), y^* \rangle - f(x)\}.$$

Then relation (38) yields

$$(43) \qquad m(x) = \bar{M}(x) = (f \circ g)^*(x), \qquad x \in X,$$

where $\bar{M}(x)$ is convex by assumption and

$$m(x) = \inf\{\liminf_a f(x_a) \mid g(x_a) = x'_a + x''_a,\, x'_a \geqslant 0,\, x''_a \to 0\},$$

$$M(x) = \inf\{f(\bar{x}) \mid g(\bar{x}) \geqslant x,\, \bar{x} \in X\},$$

$\bar{M}(x)$ being defined by relation (36).

If $g = I$ is the identity mapping of X with $K = \{0\}$, then $f \circ I = f^*$ and relation (43) yields $\bar{f} = f^{**}$ provided that $\bar{f}$ is convex. Thus, we see that Corollary 2 is a particular case of relation (43), where $g = I$ and $K = \{0\}$.

We shall now investigate some relations between the operations $\circ$ and $*$. Let X be a linear space and $f(x)$ be a real-valued convex function on X. Let Z be a linear topological space with an ordering relation defined by the convex cone K and let g be a concave mapping $g\colon X \to Z$. Suppose that B is a convex bounded set contained in Z. We shall assume that either K or B has a non-empty interior and that B contains the zero-element. But the existence of such B with a non-empty interior is very restrictive for Z. Put

(44) $$q(x) = \inf\{\liminf_{\alpha} f(x_\alpha) \mid g(x_\alpha) = g(x) + z'_\alpha + z''_\alpha;\ z'_\alpha \geqslant 0,\ z''_\alpha \to 0\},$$

where inf is over all generalized sequences $\{x_\alpha\}$, $\alpha \in A$ (directed set). It is easy to see that if $Q(x) = \inf\{f(\bar{x}) \mid \bar{x} \geqslant x,\ \bar{x} \in X\}$, then

$$q(x) = \bar{Q}(x) = \inf\{\liminf_{\alpha} Q(x_\alpha) \mid x_\alpha \to x\},$$

provided that $Z = X$ and $g = I$ is the identity mapping of X. Thus, in the particular case where $Z = X$ is a normed space with $K = \{0\}$ and $g = I$ is the identity mapping of X, we have $g = \bar{f}$, $\bar{f}$ being defined by relation (34). For the linear continuous functional z^* on Z let us put

(45) $$(f \circ g)(z^*) = \sup_{x \in X}\{\langle g(x), z^*\rangle - f(x)\}.$$

Then the following theorem is true:

THEOREM 10. *The following relation holds*:

(46) $$q(x) = (f \circ g)^*\big(g(x)\big),$$

where f is convex, g is concave, and where

(47) $$(f \circ g)^*(z) = \sup_{z^* \geqslant 0}\{\langle z, z^*\rangle - (f \circ g)(z^*)\}, \qquad z \in Z.$$

Proof. For the positive numbers t and $\varepsilon(t)$ let us consider the set W_t of elements (u, v),

$$u = f(y) - q(x) + t + s \quad \text{and} \quad v = -g(y) + g(x) + z_K + \varepsilon(t) z$$

for fixed x of X and $t > 0$, and where y, z_K, z and s are running over the sets X, K, B and the set of non-negative numbers, respectively. We shall show that for every positive t there exists a positive $\varepsilon(t)$ such that $(0, 0)$ is not in W_t. If this is not the case, then there are a positive $\hat{t}$ and two

sequences of positive numbers α and s_α sequences of elements $z_\alpha \in B$, $z_K^\alpha \in K$ and y_α such that

$$q(x) = f(y_\alpha) + \hat{t} + s_\alpha, \qquad g(y_\alpha) = g(x) + z_K^\alpha + \alpha z_\alpha \quad \text{with } \alpha \to 0.$$

Hence, it follows that

$$f(y_\alpha) \leqslant q(x) - \hat{t} \quad \text{and} \quad g(y_\alpha) = g(x) + z'_\alpha + z''_\alpha,$$

where $z'_\alpha = z_K^\alpha \geqslant 0$ and $z''_\alpha = \alpha z_\alpha \to 0$, since B is bounded. Thus we infer by (44) that

$$\inf \liminf_{\alpha} f(y_\alpha) \leqslant q(x) - \hat{t}.$$

This contradiction shows that $(0, 0)$ is not in W_t. Since W_t is a convex set with a non-empty interior, it follows from the separation theorem that there exist a negative number c_t and a linear continuous functional $z_t^* \leqslant 0$ such that

$$c_t[f(y) - q(x) + t] + z_t^*[-g(y) + g(x)] \leqslant 0 \quad \text{for any } y \text{ in } X.$$

Putting $z_t^* = c_t^{-1} z_t^* \geqslant 0$, we obtain

$$-[f(y) - q(x) + t] - z_t^*[-g(y) + g(x)] \leqslant 0,$$

i.e.,

$$\langle g(y), z_t^*\rangle - f(y) - t \leqslant \langle g(x), z_t^*\rangle - q(x).$$

Hence, it follows from definition (45) that

$$(f \circ g)(z_t^*) - t \leqslant \langle g(x), z_t^*\rangle - q(x),$$

i.e.

$$q(x) - t \leqslant \sup_{z^* \geqslant 0}\{\langle g(x), z^*\rangle - (f \circ g)(z^*)\}.$$

Since $t > 0$ is arbitrary, we have, by (47), $q(x) \leqslant (f \circ g)^*\big(g(x)\big)$. Suppose now that $b = q(x) < (f \circ g)^*\big(g(x)\big) = a$. Hence, it follows from definition (47) that for a positive ε_0 there exists a continuous linear functional $z_{\varepsilon_0}^* \geqslant 0$ such that

$$b = \langle g(x), z_{\varepsilon_0}^*\rangle - (f \circ g)(z_{\varepsilon_0}^*) + \varepsilon_0 + (b - a).$$

For $\varepsilon_1 > 0$ in virtue of (44) there exists a generalized sequence $\{x_\alpha\}$ such that $q(x) = \liminf_{\alpha} f(x_\alpha) - \varepsilon_1$ and $g(x_\alpha) = g(x) + z'_\alpha + z''_\alpha$ with $z'_\alpha \geqslant 0$ and $z''_\alpha \to 0$. Thus, for $\varepsilon_2 > 0$ there exist elements x_α of $\{x_\alpha\}$ such that $q(x) = f(x_\alpha) - \varepsilon_2 - \varepsilon_1$. Thus, we obtain

(48) $$b = f(x_\alpha) - \varepsilon_2 - \varepsilon_1 = \langle g(x), z_{\varepsilon_0}^*\rangle - (f \circ g)(z_{\varepsilon_0}^*) + \varepsilon_0 + (b - a).$$

It follows from (45) that

$$(f\circ g)(z^*_{\varepsilon_0}) \geqslant \langle g(x_a), z^*_{\varepsilon_0}\rangle - f(x_a)$$
$$= \langle g(x)+z'_a+z''_a, z^*_{\varepsilon_0}\rangle - f(x_a) \geqslant \langle g(x)+z''_a, z^*_{\varepsilon_0}\rangle - f(x_a).$$

Hence, in virtue of (48), we obtain

(49) $$f(x_a)-\varepsilon_2-\varepsilon_1 \leqslant \langle z''_a, z^*_{\varepsilon_0}\rangle + f(x_a)+\varepsilon_0+(b-a).$$

If ε_0, ε_1, ε_2 and a are chosen so as to satisfy the inequality

$$\langle z''_a, z^*_{\varepsilon_0}\rangle + \varepsilon_0+\varepsilon_1+\varepsilon_2 < \frac{a-b}{2},$$

then relation (49) yields $0<(b-a)/2$. This contradiction shows that relation (46) is true.

We shall now investigate another relation between the operations $\circ$ and $*$. Let $f(x)$, $x\in X$, be a real-valued convex function defined on the linear space X and let Z be a linear space with an ordering relation defined by the convex cone K. Given the convex mapping $g\colon X\to Z$.

Let us define the set W of elements (u,v), where

$$u = f(y)+t \quad\text{and}\quad v = g(y)+z_K, \quad y\in X, z_K\in K,$$

for y, z_K and t running over the sets X, K and the set of positive numbers, respectively. For the linear functional z^* on Z put

(50) $$(f\circ g)(z^*) = \sup_{x\in X}\{\langle g(x), z^*\rangle - f(x)\}$$

and

(51) $$(f\circ g)^*(z) = \sup_{z^*\geqslant 0}\{\langle z, z^*\rangle - (f\circ g)(z^*)\}, \quad z\in Z.$$

The following theorem can be considered as a generalization of Theorem 8:

THEOREM 11. *Let us assume that the set W for convex f and g has an internal point. If $x, y\in X$, $g(x)\geqslant g(y)$ implies $f(x)\leqslant f(y)$, and for every x and $z^*\geqslant 0$ there exists an element y of X such that*

(52) $$z^*[g(y)] < z^*[g(x)] \quad (y \text{ depends on } x \text{ and on } z^*),$$

then

(53) $$(f\circ g)^*\big(g(x)\big) = f(x), \quad x\in X.$$

Proof. For arbitrary but fixed x of X let us consider the set W_x of elements (u,v), where $u=f(y)-f(x)+t$ and $v=g(y)-g(x)+z_K$ and where y, z_K and t run over the sets X, K and the set of positive numbers, respectively. It is clear that the set W_x is convex. It is easy to verify that $(0,0)$ is not in W_x. For $g(y)-g(x)+z_K=0$ implies $g(x)\geqslant g(y)$.

Hence, by assumption, $f(x)\leqslant f(y)$ and, consequently, $f(y)-f(x)+t>0$ if $t>0$. In virtue of the basic separation theorem, since W_x has an internal point, there exist a number $c\leqslant 0$ and a linear functional z_0^* on Z such that

(54) $$c[f(y)-f(x)]+z_0^*[g(y)-g(x)] \leqslant 0 \quad \text{for all } y \text{ of } X.$$

Since $z_0^*\leqslant 0$, it follows from (52) that $c\neq 0$ and, consequently, $c<0$. Putting $\bar{z}_0^* = c^{-1}z_0^*\geqslant 0$, we obtain

$$-[f(y)-f(x)]+\bar{z}_0^*[g(y)-g(x)]\leqslant 0$$

for all $y\in X$. Hence, it follows that

$$\langle g(y), \bar{z}_0^*\rangle - f(y) \leqslant \langle g(x), \bar{z}_0^*\rangle - f(x) \quad \text{for all } y\in X.$$

Thus, in virtue of (50) we have the relation

$$(f\circ g)(\bar{z}_0^*) \leqslant \langle g(x), \bar{z}_0^*\rangle - f(x),$$

which yields, by (51),

(54a) $$f(x)\leqslant \langle g(x), \bar{z}_0^*\rangle - (f\circ g)(\bar{z}^*) \leqslant (f\circ g)^*\big(g(x)\big).$$

On the other hand, we have, by (50), $f(x)\geqslant \langle g(x), z^*\rangle - (f\circ g)(z^*)$. Since z^* is an arbitrary linear functional on Z, we obtain in virtue of definition (51)

$$f(x)\geqslant (f\circ g)^*\big(g(x)\big).$$

This inequality and inequality (54a) imply relation (53) and the proof is completed.

Let us observe that in the particular case where $Z=X$ and $K=\{0\}$, all the hypotheses of Theorem 11 are obviously fulfilled provided that $g=I$ is the identity mapping of X. Since $f\circ I = f^*$, relation (53) coincides with (32). Thus, Theorem 8 is a particular case of Theorem 11.

COROLLARY 3. *If Z is a linear topological space, Theorem 11 remains true. In this case it is assumed that the set W has an interior point and the linear functionals z^* on Z are everywhere replaced by linear continuous functionals z^* on Z.*

The proof is exactly the same as that of Theorem 11. Theorem 11 has been formulated as a relation between the operations $\circ$ and $*$. However, this relation yields actually a property of the Lagrange function for the convex functional f and the convex mapping g. Since no topology is needed to prove it, this relation is rather of algebraic character. On the other hand, Theorem 8 proves the duality property of convex functions. Since this theorem is a particular case of Theorem 11, it follows that the duality property of a convex function is a particular case of a more general property of the Lagrange function.

8. Convex-concave functions. It is the purpose of this section to generalize two theorems concerning a real-valued function of two abstract variables. The first theorem concerns the commutativity of the infsup and has been proved by Moreau (see [7], p. 79). The second one concerns the existence of a saddle-point and is due to Joffe-Tihomirov ([7], p. 80). Both theorems are proved for the case of two pairs of dual spaces. However, following the argument in [7] it is easily seen that the generality of these theorems depends only on the validity of the duality property of a convex function. Owing to this fact and on the basis of Theorem 8 we are now in a position to give a significant generalization of both theorems mentioned above. New theorems are also presented.

Let X and U be two real linear spaces. For the real-valued function $f(x, u)$ defined on the product space $X \times U$ let us introduce the following notation

$$\operatorname{dom} f = [x, u\colon\ |f(x, u)| < \infty].$$

The projections of $\operatorname{dom} f$ into X and U are denoted by $\operatorname{dom}_X f$ and $\operatorname{dom}_U f$, respectively.

For every $u \in \operatorname{dom}_U f$, $f_u(x) = f(x, u)$ is a function on X and in this sense we shall use the notation $f_u^*(y) = f_u^*(y, u)$, where y is a linear functional on X. We shall say that the function $f(x, u)$ is *convex* with respect to x if for every $u \in \operatorname{dom}_U f$ the function $f(x, u)$ is convex. In the same sense we use the expression that the function $f(x, u)$ is *concave* with respect to u. The function $f(x, u)$ is said to be *convex-concave* if it is convex with respect to x and concave with respect to u. The following lemma is evident:

LEMMA 4. *If the function $f(x, u)$ is concave with respect to u, then*

$$f_u^*(y, u) = \sup_x \{\langle x, y\rangle - f(x, u)\}, \qquad y \in Y,$$

is convex on $Y \times U$, i.e. with respect to both variables y and u jointly, where Y denotes the linear space of all real-valued linear functionals defined on X.

Put

$$h(y) = \inf_{u \in \operatorname{dom}_U f} f_u^*(y, u), \tag{55}$$

where $y \in Y$, the linear space of all real-valued linear functionals defined on X.

THEOREM 12. *Suppose that the function $f(x, u)$ is convex with respect to x for every u in $\operatorname{dom}_U f$. Then*

$$\inf_{x \in \operatorname{dom}_X f} \sup_{u \in \operatorname{dom}_U f} f(x, u) = \sup_{u \in \operatorname{dom}_U f} \inf_{x \in \operatorname{dom}_X f} f(x, u)$$

if and only if $h(0) = h^{**}(0)$.

Proof. If u is in $\operatorname{dom}_U f$ and x is not in $\operatorname{dom}_X f$, then $f(x, u) = \infty$. Hence, it follows that

$$\inf_{x \in \operatorname{dom}_X f} \sup_{u \in \operatorname{dom}_U f} f(x, u) = \inf_x \sup_{u \in \operatorname{dom}_U f} f(x, u),$$

$$\sup_{u \in \operatorname{dom}_U f} \inf_{x \in \operatorname{dom}_X f} f(x, u) = \sup_{u \in \operatorname{dom}_U f} \inf_x f(x, u).$$

Since $f(x, u)$ is convex with respect to x, we have in virtue of Theorem 8

$$f(x, u) = \sup \{\langle x, y\rangle - f_u^*(y, u)\}, \qquad y \in Y.$$

Thus, in virtue of (55), we obtain the following relations:

$$\begin{aligned}\inf_x \sup_{u \in \operatorname{dom}_U f} f(x, u) &= \inf_x \sup_{u \in \operatorname{dom}_U f} \sup_y \{\langle x, y\rangle - f_u^*(y, u)\}\\ &= \inf_x \sup_y \sup_{u \in \operatorname{dom}_U f} \{\langle x, y\rangle - f_u^*(y, u)\}\\ &= \inf_x \sup_y \{\langle x, y\rangle - h(y)\} = \inf_x h^*(x) = -h^{**}(0).\end{aligned}$$

On the other hand, we have

$$\inf_x f(x, u) = -f_u^*(0, u)$$

and, consequently, we obtain

$$\sup_{u \in \operatorname{dom}_U f} \inf_x f(x, u) = -h(0),$$

in virtue of (55).

The theorem of Moreau (see [7], p. 79) for $f(x, u)$ defined on $X \times U$ concerns the case of two pairs of dual spaces: (X, Y) and (U, V).

Let us remark that if in Theorem 12 the linear space X is a linear topological one, then the linear space Y of all linear functionals y can be replaced by the linear space X^* of all linear continuous functionals defined on X. The proof is obviously the same. However, in this case we must assume additionally that the function $f(x, u)$ is continuous with respect to the variable u. Under this assumption, Theorem 8 can be replaced by Remark 9a. In the case of dual spaces considered by Moreau it is sufficient to assume lower semi-continuity instead of continuity. The same is true for general locally convex linear topological spaces, in virtue of Remark 9b.

Let f be a real-valued function defined on the linear space X. The function f^* defined on the linear space Y of all linear functionals on X by relation (31) will be called the *algebraic conjugate* of f. If X is a linear topological space and $Y = X^*$ is the linear space of all continuous linear functionals on X, then f^* is the conjugate of f.

Definition. The linear functional $y_0 \in Y$ is called the *algebraic subgradient* of $f(x)$ at x_0 if the following relation holds:

$$f(x)-f(x_0) \geqslant \langle x-x_0, y_0\rangle \quad \text{for all } x \text{ in } X. \tag{56}$$

If y_0 is a continuous linear functional $y_0 = y_0^* \in X^*$, then it is called the *subgradient* of $f(x)$ at the point x_0. The set of all algebraic subgradients of $f(x)$ at x_0 is called the *algebraic subdifferential* of $f(x)$ at the point x_0 and it is denoted by $\vartheta f(x_0)$. The set of all subgradients of $f(x)$ at x_0 is called the *subdifferential* of $f(x)$ at x_0 and it is denoted by $\partial f(x_0)$. If $f(x)$ is convex, then $\vartheta f(x) \neq \emptyset$. If $f(x)$ is convex and continuous, then $\vartheta f(x) \neq \emptyset$. These assertions follow from the proof of Theorem 8 and Remark 9a.

If f^* is the conjugate of f, then the inequality $f(x)+f^*(y^*) \geqslant \langle x, y^*\rangle$ holds for arbitrary x of X and y^* of X^*. The same inequality holds for the algebraic conjugate function replacing y^* of X^* by y of Y. The question arises when the inequality becomes an equality. The following theorem gives the answer to this question

THEOREM 13. *The equality*

$$f(x_0)+f^*(y_0^*) = \langle x_0, y_0^*\rangle \quad or \quad f(x_0)+f^*(y_0) = \langle x_0, y_0\rangle$$

is true if and only if $y_0^* \in \partial f(x_0)$ *or* $y_0 \in \vartheta f(x_0)$, *respectively.*

Proof. Suppose that the first equality is satisfied. Hence, in virtue of the definition of the conjugate function, we have $\langle x, y^*\rangle - f(x) \leqslant \langle x_0, y_0^*\rangle - f(x_0)$ for all x in X. Thus, $y_0^* \in \partial f(x_0)$, by (56). Let us assume that $y_0^* \in \partial f(x_0)$. Thus relation (56) yields $\langle x, y_0^*\rangle - f(x) \leqslant \langle x_0, y_0^*\rangle - f(x_0)$ and, consequently, $f^*(y_0^*) \leqslant \langle x_0, y_0^*\rangle - f(x_0)$. Since the opposite inequality is always satisfied, we obtain the required equality. The proof for the algebraic conjugate function is exactly the same.

COROLLARY 4. *If* $\partial f(x_0) \neq \emptyset$, *then* $f^{**}(x_0) = f(x_0)$. *The same assertion holds for the algebraic conjugate function, if* $\vartheta f(x_0) \neq \emptyset$.

Proof. In virtue of Theorem 13, the relation $y_0^* \in \partial f(x_0)$ implies

$$f(x_0) = \langle x_0, y_0^*\rangle - f^*(y_0^*) \leqslant f^{**}(x_0).$$

But $f(x_0) \geqslant f^{**}(x_0)$, by Lemma 1, and we obtain the equality.

Let us observe that Corollary 4 shows that if $y_0^* \in \partial f(x_0)$, then

$$f(x_0) = \langle x_0, y_0^*\rangle - f^*(y_0^*) = \max_{y^* \in X^*}\{\langle x_0, y^*\rangle - f^*(y^*)\} = f^{**}(x_0). \tag{57}$$

The same is valid for the algebraic conjugate functions.

Let us introduce the following notation. If f^* is the algebraic conjugate of f, then $x_0 \in \vartheta f^*(y_0)$, $y_0 \in Y$, means $f^*(y) - f^*(y_0) \geqslant \langle x_0, y-y_0\rangle$ for all y of Y. Such a notation is justified, since $x_0(y) = \langle x_0, y\rangle$, $y \in Y$, is a linear functional on Y. Similarly, if f^* is the conjugate of f, then $x_0 \in \partial f^*(y_0^*)$ means

$$f^*(y^*) - f^*(y_0^*) \geqslant \langle x_0, y^*-y_0^*\rangle \quad \text{for all } y^* \text{ of } X^*.$$

Using this notation we shall discuss the minimum problem of a real-valued function $f(x)$ defined on X. For the conjugate functions Theorem 13 can be formulate as follows:

THEOREM 13*. *The equality*

$$f^*(y_0)+f^{**}(x_0) = \langle x_0, y_0\rangle \quad \text{or} \quad f^*(y_0^*)+f^{**}(x_0) = \langle x_0, y_0^*\rangle$$

is valid if and only if $x_0 \in \vartheta f^*(y_0)$ *or* $x_0 \in \partial f^*(y_0^*)$, *respectively.*

Proof. Suppose that the first equality is satisfied. Then, in virtue of the definition of the algebraic conjugate function, we have

$$f^{**}(x_0) = \langle x_0, y_0\rangle - f^*(y_0) \geqslant \langle x_0, y\rangle - f^*(y)$$

for all y of Y. Hence, it follows that $x_0 \in \vartheta f^*(y_0)$. If $x_0 \in \vartheta f^*(y_0)$, then $\langle x_0, y\rangle - f^*(y) \leqslant \langle x_0, y_0\rangle - f^*(y_0)$ for all y of Y and, consequently,

$$f^{**}(x_0) = \sup_{y \in Y}\{\langle x_0, y\rangle - f^*(y)\} = \langle x_0, y_0\rangle - f^*(y_0).$$

The proof of the second assertion concerning the case where X is a linear topological space is exactly the same.

Let us remark that in all cases where X is a locally convex linear topological space the condition of continuity of f can be replaced by the lower semi-continuity of the convex function f, since we then have $f^{**} = f$, in virtue of Remark 9b.

THEOREM 14. *If* $f(x_0)$ *is a minimum of* $f(x)$ *on the linear space* X, *then the algebraic conjugate* f^* *satisfies the following necessary condition:* $x_0 \in \vartheta f^*(0)$.

If $f^{**}(x_0) = f(x_0)$ *or* $\vartheta f(x_0) \neq \emptyset$, *then this condition is also sufficient. Similarly, if* $f(x_0)$ *is a minimum of* $f(x)$ *on the linear topological space* X, *then the conjugate* f^* *satisfies the following necessary condition:* $x_0 \in \partial f^*(0)$.

If $f^{**}(x_0) = f(x_0)$ *or* $\partial f(x_0) \neq \emptyset$, *then this condition is also sufficient.*

Proof. Suppose that $f(x)$ achieves its minimum at the point x_0. It follows from the definition of the algebraic conjugate function that $f^*(y) \geqslant \langle x_0, y\rangle - f(x_0)$ for all y of Y. Since $-f^*(0) = \inf f(x) = f(x_0)$, we obtain $f^*(y) - f^*(0) \geqslant \langle x_0, y\rangle$ for all y of Y, i.e. $x_0 \in \vartheta f^*(0)$. Let us prove the sufficiency. In virtue of Theorem 13* the condition $x_0 \in \vartheta f^*(0)$ implies $f^*(0)+f^{**}(x_0) = 0$. If $\vartheta f(x_0) \neq \emptyset$, then it follows from Corollary 4 that $f^{**}(x_0) = f(x_0)$. Hence, we obtain $f(x_0) = -f^*(0) = \inf f(x)$. The proof in the case of a linear topological space X is exactly the same. The only change is the replacing Y by X^* and the symbol ϑ by ∂.

COROLLARY 5. *If* f *is a convex function on a linear space* X, *then* $f(x_0)$ *is a minimum of* f *on* X *if and only if* $x_0 \in \vartheta f^*(0)$. *If* f *is a convex function on a linear topological space* X *and* f *is continuous at* x_0, *then* $f(x_0)$ *is a minimum of* f *on* X *if and only if* $x_0 \in \partial f^*(0)$. *If* X *is a locally convex linear topological space, then it is supposed that* f *is lower semi-continuous.*

The proof follows immediately from Theorem 14, taking into account the fact that if f is convex, then $\vartheta f(x_0) \neq \emptyset$, and if f is convex and continuous at x_0, then $\partial f(x_0) \neq \emptyset$, in virtue of Theorem 8 and Remark 9a or $f^{**}(x_0) = f(x_0)$, by Remark 9b.

Let us observe that in virtue of (57) we infer that

$$\textit{relation } y_0 \in \vartheta f(x_0) \textit{ implies } x_0 \in \vartheta f^*(y_0),$$

$$\textit{relation } y_0^* \in \partial f(x_0) \textit{ implies } x_0 \in \partial f^*(y_0^*).$$

If $f^{**}(x_0) = f(x_0)$ or $\vartheta f(x_0) \neq \emptyset$, $\partial f(x_0) \neq \emptyset$, then the opposite implication is also true, in virtue of Corollary 4 and Theorem 13*.

Thus, Corollary 5 says that if f is a convex function on a linear space X, then the following three relations are equivalent: $f(x_0)$ is a minimum of f on X; $0 \in \vartheta f(x_0)$ and $x_0 \in \vartheta f^*(0)$. Analogously, if f is a convex function continuous at x_0 on a linear topological space X, then the following three relations are equivalent: $f(x_0)$ is a minimum on X; $0 \in \partial f(x_0)$ and $x_0 \in \partial f^*(0)$. If X is a locally convex linear topological space, then f is assumed to be lower semi-continuous on X.

The existence of continuous Lagrangian multipliers in the duality theorem for programming in linear topological spaces is obtained in section 7 under the assumption that the ordering convex cone has certain structural properties. This result is proved on the basis of Corollary 1*. We shall now investigate the same problem in the case where no restrictions are made concerning the ordering convex cone. Thus, necessary and sufficient conditions are obtained in virtue of the duality theorem based on Theorem 9.

Let X be a locally convex linear topological space with an ordering relation defined by the convex cone K. The functions $M(x)$ and $\psi(y^*)$ are determined by relations (35) and (37), respectively.

LEMMA 5. *The linear continuous functional y_0^* is a Lagrangian multiplier if and only if $0 \in \partial M^*(y_0^*)$.*

Proof. Suppose that y_0^* is a Lagrangian multiplier, i.e.

$$\inf_{u \in U}\{f(u) - \langle g(u), y_0^*\rangle\} = \sup_{y^* \geqslant 0} \inf_{u \in U}\{f(u) - \langle g(u), y^*\rangle\}.$$

It follows from the proof of Theorem 9 that

$$M^*(y^*) = \begin{cases} \infty & \text{if } y^* \text{ is non} \geqslant 0, \\ \psi(y^*) & \text{if } y^* \geqslant 0. \end{cases}$$

Thus, in virtue of (37) we obtain

$$-M^*(y_0^*) = -\psi(y_0^*) = \sup_{y^* \geqslant 0}\{-\psi(y^*)\} = \sup_{y^*}\{-M^*(y^*)\},$$

i.e. $-M^*(y_0^*) = M^{**}(0)$. Hence, we conclude, by Theorem 13*, that $0 \in \partial M^*(y_0^*)$.

Suppose now that $0 \in \partial M^*(y_0^*)$. Then using again Theorem 13* we obtain $-M^*(y_0^*) = M^{**}(0)$. But it follows from the proof of the necessity that the last relation and the assertion that y_0^* is a Lagrangian multiplier are equivalent.

We are now in a position to formulate the following

CRITERION OF THE EXISTENCE OF LAGRANGIAN MULTIPLIERS. *If $\partial M(0) \neq \emptyset$, then every $y_0^* \in \partial M(0)$ is a Lagrangian multiplier. If $\overline{M}(0) = M(0)$ and $\overline{M}(x)$ is convex, then there exists a Lagrangian multiplier if and only if $\partial M(0) \neq \emptyset$.*

Proof. Suppose that $\partial M(0)$ is not empty. Then $y_0^* \in \vartheta M(0)$ implies $0 \in \partial M^*(y_0^*)$ and, in virtue of Lemma 5, y_0^* is a Lagrangian multiplier. Suppose now that y_0^* is a Lagrangian multiplier, then $0 \in \partial M^*(y_0^*)$ and, by Theorem 13*, $M^*(y_0^*) + M^{**}(0) = 0$. But in virtue of Theorem 9 we have $M^{**}(0) = \psi^*(0) = \overline{M}(0)$. Thus, we obtain $M^*(y_0^*) + M(0) = 0$. Hence, we conclude, by Theorem 13, that $y_0^* \in \partial M(0) \neq \emptyset$.

Let X and U be two real linear spaces. For the real-valued function $f(x, u)$ defined on the product space $X \times U$, the point (x_0, u_0) is called a *saddle-point* of f if the following relation is satisfied:

$$(58) \qquad f(x_0, u) \leqslant f(x_0, u_0) \leqslant f(x, u_0) \quad \text{for all } x \text{ of } X \text{ and } u \text{ of } U.$$

For fixed u of U, the linear functional $y_0 \in Y$ defined on X is called the *partial algebraic subgradient* of $f_u(x) = f(x, u)$ at the point (x_0, u) if the following relation holds:

$$f(x, u) - f(x_0, u) \geqslant \langle x - x_0, y_0\rangle \quad \text{for all } x \text{ of } X.$$

The set of all partial algebraic subgradients of $f(x, u)$ at the point (x_0, u) is called the *algebraic partial subdifferential* of $f(x, u)$ at the point (x_0, u) and it is denoted by $\vartheta_x f(x_0, u)$. If X is a linear topological space, then the partial subdifferential $\partial_x f(x_0, u)$ can be introduced in the same way through replacing Y by X^*. Analogously, for fixed x of X, one can introduce the algebraic partial subdifferential $\vartheta_u f(x, u_0)$ of $f(x, u)$ at the point (x, u_0) and the partial subdifferential $\partial_u f(x, u_0)$ of $f(x, u)$ at the point (x, u_0). Similar notions can be introduced for the conjugate functions. Put

$$f_u^*(y, u) = \sup_{x \in X}\{\langle x, y\rangle - f(x, u)\} \quad \text{for } y \text{ of } Y,$$

$$f_x^*(x, v) = \sup_{u \in \mathrm{dom}_U f}\{\langle u, v\rangle + f(x, u)\} \quad \text{for } v \text{ of } V,$$

where V is the linear space of all linear functionals defined on U. For fixed u of U, y_0 of Y and x_0 of X, $x_0 \in \vartheta_y f_u^*(y_0, u)$ means that

$$f_u^*(y, u) - f_u^*(y_0, u) \geqslant \langle x_0, y - y_0 \rangle \quad \text{for all } y \text{ of } Y.$$

Similarly, $x_0 \in \partial_y f_u^*(y_0^*, u)$ means that

$$f_u^*(y^*, u) - f_u^*(y_0^*, u) \geqslant \langle x_0, y^* - y_0^* \rangle \quad \text{for all } y^* \text{ of } X^*.$$

Analogously, for $f_x^*(x, v)$, $u_0 \in \vartheta_v f_x^*(x, v_0)$ means that

$$f_x^*(x, v) - f_x^*(x, v_0) \geqslant \langle u_0, v - v_0 \rangle \quad \text{for all } v \text{ of } V.$$

Similarly, $u_0 \in \partial_v f_x^*(x, v_0^*)$ means that

$$f_x^*(x, v^*) - f_x^*(x, v_0^*) \geqslant \langle u_0, v^* - v_0^* \rangle \quad \text{for all } v^* \text{ of } U^*,$$

where U^* denotes the linear space of all continuous linear functionals on the linear topological space U. Let us observe that $f_x^*(x, v)$ is actually the algebraic conjugate of the function $-f(x, u)$ for fixed x instead of $f(x, u)$. We use this notation, since we are interested in the maximum with respect to u. In virtue of Theorem 14 we obtain the following

THEOREM 15. *If the point (x_0, u_0) is a saddle-point of the function $f(x, u)$, then the following necessary conditions are satisfied: $x_0 \in \vartheta_y f_u^*(0, u_0)$ and $u_0 \in \vartheta_v f_x^*(x_0, 0)$.*

If $\vartheta_x f(x_0, u_0) \neq \emptyset$ and $\vartheta_u f(x_0, u_0) \neq \emptyset$ or if $f(x, u)$ is convex with respect to x and concave with respect to u, then the necessary conditions are also sufficient. If X and U are linear topological spaces, then the following conditions are necessary: $x_0 \in \partial_y f^*(0, u_0)$ and $u_0 \in \partial_v f^*(x_0, 0)$.

If $\partial_x f(x_0, u_0) \neq \emptyset$ and $\partial_u f(x_0, u_0) \neq \emptyset$ or if $f_u(x) = f(x, u)$ is convex and continuous at x_0 and $f_x(u) = f(x, u)$ is concave and continuous at u_0, then the necessary conditions are also sufficient. If X and U are locally convex linear topological spaces, then the continuity condition is replaced by the lower semi-continuity of $f_u(x)$ and $f_x(u)$ on X and U, respectively.

Proof. Since $f(x_0, u_0)$ minimizes $f(x, u_0)$ on X and maximizes $f(x_0, u)$ on U, the proof immediately follows from Theorem 14 and Corollary 5.

For the function $f(x, u)$ let us define

$$g(y, v) = \sup_{u \in \mathrm{dom}_U f} \{\langle u, v \rangle - f_u^*(y, u)\} = \sup_{u \in \mathrm{dom}_U f} \inf_{x \in \mathrm{dom}_X f} \{f(x, u) + \langle u, v \rangle - \langle x, y \rangle\}. \tag{59}$$

Then we have

$$g(y, 0) = \sup_u \inf_x \{-\langle x, y \rangle + f(x, u)\} = \sup_u \{-f_u^*(y, u)\} = -h(y). \tag{60}$$

Put

$$k(v) = g(0, v) = \sup_u \{\langle u, v \rangle - f_u^*(0, u)\} = [f_u^*(0, \cdot)]^*(v). \tag{61}$$

THEOREM 16. *If (x_0, u_0) is a saddle-point of $f(x, u)$, then the following conditions are necessary:*

$$x_0 \in \vartheta h(0) \quad \text{and} \quad u_0 \in \vartheta k(0). \tag{62}$$

If $f(x, u)$ is convex with respect to x and concave with respect to u, then the necessary conditions are also sufficient. If X and U are linear topological spaces, then the following conditions are necessary:

$$x_0 \in \partial h(0) \quad \text{and} \quad u_0 \in \partial k(0). \tag{63}$$

If $f(x, u)$ is convex with respect to x and concave with respect to u and, in addition, $f_u(x) = f(x, u)$ is continuous at x_0 and $f_u^*(0, u)$ is continuous at u_0, then necessary conditions (63) are also sufficient.

Proof. We shall prove that conditions (62) are necessary. It follows from the second inequality of (59) that

$$f_u^*(0, u_0) = \sup_x \{-f(x, u_0)\} = -f(x_0, u_0).$$

Since $f(x_0, u) + f_u^*(0, u) \geqslant \langle 0, x_0 \rangle = 0$, it follows from the first inequality of (58) that

$$f_u^*(0, u) \geqslant -f(x_0, u) \geqslant -f(x_0, u_0).$$

Thus, we obtain

$$h(0) = \inf_u f_u^*(0, u) = f_u^*(0, u_0) = -f(x_0, u_0).$$

Hence, it follows, in virtue of (60), (58),

$$\begin{aligned} -h(y) = g(y, 0) &= \sup_u \inf_x \{-\langle x, y \rangle + f(x, u)\} \\ &\leqslant \sup_u \{-\langle x_0, y \rangle + f(x_0, u)\} = -\langle x_0, y \rangle + f(x_0, u_0) \\ &= -\langle x_0, y \rangle - h(0). \end{aligned}$$

Thus

$$h(y) - h(0) \geqslant \langle x_0, y \rangle, \quad \text{i.e. } x_0 \in \vartheta h(0). \tag{64}$$

Similarly, we obtain in virtue of (59), (58),

$$\begin{aligned} k(v) = g(0, v) &= \sup_u \inf_x \{\langle u, v \rangle + f(x, u)\} \\ &\geqslant \inf_x \{\langle u_0, v \rangle + f(x, u_0)\} = \langle u_0, v \rangle + f(x_0, u_0). \end{aligned}$$

But, in virtue of (61), $k(0) = -h(0) = f(x_0, u_0)$. Hence, we obtain

$$k(v)-k(0) \geqslant \langle u_0, v\rangle, \quad \text{i.e. } u_0 \in \vartheta k(0). \tag{65}$$

Let us prove that conditions (62) are sufficient. Since the algebraic conjugate function $f_u^*(y, u)$ is convex, by Lemma 4, it follows from Theorem 8 and (61) that

$$f_u^*(0, u_0) = \sup_v \{\langle u_0, v\rangle - [f_u^*(0, \cdot)]^*(v)\}$$
$$= \sup_v \{\langle u_0, v\rangle - g(0, v)\} = \sup_v \{\langle u_0, v\rangle - k(v)\} \leqslant -k(0),$$

in virtue of (65). Thus, we have

$$f_u^*(0, u_0) = -k(0) = h(0).$$

Since $f_u^*(y, u) \geqslant h(y)$, relation (64) implies

$$f_u^*(y, u) - f_u^*(0, u_0) \geqslant \langle x_0, y\rangle \tag{66}$$

for arbitrary (y, u). Since $f(x, u)$ is convex with respect to x, we obtain, by Theorem 8,

$$f(x_0, u) = \sup_y \{\langle x_0, y\rangle - f_u^*(y, u)\} \leqslant -f_u^*(0, u_0),$$

by relation (66). For $u = u_0$, the last relation yields $f(x_0, u_0) = -f_u^*(0, u_0)$. Hence, $f(x_0, u) \leqslant -f_u^*(0, u_0) = f(x_0, u_0)$. Since $f_u^*(0, u_0) = -\inf_x f(x, u_0)$, we have

$$f(x, u_0) \geqslant -f_u^*(0, u_0) = f(x_0, u_0)$$

for arbitrary x of X and relation (58) is satisfied.

If X and U are linear topological spaces, then replacing Theorem 8 by Remark 9a the proof remains exactly the same.

In the case where (X, Y) and (U, V) are two pairs of dual spaces, Theorem 16 is proved by Joffe and Tihomirov ([7], p. 80).

Let us observe that the functions $h(y)$ and $k(v)$ involved in the assumptions of Theorem 16 are not symmetric with regard to the variables x and u. We shall show that it is possible to formulate a theorem on saddle-points of $f(x, u)$ so that the corresponding functions will be symmetric. For this purpose let us put

$$f_x^*(x, v) = \sup_{u \in \mathrm{dom}_U f} \{\langle u, v\rangle + f(x, u)\}, \quad v \in V, \tag{67}$$

$$l(v) = \inf_{x \in \mathrm{dom}_X f} f_x^*(x, v) = \inf_{x \in \mathrm{dom}_X f} \sup_{u \in \mathrm{dom}_U f} \{\langle u, v\rangle + f(x, u)\}. \tag{68}$$

In the case of linear topological spaces X and U the space V of linear functionals is replaced by U^*.

THEOREM 17. *If (x_0, u_0) is a saddle-point of $f(x, u)$, then the following conditions are necessary:*

$$x_0 \in \vartheta h(0) \quad \text{and} \quad u_0 \in \vartheta l(0). \tag{69}$$

If $f(x, u)$ is convex with respect to x and concave with respect to u, then the necessary conditions are also sufficient. If X and U are linear topological spaces, then the following conditions are necessary:

$$x_0 \in \partial h(0) \quad \text{and} \quad u_0 \in \partial l(0). \tag{70}$$

If $f(x, u)$ is convex with respect to x and concave with respect to u and, in addition, $f_u(x) = f(x, u)$ is continuous at x_0 and $f_x(u) = f(x, u)$ is continuous at u_0, then the necessary conditions (69) are also sufficient.

Proof. If (x_0, u_0) is a saddle-point of $f(x, u)$, then using the same argument as in the proof of Theorem 16 we infer that condition (64) is satisfied. Repeating this argument for the function $-f(x, u)$ we obtain

$$u_0 \in \vartheta l(0), \quad \text{i.e. } l(v) - l(0) \geqslant \langle u_0, v\rangle. \tag{71}$$

The proof that conditions (69) are satisfied in the case of linear topological spaces X and U is exactly the same. Let us prove that conditions (69) are sufficient. If condition (64) is satisfied, then we have, by (55), for all y, u,

$$f_u^*(y, u) - h(0) \geqslant \langle x_0, y\rangle, \quad \text{i.e. } \langle x_0, y\rangle - f_u^*(y, u) \leqslant -h(0).$$

Hence, using Theorem 8 we obtain

$$\sup_y \{\langle x_0, y\rangle - f_u^*(y, u)\} = f(x_0, u) \leqslant -h(0).$$

In particular, we have for $y = 0$

$$-f_u^*(0, u) \leqslant f(x_0, u) \leqslant -h(0) \quad \text{for all } u \text{ of } U. \tag{72}$$

Thus, we have, by (55) and (67),

$$-h(0) \leqslant f_x^*(x_0, 0) \leqslant -h(0)$$

and, by (72),

$$-f_u^*(0, u_0) \leqslant f(x_0, u_0) \leqslant -h(0) = f_x^*(x_0, 0). \tag{73}$$

On the other hand, relation (71) implies, by (68),

$$f_x^*(x, v) - l(0) \geqslant \langle u_0, v\rangle, \quad \text{i.e. } \langle u_0, v\rangle - f_x^*(x, v) \leqslant -l(0)$$

for all x, v. Hence, applying Theorem 8 to the last inequality we obtain

$$\sup_v \{\langle u_0, v\rangle - f_x^*(x, v)\} = -f(x, u_0) \leqslant -l(0)$$

for arbitrary x and v, and for $v = 0$ we have

$$-f_x^*(x, 0) \leqslant -f(x, u_0) \leqslant -l(0).$$

Hence, it follows from (68) that

$$-l(0) \leqslant f_u^*(0, u_0) \leqslant -l(0).$$

Thus, we infer from (73) that

(74) $$l(0) = -f_u^*(0, u_0) \leqslant f(x_0, u_0) \leqslant f_x^*(x_0, 0) = -h(0).$$

Hence, it follows from (68) and (60) that

$$\inf_x \sup_u f(x, u) = l(0) \leqslant -h(0) = \sup_u \inf_x f(x, u).$$

Since the opposite inequality is evident, we conclude that $l(0) = -h(0)$ and

$$-f_u^*(0, u_0) = f(x_0, u_0) = f_x^*(x_0, 0).$$

Thus, inequalities (58) result from the relations

$$-f_u^*(0, u_0) = \inf_x f(x, u_0) \quad \text{and} \quad f_x^*(x_0, 0) = \sup_u f(x_0, u).$$

If X and Y are linear topological spaces, then on replacing Theorem 8 by Remark 9 the proof remains exactly the same.

Remark 10. If X and U are locally convex linear topological spaces, then we assume that $f_u(x) = f(x, u)$ and $f_x(u) = f(x, u)$ are lower semi-continuous on X and U, respectively. The proof remains without change by using Remark 9b. This remark is valid for Theorem 16 as well as for Theorem 17.

Other properties of conjugate functions as well as some applications will be discussed separately.

APPENDIX

Let X be a linear space containing the convex cone K. The order relation $\geqslant$ defined in X by the cone K means that $x \geqslant y$ $(y \leqslant x)$ if and only if $x - y \in K$, where $x, y \in X$. Thus, $x \geqslant 0$ is equivalent to $x \in K$. If y^* is a linear functional on X, then $y^* \geqslant 0$ means that $y^*(x) \geqslant 0$ for all x of K.

Definition ([3], p. 410). If M is the subset of the linear space X, then $p \in M$ is called an *internal point* of M if, for each $x \in X$, there exists an $\varepsilon > 0$ such that $p + \delta x \in M$ for $|\delta| \leqslant \varepsilon$.

Basic separation theorem ([3], p. 412). *Let M and N be disjoint convex subsets of a linear space X, and let M have an internal point. Then there exists a non-zero linear functional f which separates M and N.*

The separation theorem in linear topological spaces ([3], p. 417). *In a linear topological space, any two disjoint convex sets, one of which has an interior point, can be separated by a non-zero continuous linear functional.*

Definition ([3], p. 26). A partially ordered set $(D, \leqslant)$ is said to be *directed* if every finite subset of D has an upper bound. A map $f\colon D \to X$ of a directed set D into a set X is called a *generalized sequence* of elements in X. If $f\colon D \to X$ is a generalized sequence in the topological space X, it is said to *converge* to the point p in X, if to every neighbourhood N of p there corresponds a $d_0 \in D$ such that $d \geqslant d_0$ implies $f(d) \in N$.

Definition. A convex set W in a real linear space L is called a *wedge* if $tW \subset W$ for arbitrary $t \geqslant 0$.

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