Since $n \in N\big(S^*(I+|T|)^2\big)$, if follows that

$$\langle (I+|T|)Su, (I+|T|)(Sz-Sw)\rangle = 0 \quad \text{for all } u \in D(|T|S).$$

Taking $u = z-w$ and recalling that $I+|T|$ is 1-1, we may conclude that $Sz = Sw$. Therefore

$$\big((I+|T|)S\big)^*|T|v = S^*(I+|T|)^2 Sw = S^*(I+|T|)^2 Sz = S^*(I+|T|)^2 d$$
$$= S^*(I+|T|)|T|v = S^*|T|(I+|T|)v.$$

Thus (iii) of lemma 11 holds.

13. Theorem. *Let T and S be closed densely defined linear operators from Hilbert space H into H. If $S^*(I+|T|)$ is closed, then $(TS)^* = S^*T^*$.*

Proof. Since $I+|T|$ is self-adjoint and surjective and $S^*(I+|T|)$ is closed, it follows from theorem 1 that

$$S^*(I+|T|) = \big(S^*(I+|T|)\big)^{**} = \big((I+|T|)S\big)^*.$$

In particular, (iii) of lemma 11 holds. Hence $(TS)^* = S^*T^*$.

References

[1] N. Dunford and J. Schwartz, *Linear operators*, part II, New York 1963.
[2] S. Goldberg, *Unbounded linear operators*, New York 1966.
[3] M. Schechter, *The conjugate of a product of closed operators* (to appear).

KATHOLIEKE UNIVERSITEIT, NIJMEGEN
UNIVERSITY OF MARYLAND AND IMPERIAL COLLEGE, LONDON

STUDIA MATHEMATICA, T. XXXVIII. (1970)
Colloquium on
Nuclear Spaces and Ideals in Operator Algebras

Linear operations, tensor products, and contractive projections in function spaces*

by

M. M. RAO (Pittsburgh)

Contents

0. INTRODUCTION

One of the main purposes of this paper is to characterize all the subspaces of general Banach function spaces admitting contractive projections onto them, and to extend some of the results when the functions

* This research was partly supported under the NSF Grants, GP-8777, GP-15632 and the Air Force Grant AFOSR-69-1647, and was largely carried out while the author was visiting the Mathem. Institut der Universität, Wien, during the Spring semester of 1969.

are also vector-valued. For this, however, it will be necessary to consider the adjoint spaces and analyze their structure. Consequently, this latter study occupies the first chapter, the work there having independent interest. The projection problem itself will be solved in Chapter II, using the preceding work. The results can be explained briefly as follows.

Let L^ϱ be the subspace of measurable scalar functions f on a general measure space (Ω, Σ, μ) such that $\varrho(f) < \infty$, where $\varrho(\cdot)$ is a *function norm*, i.e., (i) it is a norm, (ii) $\varrho(f) = \varrho(|f|)$, and (iii) $0 \leqslant f_1 \leqslant f_2$, a.e., implies $\varrho(f_1) \leqslant \varrho(f_2)$. Such a space L^ϱ is termed a *normed function space*, or, if complete, a *Banach function space* (BFS). (Other names are a *normed Riesz space*, or a *normed Köthe-Toeplitz space*.) Recently Gretsky [13] has analyzed the adjoint space $(L^\varrho)^*$ of L^ϱ and of certain subspaces of $B(L^\varrho, \mathscr{X})$, the space of bounded linear operators on L^ϱ to a Banach (or B-) space $\mathscr{X}$, assuming (a) μ is σ-finite, and (b) certain conditions called (I) and (J) on the norm ϱ. The aim of Chapter I, in part, is to complete and complement [13] without these conditions, to represent $B(L^\varrho, \mathscr{X})$ itself, and to give some results on the tensor products $L^\varrho \otimes_\gamma \mathscr{X}$ and $L^\varrho \otimes_\lambda \mathscr{X}$, where $\otimes_\gamma$ and $\otimes_\lambda$ are the greatest and least crossnorms, [33]. These results are needed for the work on projections. The basic theory of [20] and [21] (cf. also [39], Ch. 15), always assumes the σ-finiteness of μ and a few of the results (to be used later) will be proved here for a general μ.

Briefly then, Chapter I includes the following. After giving a characterization of $(L^\varrho)^*$, the most general condition for the reflexivity of L^ϱ-spaces is obtained in Section 1.2. Then integral representation of $B(L^\varrho, \mathscr{X})$ relative to vector measures and some characterizations of (weakly) compact operators are given in the next two sections. It is then shown that both L^ϱ and $(L^\varrho)^*$ enjoy the metric approximation property, i.e., the identity operator can be approximated uniformly by the degenerate ones on precompact sets. The final section contains tensor products of L^ϱ and a B-space $\mathscr{X}$, subsuming some results in [14]. It is shown that $L^\varrho \otimes_\gamma \mathscr{X} \subset L^\varrho_{\mathscr{X}} \subset W^\varrho_{\mathscr{X}} = L^\varrho \otimes_\lambda \mathscr{X}$, where $L^\varrho_{\mathscr{X}}$ is the L^ϱ-space of strongly measurable $\mathscr{X}$-valued f on Ω, with $\varrho(f) = \varrho(|f|)$, $|\cdot|$ being the norm of $\mathscr{X}$, and $W^\varrho_{\mathscr{X}}$ is the closure of weakly measurable $\mathscr{X}$-valued functions, under a natural norm. If, in particular, $L^\varrho = L^p, 1 < p \leqslant \infty$, then $L^\varrho \otimes_\gamma \mathscr{X}$ is also dense in $L^\varrho_{\mathscr{X}}$ and equality holds if $L^\varrho = L^1$. Some of the results of this chapter were given in [30], without details of proofs.

The projection problem is considered in Chapter II. This seems to have been first treated in [15] if $L^\varrho = L^1$ (using certain general theorems on the injective envelopes of B-spaces), and a more detailed analysis of the same case, on a finite measure space, has been considered in [8] independently of [15]. These results were complemented for the L^p-spaces, with $0 < p < \infty$ in [1], $\mu(\Omega) < \infty$, and they were extended for the L^Φ-spaces of Orlicz, for σ-finite μ, with $\Phi(2x) \leqslant C\Phi(x), x \geqslant 0$, in [29].

The result for the L^p, $p \geqslant 1$, has been extended in ([19], p. 309) when μ is σ-finite, and this was recently generalized for arbitrary μ in [36]. The present paper contains all these results, in the context of BFS' s, and the final characterization is given in Theorem 2.10 for general measure spaces. The latter was extended, through the theory of tensor products above, to certain $L^\varrho_{\mathscr{X}}$ spaces. ($L^1_{\mathscr{X}}$ was treated in [15].) The solution is based on several reductions of the measure spaces, using the isomorphism results of [34], in addition to the work of Chapter I. The first three sections of Chapter II, contain all this work. The uniqueness of projections onto subspaces, and their form were discussed in Section 2.4. The projection problem has many important applications. This is already clear from the work of [19]. In the non-linear prediction (and approximation) theory the prediction operator P relative to a Tshebyshev subspace $\mathscr{M} \subset L^\varrho$ is linear iff (= if and only if) $I - P = Q$ is a contractive projection, and the above characterizations will be useful in this work. This application is treated in Section 2.5. Finally some remarks and open problems are included. An account of some of the results of Chapter II was given in [31] without complete details.

I. LINEAR OPERATIONS AND TENSOR PRODUCTS

1.1. Linear functionals. Let L^ϱ be a normed function space on (Ω, Σ, μ) as described in the Introduction. Without further conditions on ϱ, the space L^ϱ is not complete (cf. [21], Example 4.9). As shown in [21] or [39], μ can be arbitrary, L^ϱ is complete iff ϱ verifies the triangle inequality for infinite sums — called the *Riesz-Fischer property*. This is not essential for much of the work below, and it will not be assumed without mention.

Let $\Sigma_0 = \{A \in \Sigma\colon \varrho(\chi_A) < \infty\}$ and $M^\varrho = \overline{sp}\{f \in L^\varrho\colon f$ bounded with support in $\Sigma_0\} = \{f \in L^\varrho\colon f$ is Σ_0-step function$\}$, where χ_A is the indicator of A. In the analysis of $(L^\varrho)^*$ it is necessary to consider two cases: (i) the elements that annihilate M^ϱ, denoted $(M^\varrho)^\perp$, and (ii) that do not, i.e. of $(L^\varrho)^*/(M^\varrho)^\perp$, the quotient space which is isometrically isomorphic to $(M^\varrho)^*$ (cf. [9], II. 4. 18). For L^p-spaces, $1 \leqslant p \leqslant \infty$, $(M^\varrho)^\perp = \{0\}$, and is already non-trivial for the Orlicz spaces (cf. [26]), and thus also for the L^ϱ-spaces. After considering (i) and (ii), the general structure of $(L^\varrho)^*$ will be determined below. The following condition was abstracted from [26] and used in [13]:

CONDITION (I). If $\tilde{L}^\varrho = \{f \in L^\varrho\colon f = \max_{1 \leqslant i \leqslant n}(f_i),\ \varrho(f_i) \leqslant 1,\ f_i \geqslant 0,\ n \geqslant 1\}$, then, in terms of the natural map $F\colon L^\varrho \mapsto L^\varrho/M^\varrho = N^\varrho$ (say), the space L^ϱ (or the norm ϱ) verifies the condition that $F(L^\varrho)$ is contained in the closed unit ball of N^ϱ, where the norm of N^ϱ is:

$$(+)\qquad d(\hat{f}) = \inf\{\varrho(f+g):\ g\in M^\varrho\},\qquad \hat{f} = f+M^\varrho \in N^\varrho.$$

In what follows, another aspect of [26] and certain other properties of $M^\perp$ will be used, without condition (I). However, the primary purpose of this section is only to outline this generalization, and thus show that the results of [13] hold. *Even if the above condition* (I) *is assumed, and the* L^ϱ*-space is understood with this added restriction, the rest of this paper contains a novel contribution and complements* [26] *and* [13]. The material of this section will be used constantly, and so the details are sketched, for completeness.

Definition 1.1. Let $B_{\varrho'}(\mu)$ be the set of purely finitely additive (pfa) set functions $\nu: \Sigma \mapsto$ scalars, vanishing on μ-null sets and with supports in the support of some $f\in L^\varrho - M^\varrho$ (set theoretic difference) such that ν has finite variation, $|\nu|(\Omega) < \infty$. (About pfa set functions, also called *pure changes*, see [9] and [39].)

The aim here is to characterize $(M^\varrho)^\perp$ to be isometrically equivalent to $B_{\varrho'}(\mu)$. This is presented in a series of lemmas.

LEMMA 1.2. *Every element of* $(M^\varrho)^\perp$ *defines an element of* $B_{\varrho'}(\mu)$, *i.e., if* $z^*\in(M^\varrho)^\perp$, *then there exists a* $\nu\in B_{\varrho'}(\mu)$ *and an* $f_0\in L^\varrho - M^\varrho$ *such that* ν *has support in that of* f_0 *and* $\nu(A) = z^*_A(f_0) = z^*(f_0\chi_A)$ *for all* $A\in\Sigma$.

Proof. Since M^ϱ is a vector lattice, so is $(M^\varrho)^\perp$ and hence it suffices to consider $z^*\geqslant 0$. (Here and below a vector lattice always means, the real functions from such, and it is self-adjoint.) It may be supposed $\|z^*\| > 0$. Then there exists $0\leqslant f_0 \in L^\varrho - M^\varrho$, with $\|z^*\| = z^*(f_0)$ (and $d(\hat{f}_0)$ is arbitrarily close to 1). Define $\nu(A) = z^*_A(f_0) = z^*(f_0\chi_A) \geqslant 0$. Then $\nu: \Sigma\mapsto R^+$, is additive, vanishes on μ-null sets and $\nu(\Omega) = \|z^*\|$. Since clearly the support of ν is in that of f_0, it is only necessary to show that it is pfa. For this by ([39], p. 57) it suffices to exhibit $A_n\in\Sigma$, $A_n\downarrow\emptyset$, and $\nu(A_n) = \nu(\Omega)$, for all n.

Let $0\leqslant f_n\uparrow f_0$, $f_n\in M^\varrho$. In fact, by the structure theorem there exist simple f_n such that $0\leqslant f_n \uparrow f_0$ and, since $f_0\in L^\varrho$ implies $f_n\in L^\varrho$ and, hence $f_n\in M^\varrho$. If B_n is now the support of f_n, and $E_n = \{\omega: f_0\chi_{B_n} < n\}$, then $E_n\in\Sigma_0$, $E_n\uparrow \operatorname{supp}(f_0)$, and let $A_n = \operatorname{supp}(f_0) - E_n$, with $A_0 = \Omega$. So $A_n\downarrow\emptyset$, and $f_0 - f_0\chi_{A_n} = f_0\chi_{E_n}\in M^\varrho$. Then

$$\nu(\Omega) = z^*(f_0) = z^*(f_0 - f_0\chi_{E_n}) = z^*(f_0\chi_{A_n}) = \nu(A_n),$$

since $z^*\in(M^\varrho)^\perp$. Thus ν is pfa, and the result follows. (That ν is determined uniquely by z^* above, will follow later.)

The general results on quotient spaces imply the following (cf. [9], and a discussion in [26], p. 559):

LEMMA 1.3. *Let* N^ϱ, *with norm* $d(\cdot)$, *be as in* (+). *Then* $(N^\varrho)^*$ *and* $(M^\varrho)^\perp$ *are isometrically isomorphic (written* $(N^\varrho)^* \simeq (M^\varrho)^\perp$), *where* N^ϱ *inherits the natural ordering from* L^ϱ, *in terms of which it is also a vector lattice and hence so is* $(N^\varrho)^*$. *The correspondence* $j: x^*\mapsto z^*\in(N^\varrho)^*$ *for* $x^*\in(M^\varrho)^\perp$ *and* $\hat{f} = f+M^\varrho\in N^\varrho$ *is given by* $z^*(\hat{f}) = x^*(f)$.

For the converse implication that every $\nu\in B_{\varrho'}(\mu)$ defines an element of $(N^\varrho)^*$, an integral has to be defined. So let $\hat{0}\leqslant\hat{f}\in N^\varrho$ and $0\leqslant\nu\in B_{\varrho'}(\mu)$, $d(f) > 0$. Then using an argument in ([26], p. 571) verbatim, where no special property of Orlicz spaces, other than the fact that the norm is a function norm, was used, one can map N^ϱ onto a (generally proper) subspace of $L^\infty(\Sigma)$ such that $\hat{f}\leftrightarrow\tilde{f}\in L^\infty$ are in correspondence, and $d(\hat{f}) = \|\tilde{f}\|_\infty$ so that it is an isometry. (The $\tilde{f}$ was $\tilde{f}_\pi$ in [26], where π is a partition. Since $\tilde{f}_\pi$ there is monotone, one can consider a cofinal sequence π_i and $\tilde{f}_{\pi_i}$ tend monotonely to $\tilde{f}$, and this limit is seen not to depend on the sequence, since two such sequences can be combined to yield the same limit.) However, using the same partitions π, on the other hand, one can define an integral for the step $\tilde{f}_\pi$ and ν, and this limit exists uniquely (as π's are refined) and the thus defined functional x^* is given by

$$(1)\qquad x^*(\hat{f}) = \int_\Omega F(f)\,d\nu,\qquad \hat{0}\leqslant\hat{f}\in N^\varrho,\ f\in\hat{f}.$$

$x^*(\cdot)$ is uniquely defined, and is linear as shown in [26]. The use of this kind of representation will be further pointed out in Remark 1.6 below, in the context of L^p-spaces as it throws more light on the problem.

The following result, proved in ([26], p. 572) and in ([13], p. 36-39), holds here without change:

LEMMA 1.4. *The functional* $x^*(\cdot)$ *of* (1) *satisfies*: (i) $x^*\in(N^\varrho)^*$, (ii) $0\leqslant x^*(\hat{f})\leqslant d(\hat{f})\nu(\Omega)$, $\hat{0}\leqslant\hat{f}\in N^\varrho$, *and* (iii) $\|x^*\| = \nu(\Omega)$.

The only point not mentioned in the above references is

$$\nu(\Omega) = \sup\Big\{\Big|\int_\Omega F(f)\,d\nu\Big|:\ d(\hat{f})\leqslant 1\Big\},$$

which is a consequence of the definition of the integral there, and the isometry noted earlier. (See, in particular, the remark on p. 574 of [26].)

Identifying $(N^\varrho)^*$ and $(M^\varrho)^\perp$, as given by Lemma 1.3, the following result will now be established:

PROPOSITION 1.5. *Let* $0\leqslant z^*\in(N^\varrho)^* = j(M^\varrho)^\perp$, *and let* ν *be the corresponding pfa given by Lemma* 1.2. *Then* ν *is uniquely determined by* z^* *and if* x^* *is the functional determined by this* ν *through* (1), *then* $x^* = z^*$. *Moreover,* $(N^\varrho)^*$ *is an* (AL)*-space, when its real elements are considered.*

Proof. By Lemmas 1.2 and 1.4, $z^*(\hat{f}_0) = \|z^*\| = \nu(\Omega) = \|x^*\|$, and $z^*(\widehat{f_0\chi_E}) = z^*_E(\hat{f}_0) = \nu(E)$. Let $\tau: \hat{f}\mapsto\tilde{f}$ be the isometric (and order preserving) map considered above, and let $\mathscr{S} = \tau(N^\varrho) \subset L^\infty$ be the closed subspace. Then $\hat{\tau}: z^* \mapsto \hat{z} \in (\mathscr{S})^*$ defined by $\hat{\tau} \circ z^*(f) = \hat{z}(\tau \hat{f})$, is an isometry and order preserving. Since z^* determines a pfa set function on (Ω, Σ, μ), so is $\hat{z} \in (\mathscr{S})^*$ and thus the latter is equivalent to a closed subspace of the pfa set functions $\mathscr{P}(\mu)$, in $ba\,(\Omega, \Sigma, \mu)$, of bounded additive set functions on Σ; vanishing on μ-null sets. If now $\nu \in \mathscr{P}(\mu)$ is determined by z^*, then the above mappings yield $\nu \leftrightarrow \hat{z} \in (\mathscr{S})^*$, and since $(\mathscr{S}^*) \subset (L^\infty)^*$, the known properties of the latter space yield $z_E^* \leftrightarrow \hat{z}_E$ and so $\|\hat{z}_E\| = \nu(E) = z_E^*(f_0)$. Since $\|\hat{z}_E\| = \|z_E^*\|$ also holds, by the above correspondence, one deduces $\|z_E^*\| = z_E^*(f_0) = \nu(E)$, $E \in \Sigma$, and ν is determined only by $z^* \in (N^\varrho)^*$. Since $(\mathscr{S})^*$ is an (AL)-space the same must be true of $(N^\varrho)^*$ due to the order preserving property of $\hat{\tau}$.

Next let $x^* \geqslant 0$ be given by (1). Then by definition of (1),

$$x^*(\hat{f}) = \lim_\pi \sum_{i=1}^n d(\hat{f\chi_{E_i}})\nu(E_i) = \lim_\pi \sum_{i=1}^n \|z_{E_i}^*\| d(\widehat{f z_i}) \geqslant z^*(\hat{f}).$$

Since $\hat{0} \leqslant \hat{f} \in N^\varrho$ is arbitrary, $x^* \geqslant z^* \geqslant 0$, and $\|x^*\| = \|z^*\| = \nu(\Omega)$. Thus $\|x^* - z^*\| = \|x^*\| - \|z^*\| = 0$, so $x^* = z^*$, as desired.

Remark 1.6. In the definition of the mappings $\tau: N^\varrho \mapsto \mathscr{S} \subset L^\infty$, and $\hat{\tau}: (N^\varrho)^* \mapsto (\mathscr{S})^*$, the fact that $(N^\varrho)^* = (M^\varrho)^\perp$, where M^ϱ is the particular vector lattice was needed only in identifying $(\mathscr{S})^*$ with a subspace of pfa set functions in $ba(\Omega, \Sigma, \mu)$. Otherwise the procedure works for any vector sublattice of L^ϱ. Thus, if M_1^ϱ is any other vector sublattice, the procedure still yields $(N_1^\varrho)^*$ being identified with a subspace $(\mathscr{S}_1)^*$ of $ba(\Omega, \Sigma_0, \mu)$ where the norm of $\nu \in (\mathscr{S}_1)^*$ is calculated using the correspondence. Thus

$$|\nu|(\Omega) = \sup\Big\{\Big|\int_\Omega F(f)\,d\nu\Big| : d(\hat{f}) \leqslant 1\Big\}.$$

For instance, if $L^\varrho = L^p$, $1 \leqslant p < \infty$, and $M_1^p = \{0\}$, so $N_1^p = L^p$ itself, (1) says for x^* $(q^{-1} + p^{-1} = 1)$

$$x^*(\hat{f}) = \int_\Omega F(f)\,d\nu, \quad \hat{f} = f \in L^p, \ \nu \in ba(\Omega, \Sigma, \mu),$$

and

$$\|x^*\| = |\nu|(\Omega) = \sup\Big\{\Big|\int_\Omega f\,d\nu\Big| : d(\hat{f}) \leqslant 1\Big\}.$$

If $d\nu/d\mu = g$ exists, then $|\nu|(\Omega) = \|g\|_q$. If $p = 1$, μ is arbitrary, then g does not exist as a measurable function. Still the above representation is meaningful. *In this point of view, however, only the subspace $\mathscr{S} \subset L^\infty(\Sigma)$ is used even in finding the norms of ν.* In the L^p-example $\mathscr{S} \subsetneqq L^\infty$ even if $\mu(\Omega) < \infty$.

Since an arbitrary $z^* \in (N^\varrho)^*$ and $\nu \in B_{\varrho'}(\mu)$ can be (Jordan-) decomposed into (four) positive parts and added, by linearity (1) can be extended. Using the preceding lemmas, the general result is given by the following

THEOREM 1.7. *If $z^* \in (N^\varrho)^*$, then there is a unique $\nu \in B_{\varrho'}(\mu)$, such that*

$$z^*(\hat{f}) = \int_\Omega F(f)\,d\nu, \quad f \in \hat{f} \in N^\varrho, \tag{2}$$

and

$$\|z_A^*\| = |\nu|(A), \quad A \in \Sigma. \tag{3}$$

Thus, $(N^\varrho)^*$ is an (AL)-space when real elements are considered.

Remark. Without using the isometric mapping $\tau: N^\varrho \mapsto \mathscr{S} \subset L^\infty$, a direct characterization of $(N^\varrho)^*$ was also proved in [26] for the Orlicz spaces. That method extends also for the L^ϱ-spaces when condition (I) is assumed, as shown in [13]. The same does not seem to work for all L^ϱ-spaces without some such condition. Note also that the singular functionals ($\in (M^\varrho)^\perp$) have arbitrarily small supports.

To obtain a representation of $(L^\varrho)^*$, a new norm ϱ' for set functions is needed. This is somewhat different from one in [13], since similarly defined ϱ' on point functions can be identically zero, if μ is arbitrary.

Definition 1.8(a). Let $A_{\varrho'}(\Omega, \Sigma_0, \mu)$, or $A_{\varrho'}(\mu)$, be the class of all additive set functions $G: \Sigma_0 \mapsto$ scalars, vanishing on μ-null sets and such that $\varrho'(G) < \infty$ where

$$\varrho'(G) = \sup\Big\{\Big|\int_\Omega f\,dG\Big| : \varrho(f) \leqslant 1, f \in M^\varrho\Big\}. \tag{4}$$

The integrals here, relative to finitely additive set functions, are taken in the sense of ([9], Ch. III, or [2]).

(b) The norm ϱ is said to be *continuous at zero*, if for every $\varepsilon > 0$, there is a $\delta_\varepsilon > 0$ such that $\mu(E) < \delta_\varepsilon$ implies $\varrho(\chi_E) < \varepsilon$. ϱ has the *Fatou property* iff $0 \leqslant f_n \uparrow f$ implies $\varrho(f_n) \uparrow \varrho(f)$.

Clearly ϱ' is a norm and $A_{\varrho'}(\mu)$ is a normed linear space. Moreover, if $dG = g\,d\mu$, then $\varrho'(G) = \varrho'(g)$, as in [20], and this gives the earlier definition. Hereafter, integrals relative to finitely additive (scalar or vector) set functions will be often used without checking the definitions of [9] or of [2]. In fact only the linearity and positivity of the maps will be used, and thus any reasonable definition will serve the purpose. For a lucid account of how such integration processes can be developed and employed, a reference should be made to [23].

THEOREM 1.9. *For each $x^* \in (M^\varrho)^*$, there is a unique $G \in A_{\varrho'}(\mu)$ such that*

$$x^*(f) = \int_\Omega f\,dG, \quad f \in M^\varrho, \quad \|x^*\| = \varrho'(G). \tag{5}$$

Moreover, if $\varrho(\cdot)$ is continuous at 0, then G is μ-continuous. Thus $A_{\varrho'}(\mu)$ is complete in any case, (being an adjoint space).

Proof. If $f = \chi_E$, $E \in \Sigma_0$, and $G(E) = x^*(\chi_E)$, then $G: \Sigma_0 \mapsto$ scalars, is an additive set function, and (5) holds, by linearity, for any simple $f \in M^\varrho$. Since such functions are dense in M^ϱ, by definition, then (5) obtains by continuity to all of M^ϱ provided $G \in A_{\varrho'}(\mu)$, from the general definition of the integral (as in [9] or [2]). Now

$$\varrho'(G) = \sup\{|x^*(f)|: \varrho(f) \leqslant 1,\ f \text{ simple}\} = \|x^*\| < \infty,$$

so that $G \in A_{\varrho'}(\mu)$. (The definition of the integral, with [2], was checked in [13], for a σ-finite μ.)

For the last part, let $G \neq 0$, and for any $\varepsilon > 0$, choose $A \in \Sigma$ with $(\mu A) < \delta_\varepsilon$ implying $\varrho(\chi_A) < (\varepsilon/\varrho'(G))$. It follows that

$$|G(A)| = \left|\int_\Omega \chi_A\, dG\right| \leqslant \varrho(\chi_A)\cdot\varrho'(G) \leqslant \varepsilon,$$

and the result obtains.

Remark. If ϱ is continuous at zero and μ has the *finite subset property* (i.e. every set of positive μ-measure has a subset of positive finite μ-measure, called FSP hereafter), then the last part implies that G is actually σ-additive. This follows from Lemma 1.15 below. If, moreover, μ is localizable (cf. [34] and [39]), then $g = dG/d\mu$, the Radon-Nikodým derivative, exists and $g \in L^{\varrho'}$, the BFS with norm ϱ', now existing non-trivially on point functions. If only μ has FSP, then for every $E \in \Sigma$, $\mu(E) < \infty$, there is a $g_E(\in L^{\varrho'})$, and the collection $\{g_E\}$, also denoted g^*, is called a *cross-section* [38], or a *quasi-function* [22] (for which integrals are defined, see [22]), and g^* determines a measurable function iff μ is localizable. (L^ϱ is complete if ϱ has the Fatou property, [39].)

Using a procedure similar to ([26], p. 567, or Cor. 6.1 there) the integral (5) can be extended from M^ϱ to L^ϱ itself. Thus,

Corollary 1.10. *If μ has FSP, and ϱ is continuous at zero, then for each $x^* \in (M^\varrho)^*$ there is a unique quasi-function g^* such that, for $f \in L^\varrho$,*

$$x^*(f) = \int_\Omega fg^*\,d\mu \left(= \int_\Omega f dG\right), \qquad \|x^*\| = \varrho'(g^*)\ \left(= \varrho'(G)\right). \tag{6}$$

Moreover, g^* is measurable iff either μ is localizable, or μ-simple functions (i.e. those with support having a finite measure) are dense in $L^{\varrho'}$. In any case $L^{\varrho'}_*$ of quasi-functions is isometrically equivalent to $A_{\varrho'}(\mu)$.

Definition 1.11. Let $\mathscr{A}_{\varrho'}(\mu) = A_{\varrho'}(\mu) \oplus B_{\varrho'}(\mu)$, where $G \in \mathscr{A}_{\varrho'}(\mu)$ iff $G = G_1 + G_2$ with $G_1 \in A_{\varrho'}(\mu)$, $G_2 \in B_{\varrho'}(\mu)$ and norm

$$\|G\|_{\varrho'} = \varrho'(G_1) + |G_2|(\Omega).$$

Proposition 1.12. *Every $x^* \in (L^\varrho)^*$ admits a unique decomposition, $x^* = y^* + z^*$, where $y^*(f) = \int_\Omega f dG$, $f \in L^\varrho$, $G \in A_{\varrho'}(\mu)$, called an absolutely continuous functional and $z \in (M^\varrho)^\perp$, called a singular functional. Also $\|x^*\| = \|y^*\| + \|z^*\|$.*

The proof given in [26], p. 575, for the Orlicz spaces, holds here verbatim. (For an alternate argument, see [13], p. 43.)

Now the main result of this section can be given as:

Theorem 1.13. *Let L^ϱ be a normed (scalar) function space on a measure space (Ω, Σ, μ). Then its adjoint space $(L^\varrho)^*$ is isometrically (and lattice) isomorphic to $\mathscr{A}_{\varrho'}(\mu)$. More explicitly, for each $x^* \in (L^\varrho)^*$, there is a unique $G \in \mathscr{A}_{\varrho'}(\mu)$ such that*

$$x^*(f) = \int_\Omega f dG\ \left(= \int_\Omega f dG_1 + \int_\Omega F(f)\, dG_2\right), \quad f \in L^\varrho, \tag{7}$$

and

$$\|x^*\| = \|G\|_{\varrho'}\ (= \varrho'(G_1) + |G_2|(\Omega)).$$

Moreover, if ϱ is continuous at the origin, then G_1 is μ-continuous.

This is an immediate consequence of Proposition 1.12, Theorems 1.7, 1.9, 1.10 and Lemma 1.15 below.

Corollary 1.14. *If $\mu(\Omega) < \infty$, then (7) can be uniquely written as*

$$x^*(f) = \int_\Omega fg\,d\mu + \int_\Omega f d\nu_0 + \int_\Omega F(f)\, dG_2, \quad f \in L^\varrho, \tag{8}$$

and

$$\|x^*\| = \varrho'(g) + |\nu_0|(\Omega) + |G_2|(\Omega),$$

where $g \in L^{\varrho'}$ and ν_0 is a pfa on Σ, $G_2 \in B_{\varrho'}(\mu)$. If also ϱ is continuous at zero, then $\nu_0 = 0 = G_2$. On the other hand, if $M^\varrho = L^\varrho$, $G_2 = 0$, but not necessarily ν_0.

The result follows from the fact that, when $\mu(\Omega) < \infty$, for the set functions of $A_{\varrho'}(\mu)$ one can apply the Yosida-Hewitt theorem ([9], p. 163) so that $G_1 = \bar{G}_1 + \nu_0$ where ν_0 is pfa, and $d\bar{G}_1 = g\,d\mu$. The rest follows from the theorem.

Lemma 1.15. *If $G \in A_{\varrho'}(\mu)$ is not a pfa, then it has an extension to Σ, and is σ-additive there. The extension is unique on $\Sigma_1 = \sigma(\Sigma_0)$, the tribe generated by the clan Σ_0.*

Proof. First note that if G is a pfa, then it is a bounded additive set function on all of Σ, and the exception is vacuous for this part. Then from general considerations, as in Lemma 1.2, it follows that there is an $f \in M^\varrho$ such that $\varrho(f\chi_{A_n}) \to 0$ as $A_n \downarrow \emptyset$, where support of f may be taken to include supp (G). Now $A_n = \{|f| \geqslant n\} \in \Sigma_0$, and $\varrho(\chi_{A_n}) \to 0$ so that

ϱ is continuous at zero, on the support of G. So by the last part of Theorem 1.9, G is μ-continuous. In fact, it is clear that $|G(E)|$ is finite for all $E \in \Sigma_0$, and for $E_n \downarrow \emptyset$, $E_n \in \Sigma_0$, $|G(E_n)| \to 0$, so that it is finite and σ-additive on Σ_0, and hence has a unique σ-additive extension to $\Sigma_1 = \sigma(\Sigma_0)$ by the Hahn extension theorem.

To see that G is also defined on Σ itself and is σ-additive there, note first that Σ_0 is an ideal in Σ, i.e., $A \in \Sigma_0$, $B \in \Sigma$ implies $A \cap B \in \Sigma_0$. Now using Carathéodory extension procedure, for any $T \in \Sigma_0$, a set $E \in \Sigma$ is a G-set iff (hereafter, for $A \subset \Omega$, $A^c = \Omega - A$)

$$G(T) = G(T \cap E) + G(T \cap E^c).$$

The (sigma-) additivity of G on Σ_0, implies that the class of G-sets is a (sigma-) field and contains Σ. It follows that G on Σ is also σ-additive, and μ-continuous. This completes the proof.

COROLLARY 1.16. *If ϱ is continuous at zero, then every element of $A_{\varrho'}(\mu)$ is defined, and σ-additive, on Σ.*

1.2. More on functionals. Using the representation, in Theorem 1.13, of $(L^\varrho)^*$, some generalizations of the results in [20] whose proofs, and even some statements, crucially use the σ-finiteness of μ (in the form of ϱ and ϱ' admissible sequences), will be given for general μ. These results will be useful later on.

Definition 2.1. (a) If $M^\varrho = L^\varrho$, then step functions are said to be *dense* in L^ϱ.

(b) If $f \in L^\varrho$, it is said to have an *absolutely continuous norm* (a.c.n.) iff $\varrho(f\chi_{A_n}) \to 0$ for all $A_n \downarrow \emptyset$, $A_n \in \Sigma$. If every element of L^ϱ has a.c.n., then L^ϱ is said to have a.c.n., or ϱ is said to be an a.c.n.

PROPOSITION 2.2. (a) *Let μ have FSP. Then $(L^\varrho)^*$ is isometrically isomorphic to $L^{\varrho'}_*$ of quasi-functions iff for each $x^* \in (L^\varrho)^*$ and $0 \leqslant f_n \in L^\varrho$ with $f_n \downarrow 0$, one has $|x^*(f_n)| \to 0$. If μ is localizable, then $L^{\varrho'}_* = L^{\varrho'}$, of measurable functions.*

(b) *An element $f \in L^\varrho$ has a.c.n. iff for every f_n with $|f| \geqslant f_1 \geqslant \ldots \to 0$, $\varrho(f_n) \downarrow 0$, where μ is arbitrary.*

Proof. (a) In Theorem 1.13, $|x^*(f_n)| \to 0$ as $f_n \downarrow 0$ implies $M^\varrho = L^\varrho$ and, moreover, $\varrho(\cdot)$ is continuous at zero. Thus $(L^\varrho)^* = A_{\varrho'}(\mu)$, and, by Corollary 1.16, every $G \in A_{\varrho'}(\mu)$ is σ-additive. Thus $A_{\varrho'}(\mu) = L^{\varrho'}_*$. The converse implication follows from the dominated convergence theorem.

(b) If the condition holds, then $f_n = |f|\chi_{A_n}$, $A_n \downarrow \emptyset$, will show that f has a.c.n. The converse is non-trivial and the corresponding proof of [20] does not seem to extend. The result will be proved using Theorem 1.13 here.

Since $0 \leqslant f_n \leqslant |f|$, and $|f|\,(= f_0$, say), has a.c.n., each f_n has a.c.n. Now by the Hahn-Banach theorem there is an $x_n^* \in (L^\varrho)^*$ such that $\varrho(f_n) = x_n^*(f_n)$, $n \geqslant 0$, $\|x_n^*\| = 1$. By Theorem 1.13, there exist $0 \leqslant G_n \in \mathscr{A}_{\varrho'}(\mu)$, such that

$$(1) \qquad x_n^*(h) = \int_\Omega h\,dG_n, \quad h \in L^\varrho, \ \|x_n^*\| = \varrho'(G_n) = 1.$$

The a.c.n. of f_n implies $|x_n^*(f_n\chi_{A_k})| \to 0$ as $A_k \downarrow \emptyset$, for each n, which in turn implies that $\varrho(\cdot)$ is continuous at 0. Thus by Lemma 1.15, G_n, $n \geqslant 0$, are σ-additive and $G_n \in A_{\varrho'}(\mu)$. Hence by the dominated convergence theorem, $|x_n^*(f_m)| \to 0$ as $m \to \infty$ for each n. It is to be shown that $x_n^*(f_n) = \varrho(f_n) \to 0$, as $n \to \infty$.

Let $S \subset A_{\varrho'}(\mu)$ be the closure of $\{G_n, n \geqslant 0\}$. Then S is a closed subset of σ-additive set functions on the unit sphere of $A_{\varrho'}(\mu)$ (and also $\mathscr{A}_{\varrho'}(\mu)$), which is compact in the $\sigma((L^\varrho)^*, L^\varrho)$-topology. For each $H \in S$, one has, moreover ($\hat{H} \in \hat{S} \subset S^{**}$),

$$\hat{f}_n(H) = \hat{H}(f_n) = \int_\Omega f_n\,dH \downarrow 0 \quad \text{as } n \to \infty.$$

Since $\hat{f}_n \in (S)^{**} \subset (L^\varrho)^{**}$, it follows that $\hat{f}_n(H) \downarrow 0$ for each $H \in S$. Since S is compact Hausdorff, the convergence is uniform. Hence $\hat{f}_n(G_n) \to 0$, so $\varrho(f_n) \to 0$, as desired.

As an immediate consequence one has

COROLLARY 3.2. *If $\varrho(\cdot)$ is continuous at zero, $g \in L^\varrho$ has a.c.n., and $0 \leqslant f_n \uparrow g$, then $\varrho(g - f_n) \to 0$, as $n \to \infty$.*

The following is a kind of dominated convergence theorem for L^ϱ-spaces, which is proved with the above result:

PROPOSITION 2.4. (a) *Let $0 \leqslant g \in L^\varrho$ have an a.c.n., and $\{f_a\} \subset L^\varrho$ be a generalized sequence such that $|f_a| \leqslant g$, a.e. If $f_a \to f$ in μ-measure, then $\varrho(f_a - f) \to 0$. Conversely, if μ has FSP on* supp(g), *and $\varrho(f_a - f) \to 0$, then $f_a \to f$ in μ-measure.*

(b) *If for every sequence $\{f_n\}$ with $|f_n| \leqslant g \in L^\varrho$, then $f_n \to f_0$ a.e. implies $\varrho(f_n - f_0) \to 0$ iff g has a.c.n.*

Proof. First suppose $\{f_a\} = \{f_n\}$ is an ordinary sequence. If $f_n \to f$, in μ-measure, then $|f| \leqslant g$ a.e. Then there is a subsequence $\{\tilde{f}_n\} \subset \{f_n\}$ with $\tilde{f}_n \to f$ a.e. If $g_n = \sup\{|f_{n+m} - f|: m \geqslant 0\}$, then $g_n \leqslant 2g$ and $g_n \downarrow 0$, so, by Proposition 2.2 (b), $\varrho(g_n) \to 0$, and so $\varrho(\tilde{f}_n - f) \to 0$. If the result is not true for the full sequence, then there exists a subsequence $\{f_{n_i}\} \subset \{f_n\}$, such that $\overline{\lim}\, \varrho(f_{n_i} - f) \geqslant \varepsilon > 0$. So there is a further subsequence, denoted by f_{n_i} itself, such that $\varrho(f_{n_i} - f) \to a \geqslant \varepsilon > 0$. But $f_{n_i} \to f$ in measure, so that it has a further subsequence $\{\bar{f}_{n_i}\} \subset \{f_{n_i}\}$ verifying $\bar{f}_{n_i} \to f$ a.e. Then by the first part, $\varrho(\bar{f}_{n_i} - f) \to 0$, and contradicts the above assumption. Hence $\varrho(f_n - f) \to 0$ for the full sequence itself. With this, the case of the generalized sequence can be deduced by an argument entirely similar to that of [9], p. 125, and the direct assertion holds.

For the converse, again consider an ordinary sequence $\{f_n\}$ such that $\varrho(f_n - f) \to 0$, $|f_n| \leqslant g \in L^\varrho$ and μ has FSP on $\mathrm{supp}(g)$. For any $\varepsilon > 0$, if $E_{n,\varepsilon} = \{\omega : |f_n - f| \geqslant \varepsilon\}$, then it should be shown that $\limsup_n \mu(E_{n,\varepsilon}) = 0$. Clearly, $E_{n,\varepsilon} \in \Sigma_0$, and let $0 \leqslant G \in S^+$, the positive unit sphere of $(L^\varrho)^*$. Then

$$0 \leqslant G(E_{n,\varepsilon}) \leqslant \frac{1}{\varepsilon} \int_\Omega |f_n - f|\, dG \leqslant \frac{1}{\varepsilon}\, \varrho(f_n - f),$$

since $\varrho'(G) = 1$. Thus $G(E_{n,\varepsilon}) \to 0$, as $n \to \infty$ uniformly in $G \in S^+$. So given $\delta > 0$, there is $n_0 = n_0(\delta)$ with $n \geqslant n_0$ implying

$$\delta > \sup\{G(E_{n,\varepsilon}) : G \in S^+\} = \sup\Big\{\int_\Omega \chi_{E_{n,\varepsilon}} dG : G \in S^+\Big\} = \varrho(\chi_{n,\varepsilon}).$$

Since $E_{n,\varepsilon} \subset \mathrm{supp}(g)$, and μ has FSP there, there is $E \in \Sigma_0$, of finite μ-measure, $E \subset \mathrm{supp}(g)$, which is "$\varrho'$-admissible", as in [20], and $\varrho'(\chi_E) < \infty$. Consequently,

$$\mu(E_{n,\varepsilon} \cap E) = \int_\Omega \chi_{E_{n,\varepsilon}} \chi_E \, d\mu \leqslant \varrho(\chi_{E_{n,\varepsilon}})\, \varrho'(\chi_E) \leqslant \delta \varrho'(\chi_E).$$

Hence $\overline{\lim}\, \mu(E_{n,\varepsilon} \cap E) \leqslant \delta\varrho'(\chi_E)$, and letting $\delta \to 0$, it follows that $\overline{\lim}\, \mu(E_{n,\varepsilon} \cap E) = 0$. Now the FSP of μ on $\mathrm{supp}(g)$ yields, since E in that support can vary on all sets of finite μ-measure, (through admissible sequences) it follows, by [39], p. 257, $\overline{\lim_n}\, \mu(E_{n,\varepsilon}) = 0$. Hence $f_n \to f$ in μ-measure. The case of generalized sequences again can be deduced from this result as in [9], p. 125.

(b) This follows from (a) and the necessity of Proposition 2.2 (b). This completes the proof.

Remark. If $G \in S^+$ as above, and $L^1(G)$ is the corresponding Lebesgue space, then $\|f_n - f\|_{1,G} \leqslant \varrho(f_n - f) \to 0$ so that the result that $f_n \to f$ in G-measure is a consequence of [9] p. 122. This is not enough to deduce the convergence in μ-measure and the above computation appears necessary. Note also that $|f_a| \leqslant g$ can be replaced by the condition: "$\mathrm{supp}(f_a)$ is contained for all a, in a measurable set A on which μ has FSP."

Corollary 2.5. *Let $L_a^\varrho \subset L^\varrho$ be the set of all elements with a.c.n. Then L_a^ϱ is a solid (or normal) closed subspace, and ϱ on L_a^ϱ has the (restricted) Fatou property, i.e., $0 \leqslant f_n \uparrow f$ a.e., and $f \in L_a^\varrho$ implies $\varrho(f_n) \uparrow \varrho(f)$ and $\varrho(f_n - f) \to 0$.*

Proof. If $|f| \leqslant |g| \in L_a^\varrho$, then $f \in L_a^\varrho$, which is normality, is obvious. If $0 \leqslant f_n \uparrow f \in L_a^\varrho$, then $0 \leqslant f - f_n \leqslant f$ and so, by Proposition 2.2 (b), $\varrho(f - f_n) \downarrow 0$, and $\varrho(f_n) \uparrow \varrho(f)$. If $\tilde{\varrho}$ is ϱ on L_a^ϱ, and $= +\infty$ otherwise, then $\tilde{\varrho}$ is a function norm with Fatou's property, so that L_a^ϱ is a BFS. (For another proof of this, see [39], p. 478.)

Remark. If μ has FSP on Ω, then ϱ' is non-trivial and so is $\varrho'' = (\varrho')'$. A result of Halperin and Luxemburg (cf. [13], for references) implies (cf. [39], p. 450 and p. 471) $L^\varrho \subset L^{\varrho''}$ and $\varrho = \varrho''$ iff ϱ has the Fatou property on L^ϱ. It can be shown in any case the set of all step functions (in particular, M^ϱ) in L^ϱ is total for $L^{\varrho'}$. If μ is only finitely additive, then even $L^1(\mu)$ will not have a.c.n.

The main result of this section is given by

Theorem 2.6. *Let ϱ be a function norm. Then L^ϱ is reflexive iff* (i) *μ has FSP,* (ii) *$L^\varrho = L_a^\varrho$,* (iii) *$L^{\varrho'} = L_a^{\varrho'}$ where ϱ' is the associate of ϱ, and* (iv) *ϱ and ϱ' have the localizable property (i.e., for each f with $\varrho(f) < \infty$, μ is localizable on* $\mathrm{supp}(f)$, *and similarly for ϱ').*

Proof. Let L^ϱ be reflexive, so that $L^\varrho = (L^\varrho)^{**}$. This implies FSP of μ. For, if this is false, then $\{0\} \neq \big(B_{\varrho'}(\mu)\big)^* \subset \big(\mathscr{A}_{\varrho'}(\mu)\big)^* = (L^\varrho)^{**} = L^\varrho$, where equalities and inclusions are understood, as usual, under the natural isomorphisms and embeddings. Hence the subspace $\big(B_{\varrho'}(\mu)\big)^*$ must also be reflexive, and this is so iff $B_{\varrho'}(\mu)$ is. But $B_{\varrho'}(\mu)$, an (AL)-space, is reflexive iff it is finite-dimensional which implies N^ϱ is also finite dimensional. This yields that $L^\varrho = M^\varrho \oplus N^\varrho$, a direct sum decomposition (since N^ϱ can be identified as a complementary manifold of M^ϱ, of finite codimension). But then, all topologies on N^ϱ being equivalent, it follows that step functions in L^ϱ are dense so that $N^\varrho = \{0\}$. This will be impossible unless $B_{\varrho'} = \{0\}$. Thus $(L^\varrho)^* = A_{\varrho'}(\mu)$ is also reflexive and (i) holds. So ϱ' exists on point functions non-trivially, and $L^{\varrho'}$ is non-trivial. Clearly, $L^{\varrho'} \subset A_{\varrho'}(\mu)$ where the indefinite integrals of $f \in L^{\varrho'}$ are considered as elements of $A_{\varrho'}(\mu)$. This is one-to-one and into the second space. (See [37], about such injunctions in case of Orlicz spaces.) To see it is onto, suppose not. Then there exist a $G_0 \in \big(A_{\varrho'}(\mu) - L^{\varrho'}\big)$ and (these two are reflexive, since L^ϱ is reflexive iff $(L^\varrho)^*$ is; so each of its subspaces is), by the Hahn-Banach theorem, an $f_0 \in \big(A_{\varrho'}(\mu)\big)^* = (L^\varrho)^{**} = L^\varrho$, such that $1 = \int_\Omega f_0\, dG_0$, and $0 = \int_\Omega f_0 g\, d\mu$, $g \in L^{\varrho'}$. The latter yields, by the remark preceding the statement of the theorem (since μ has FSP), $\varrho''(f_0) = 0$ because $L^\varrho \subset L^{\varrho''}$. Hence $f_0 = 0$ a.e. and contradicts the first equation. Thus $L^{\varrho'} = A_{\varrho'}(\mu)$ must hold. This means each element of $A_{\varrho'}(\mu)$ is σ-additive and has the (generalized) Radon-Nikodým property which is true iff ϱ' has the localizable property by [34] and [38]. On the other hand, $(L^{\varrho'})^* = \big(A_{\varrho'}(\mu)\big)^* = (L^\varrho)^{**} = (L^\varrho)$ implies, with the first part of this proof, that

$$(L^\varrho) = (L^{\varrho'})^* = A_{\varrho''}(\mu). \tag{2}$$

Hence the elements of $A_{\varrho''}(\mu)$ are σ-additive, and have Radon-Nikodým derivatives. This is true iff ϱ'' has the localizable property, and thus (2) implies also $L^\varrho = L^{\varrho''} = A_{\varrho''}(\mu)$, with the isometric equivalence.

It follows that $\varrho = \varrho''$ and (iv) holds. Thus ϱ also has the Fatou property! (This would also follow from (ii) after it is proved.)

It remains to prove (ii) and (iii). Let $0 \leqslant f_0 \in L^\varrho$ be arbitrary. Then there is an $x_0^* \in (L^\varrho)^*$, with $\varrho(f_0) = x_0^*(f_0)$, and a unique $0 \leqslant G_0 \in A_{\varrho'}(\mu)$, such that

$$\varrho(f_0) = x_0^*(f_0) = \int_\Omega f_0 dG_0.$$

If $0 \leqslant f_n \leqslant f_0$ and $f_n \downarrow 0$ is any sequence, then by Proposition 2.2 (b), since the elements of $A_{\varrho'}(\mu)$ are σ-additive, $\hat{f}_n(G) \downarrow 0$ for all $G \in S^+$ the positive part of the unit ball of $A_{\varrho'}(\mu)$. Using the same argument of that proof, one proves $\varrho(f_n) \downarrow 0$, so that f_0 has a.c.n. Hence $L^\varrho = L_a^\varrho$. Similarly, $L^{\varrho'} = L_a^{\varrho'}$. This proves the necessity.

For the converse implication, let (i)-(iv) hold. Then by Proposition 2.2 (a) (and the fact that $\varrho = \varrho''$ following from (ii) now)

$$(L_a^\varrho)^* = A_{\varrho'}(\mu) = L^{\varrho'}(\mu) = L_a^{\varrho'}.$$

Similarly, $(L_a^{\varrho'})^* = L_a^\varrho$. Consequently,

$$(L^\varrho)^{**} = (L_a^\varrho)^{**} = (L^{\varrho'})^* = (L_a^{\varrho'})^* = L_a^\varrho = L^\varrho, \tag{3}$$

and L^ϱ is reflexive, and $(L^\varrho)^* = L^{\varrho'}$, (so that $L^{\varrho'}$ is also reflexive). This completes the proof.

Remark. In [11], the norm $\varrho(\cdot)$ was more restricted than here, (it has the Fatou property, among others, in its definition). So the above result extends the result of [16]. The conditions (and methods of proof) of [16] are of a somewhat different kind. The above theorem is the best possible for BFS's.

The final result is a characterization of the second conjugate $(L^\varrho)^{**}$ of L^ϱ, with Theorem 1.13.

THEOREM 2.7. *Let μ have FSP, $M^\varrho = L_a^\varrho$, ϱ' have the localizable por-p erty, and ϱ the Fatou property. Then $(L^\varrho)^{**} \cong A_\varrho(\mu) \oplus B_\varrho(\mu) \oplus C_\varrho(\mu)$, where $C_\varrho(\mu)$ is an (AM)-space conjugate to $B_{\varrho'}(\mu)$. If $G = G_1 + G_2 + G_3$, $G_1 \in A_\varrho(\mu)$, $G_2 \in B_\varrho(\mu)$, and $G_3 \in C_\varrho(\mu)$, then $x^{**} \in (L^\varrho)^{**}$ corresponds uniquely to such a G and*

$$\|x^{**}\| = \|G\|_{\tilde{\varrho}} = \max\big(\varrho(G_1) + |G_2|(\Omega), \|G_3\|\big), \tag{4}$$

where $\|G_3\|$ is the (AM)-norm of $C_\varrho(\mu)$.

Remark. The actual representation $\langle x^{**}, x^*\rangle$ can be written down, for all $x^* \in (L^\varrho)^*$, using the results of [18]. It will be omitted here.

Proof. By Theorem 1.13, $(L^\varrho)^* = A_{\varrho'}(\mu) \oplus B_{\varrho'}(\mu)$. Since $M^\varrho = L_a^\varrho$, and μ has FSP, $\varrho'(\cdot)$ on point functions is non-trivial and $A_{\varrho'}(\mu) = L^{\varrho'}$, by Proposition 2.2 (a) and the localizability of ϱ'. Hence $(L^\varrho)^* = L^{\varrho'} \oplus B_{\varrho'}(\mu)$. Applying the same result again to the conjugate space of the right side, and remembering that $B_{\varrho'}(\mu)$ is an (AL)-space and $\varrho'' = \varrho$, one obtains

$$(L^\varrho)^{**} = (L^{\varrho'})^* \oplus C_\varrho(\mu) = A_\varrho(\mu) \oplus B_\varrho(\mu) \oplus C_\varrho(\mu), \tag{5}$$

and the norm equation is deduced from the well-known results, completing the proof.

Remark. If ϱ is continuous at zero (or $M^\varrho = L_a^\varrho$) and $\mu(\Omega) < \infty$, then $B_{\varrho'}(\mu) = \{0\}$, and $C_\varrho(\mu) = 0$. The condition of the hypothesis is weaker than this. If L^ϱ is merely a BFS, and the Fatou property is dropped, then also the above result holds, but the isometric equality in (4) will then be a topological equivalence (L^ϱ and $L^{\varrho''}$ have that property).

1.3. Representation of linear operators. The general structure of $(L^\varrho)^*$ enables a determination of the structure of $B(L^\varrho, \mathscr{X})$, the space of bounded linear maps on L^ϱ to a B-space $\mathscr{X}$. This completes the work of [13] where $B(M^\varrho, \mathscr{X})$ was analyzed directly. This general representation will be needed for the work in the following sections.

Definition 3.1. (a) Let $\mathscr{W}_\varrho(\mu)$ be the class of additive set functions $\nu: \Sigma_0 \mapsto \mathscr{X}$, vanishing on μ-null sets and $n_\varrho(\nu) < \infty$ where the norm $n_\varrho(\cdot)$ is given by: $n_\varrho(\nu) = \sup\{\varrho(x^*\nu): \|x^*\| \leqslant 1, x^* \in \mathscr{X}^*\}$.

(b) Let $\mathscr{P}_\varrho(\mu)$ be the class of additive set functions $\nu: \Sigma \mapsto \mathscr{X}$, vanishing on μ null sets, with support contained in that of an element of $L^\varrho - M^\varrho$ (set theoretical difference), and such that for each $x^* \in \mathscr{X}^*$, $x^*\nu$ is a pfa, with $\|\nu\|(\Omega) = \operatorname{supp}\{|x^*\nu|(\Omega): \|x^*\| \leqslant 1\}$, i.e., of finite weak semi-variation, in the notation of [7]. (Note that $\operatorname{supp}(\nu) \in \Sigma$; does not depend on $\mathscr{X}$.)

(c) Let $\mathscr{U}_\varrho(\mu) = \mathscr{W}_\varrho(\mu) \oplus \mathscr{P}_\varrho(\mu)$, where $G \in \mathscr{U}_\varrho(\mu)$ means $G = G_1 + G_2$ with $|||G|||_\varrho = \sup\{\varrho(x^*G_1) + |x^*G_2|(\Omega): \|x^*\| \leqslant 1\}$.

It is clear that $|||G|||_\varrho \leqslant n_\varrho(G_1) + \|G_2\|(\Omega)$ and, moreover, $|||G|||_\varrho < \infty$ iff $n_\varrho(G_1) < \infty$ and $\|G_2\|(\Omega) < \infty$. The definition in (a) above is motivated by the work in [37], p. 52.

The main representation can be given by the following

THEOREM 3.2. *In the above notations, $B(L^\varrho, \mathscr{X}) \cong \mathscr{U}_{\varrho'}(\mu)$. More explicitly, for each $T \in B(L^\varrho, \mathscr{X})$, there is a unique $G \in \mathscr{U}_{\varrho'}(\mu)$ such that (ϱ' is the associate norm of ϱ)*

$$Tf = \int_\Omega f dG = \int_\Omega f dG_1 + \int_\Omega F(f) dG_2, \quad f \in L^\varrho, \tag{1}$$

where the first integral on the right is similar to that in [9], IV. 10, on Σ_0 (instead of Σ), and the second one, (as well as the first) as a weak integral, extending that of Section 1.1 above. The isometry is given by

$$\|T\| = |||G|||_{\varrho'}. \tag{2}$$

If ϱ is continuous at zero, then $G_2 = 0$ and G_1 is (strongly) μ-continuous.

Proof. If $T^*\colon \mathscr{X}^* \mapsto (L^\varrho)^*$ is the adjoint of T, and $x^* \in \mathscr{X}^*$, let $T^* x^* = y^*_{x^*}$. So $y^*_{x^*} \in (L^\varrho)^*$ and, by Theorem 1.13, there exists uniquely $G_{x^*} \in \mathscr{A}_{\varrho'}(\mu)$, $G_{x^*} = G_{1x^*} + G_{2x^*}$ in the notation there, such that

$$(3)\qquad x^*(Tf) = y^*_{x^*}(f) = \int_\Omega f dG_{x^*} = \int_\Omega dG_{1x^*} + \int_\Omega F(f) G_{2x^*}, \qquad f \in L^\varrho,$$

and

$$(4)\qquad \|y^*_{x^*}\| = \varrho'(G_{1x^*}) + |G_{2x^*}|(\Omega).$$

But the mappings $x^* \mapsto G_{ix^*}$, $i = 1, 2$, are linear and

$$\|G_{x^*}\|_{\varrho'} = \|y^*_{x^*}\| = \|T^* x^*\| \leqslant \|T\| \cdot \|x^*\| < \infty,$$

so that they are also bounded. Consequently, there exist $G_i\colon \Sigma_0 \mapsto \mathscr{X}^{**}$, such that $G_{ix^*} = G_i x^*$, $i = 1, 2$, where G_i are additive and vanish on μ-null sets and whose bounds are given by the above equations. It is to be shown that (i) $G = G_1 + G_2$ takes values in $\mathscr{X}$, (ii) $|||G|||_{\varrho'} = \|T\|$, and (iii) the representation (1) obtains.

To see (i), let $f = \chi_A$, $A \in \Sigma_0$. Then in (3) the second term drops out and $T\chi_E \in \mathscr{X}$, so that letting $\hat{\mathscr{X}} \subset \mathscr{X}^{**}$ the natural image,

$$\widehat{T\chi_E}(x^*) = x^*(T\chi_E) = \int_\Omega \chi_E \, d(G_1 x^*) = G_1(E) x^*, \qquad x^* \in \mathscr{X}^*.$$

Since $E \in \Sigma_0$ is arbitrary, it follows that $G_1(E) = \widehat{T\chi_E} \in \hat{\mathscr{X}}$, and so $G_1\colon \Sigma_0 \mapsto \mathscr{X}$, and

$$\varrho'(G_1) = \sup\{\varrho'(x^* G_1)\colon \|x^*\| \leqslant 1\} = \sup\{\|y^*_{x^*} | M^\varrho\|\colon \|x^*\| \leqslant 1\}.$$

If $T_1 = T | M^\varrho$ is the restriction of T to M^ϱ, then $\varrho'(G_1) = \|T_1^*\|$. By considering the definition of the second integral, it can be similarly deduced that $G_2\colon \Sigma \mapsto \mathscr{X}$ and $\|G_2\|(\Omega) = \|T_2^*\|$, where $T_2 = T - T_1\colon N^\varrho \mapsto \mathscr{X}$. Such a decomposition, given for functionals in Proposition 1.12, can be obtained for the present case without difficulty. Thus $\|T_i\| \leqslant \|T\| < \infty$, so that (i) holds and (iii) is immediate for finite sums, and the general case then follows.

To prove (ii), a simple computation is needed. By the definitions of $|||\cdot|||_{\varrho'}$, letting $y^*_{ix^*}$ be the corresponding elements,

$$\begin{aligned} |||G|||_{\varrho'} &= \sup\{\varrho'(x^* G_1) + |x^* G_2|(\Omega)\colon \|x^*\| \leqslant 1\} \\ &= \sup\{\|y^*_{1x^*}\| + \|y^*_{2x^*}\|\colon \|x^*\| \leqslant 1\} \\ &= \sup\{\|y^*_{x^*}\|\colon \|x^*\| \leqslant 1\}, \qquad \text{by Proposition 1.12,} \\ &= \|T^*\| = \|T\|. \end{aligned}$$

This proves the isometry and the representation. If, conversely, T is defined by (1) it immediately follows that $T \in B(L^\varrho, \mathscr{X})$ since $G \in \mathscr{U}_{\varrho'}(\mu)$.

If, moreover, ϱ is continuous at zero, then $G_2 = 0$ and G_1 is μ-continuous on Σ_0. By Corollary 1.16, it follows that G_1 is weakly σ-additive on Σ, and hence, by [9], IV. 10. 1, it is σ-additive. This completes the proof.

Remark. It should be noted that, in the above representation, in general the measures G's will not be regular in any reasonable sense. The above result was proved directly in [13], if L^ϱ is replaced by M^ϱ (so that $G_2 = 0$), if ϱ verifies a "Jensen's condition" (J) and if μ is σ-finite.

Some consequences of the above representation will be given. The next result is closely related to a theorem of Singer [35] (compare with [12], p. 775 also).

THEOREM 3.3. *Let S be a compact Hausdorff space, $C(S)$ the space of real continuous functions on S and $T \in B(C(S), \mathscr{X})$, where $\mathscr{X}$ is a B-space. Then there exists a unique vector measure $\nu\colon \Sigma \mapsto \mathscr{X}$, Σ the σ-field of Borel sets of S, such that*

$$(5)\qquad Tf = \int_S f d\nu, \qquad f \in C(S), \ \|T\| = \|\nu\|(S),$$

where $\|\nu\|(S)$ is the semi-variation of ν (cf. Definition 3.1(b)).

Proof. The adjoint space $(C(S))^*$ of $C(S)$ is an (AL)-space and by Kakutani's theorem [18], there is a compact Hausdorff (totally disconnected) space $\tilde{S}$ and a regular finite Borel measure μ on the σ-field of closed sets of $\tilde{S}$ such that $(C(S))^* \cong L^1(\tilde{S}, \mu)$. It follows that $C(S) \subset (C(S))^{**} \cong L^\infty(\tilde{S}, \mu)$. Thus there is a closed set $S_1 \subset \tilde{S}$ which is homeomorphic to S and such that $C(S)$ and $C(S_1)$ are isometrically (lattice) isomorphic, by the Banach-Stone theorem. If $\varrho(\cdot) = \|\cdot\|_\infty$, and $\tilde{\varrho} = \varrho | C(S_1)$ and $\tilde{\varrho} = +\infty$ on $L^\infty(\tilde{S}, \mu) - C(S_1, \mu)$, then $\tilde{\varrho}$ is a function norm and $L^{\tilde{\varrho}}(S_1, \mu)$ and $C(S_1)$ are such that every element of the former is equal, μ almost everywhere, to an element of the latter. Using the lifting map (cf. [7]), identify these two spaces. Let $S_1 = \tau(S)$, where τ is the homeomorphic mapping onto the subspace S_1 noted above. Thus, identifying $B(C(S), \mathscr{X})$ and $B(L^{\tilde{\varrho}}, \mathscr{X})$, one has (by Theorem 3.2), $\nu_1\colon \Sigma_1 \mapsto \mathscr{X}$ which is additive and vanishes on μ-null sets, and since $\tilde{\varrho}'(\cdot) = \|\cdot\|_1$ and $\Sigma_0 = \Sigma_1$ is the σ-field of Borel sets of S_1, T_1 corresponds to T,

$$(6)\qquad T_1 \tilde{f} = \int_{S_1} \tilde{f} d\nu_1, \quad \tilde{f} \in L^{\tilde{\varrho}},$$

$$\|T_1\| = |||\nu_1|||_{\tilde{\varrho}'} = \sup\{\tilde{\varrho}'(x^*\nu_1)\colon \|x^*\| \leqslant 1\} = \|\nu_1\|(S_1).$$

Let $\nu = \nu_1 \circ \tau^{-1}$ so that ν is a vector measure on the Borel sets of S to $\mathscr{X}$ and $\|\nu\|(S) = \|\nu_1\|(S_1)$. Then (b) becomes

$$(7)\qquad Tf = \int_S f d\nu, \qquad f \in C(S), \ \|T\| = \|\nu\|(S).$$

It remains to show σ-additivity of ν. Since S is compact, if $f_n \downarrow 0$ pointwise, $f_n \in C(S)$, then the convergence is also uniform so that one has, for each $x^* \in \mathscr{X}^*$,

$$|x^* T(f_n)| = |y^*_{x^*}(f_n)| \leqslant \|y^*_{x^*}\| \, \|f_n\| \to 0, \qquad \text{as } n \to 0.$$

Hence, by Proposition 2.2, $x^*\nu$ is μ-continuous and is thus σ-additive. This, by [9], IV. 10. 1, completes the proof.

Remark. Again the measure ν of (5) is in general not regular. In order to have such a property, one has to enlarge the range space of ν from $\mathscr{X}$ to $\mathscr{X}^{**}$ (cf. [9]; for a general discussion on this point, see [6]). It should also be remarked that the Radon-Nikodym theorem cannot be applied to (ν, μ) without further conditions on $\mathscr{X}$.

The following consequence has some independent interest. Recall that a B-space $\mathscr{X}$ is said to have the *Radon-Nikodým* (R-N) *property* relative to a measure space (Ω, Σ, μ), if every μ-continuous vector measure, on Σ, is the indefinite integral of a (strongly) measurable function relative to μ. (Examples of such $\mathscr{X}$'s are the reflexive B-spaces or the conjugate spaces of B-spaces, cf. [7].)

THEOREM 3.4. *Let (Ω, Σ, μ) be a localizable space, and $\mathscr{X} = L^{\varrho_2}(S, \mathscr{B}, \lambda)$ with the R-N and Fatou properties. If ϱ_1 is a function norm, continuous at zero, then for each $T \in B(L^{\varrho_1}, \mathscr{X})$ there is a "kernel", $K(\cdot, \cdot)$: $\mathscr{B} \times \Omega \mapsto$ scalars, with the following properties:*

(i) *$K(\cdot, \omega)$ is σ-additive and λ-continuous on $\mathscr{B}$ such that if $g(\omega) = n_{\varrho_2}(K(\cdot, \omega)) = \sup\{\varrho_2(x^* K(\cdot, \omega)): \|x^*\| \leqslant 1\}$, $\varrho_1'(g) = M < \infty$;*

(ii) *$K(E, \cdot)$ is measurable on Ω;*

(iii) *for each $A \in \Sigma$, $\mu(A) < \infty$, $\nu(E) = \int_A K(E, \omega)\, d\mu$, $E \in \mathscr{B}$, then $\nu(\cdot)$ is σ-additive and λ-continuous on $\mathscr{B}$ for which the R-N derivative relative to λ exists (this is so if λ is localizable);*

(iv) $Tf = \dfrac{d}{d\lambda} \displaystyle\int_\Omega K(\cdot, \omega) f(\omega)\, d\mu(\omega)$, $f \in L^{\varrho_1}(\Sigma)$;

(v) $\|T\| = M$.

Conversely, a $K(\cdot, \cdot)$: $\mathscr{B} \times \Omega \mapsto$ scalars, with properties (i) *to* (iii) *defines a $T \in B(L^{\varrho_1}, \mathscr{X})$ by* (iv) *with norm bound of* (v). *ν of* (iii) *is defined on Σ_0.*

Proof. By Theorem 3.2, one has $Tf = \int_\Omega f d\nu$, $\|T\| = n_{\varrho'_1}(\nu)$, where ν: $\Sigma_0 \mapsto \mathscr{X} = L^{\varrho_2}(\lambda)$. Since μ is localizable and $\mathscr{X}$ has the R-N property relative to μ, and ϱ_1 is continuous at zero, one has $(d\nu/d\mu)(\omega) = h(\cdot, \omega) \in \mathscr{X}$ (since then ν is σ-additive also). Let $K(E, \omega) = \int_E h(t, \omega)\, d\lambda(t)$, so that $K(\cdot, \omega) \in A_{\varrho''_2}(\lambda) = A_{\varrho_2}(\lambda)$ and satisfies (i)-(iii). Moreover,

$$Tf = \int_\Omega f(\omega) h(\cdot, \omega)\, d\mu(\omega) = \frac{d}{d\lambda} \int_\Omega f(\omega) K(\cdot, \omega)\, d\mu(\omega)$$

by the chain rule for R-N derivatives (cf. [9], III. 10). Since $\|T\| = n_{\varrho'_1}(\nu) = M$, the direct part follows. The converse is obtained by a similar argument.

Remark. If $L^{\varrho_1}(\Sigma)$ is a Lebesgue space, μ is σ-finite and $L^{\varrho_2}(\mathscr{B}) = L^1(S, \mathscr{B}, \lambda)$, where S is a compact Hausdorff space and λ is a regular measure (and with some variations of this hypothesis), such results were proved in [9], p. 506 ff. All these are obtainable from the above, by choosing ϱ_1 and ϱ_2 appropriately. It also includes [20], Thm. 6.1, by specializing $K(\cdot, \cdot)$ to a point function. Moreover, a problem of disintegration of measures (cf. [7]) can be reformulated with the above result.

1.4. Compactness and metric approximation. In view of the preceding analysis, and the work to follow in the later sections, it will be useful to characterize the (weakly) compact operators in $B(L^\varrho, \mathscr{X})$. This is sketched in the following two results.

THEOREM 4.1. *Let $T \in B(L^\varrho, \mathscr{X})$. Then T is weakly compact iff the pair of set functions G_1 and G_2 in its representation* (*Theorem* 3.2) *satisfies the following conditions:*

(i) *The sets* (a) *$\{x^* G_1(E \cap \cdot): \|x^*\| \leqslant 1, \varrho(\chi_E) \leqslant 1\}$ and* (b) *$\{x^* G_2: \|x^*\| \leqslant 1\}$ are relatively sequentially compact in $ba(\Omega, \Sigma, \mu)$, the space of bounded additive set functions vanishing on μ-null sets. If, moreover, ϱ is continuous at zero, and μ is localizable, then* (G_1 *is σ-additive and* $G_2 = 0$) (i) *can be replaced by*

(ii) *there is* (*uniquely*) *a strongly measurable $g \in L^{\varrho'}_{\mathscr{X}}$ such that*

$$Tf = \int_\Omega f g\, d\mu, \qquad \|T\| = |||G|||_{\varrho'} = \sup\{\varrho'(x^* g): \|x^*\| \leqslant 1\}, \tag{1}$$

and $\int_E g d\mu$ takes its values in a weakly compact subset of $\mathscr{X}$, except for a μ-null set, for $\varrho(\chi_E) \leqslant 1$.

Proof. By a result of Gantmacher ([9], p. 485), T: $L^\varrho \mapsto \mathscr{X}$ is weakly compact iff its adjoint T^* is. But by Theorem 3.2,

$$Tf = \int_\Omega f dG_1 + \int_\Omega F(f)\, dG_2, \qquad f \in L^\varrho, \ \|T\| = |||G|||_{\varrho'},$$

and T^*: $x^* \mapsto x^* G$, is a bounded map. Since $E \in \Sigma_0$, $\varrho(\chi_E) \leqslant 1$ and $\|x^*\| \leqslant 1$ implies $|x^* G(E)| \leqslant \varrho(\chi_E) n_{\varrho'}(G_1)$, it follows by [9], p. 97, that the set in (i)(a) is in $ba(\Sigma_0, \mu)$. Now using the decomposition $T = T_1 + T_2$, $T_1 \in B(M^\varrho, \mathscr{X})$, $T_2 \in B(N^\varrho, \mathscr{X})$ it follows that, by the above quoted result, T_i (and hence T) will be weakly compact iff (i) holds. Thus the first part follows.

If ϱ is continuous at zero, then $G_2 = 0$, and by Corollary 1.16 and [9], p. 318, G_1 is μ-continuous. The weak compactness of T ($= T_1$) implies that, due to the localizability of μ, G_1 has the R-N property, and by the vector R-N theorem (cf. [7] or [9], p. 541) there is a stongly measurable $g\ (= dG_1/d\mu)$, so that (1) holds. Conversely, if $dG = g d\mu$, then, from well-known results, (1) is shown to define a weakly compact T on L^ϱ to $\mathscr{X}$, as desired.

For compact T, the above result takes the following form:

THEOREM 4.2. *An operator $T \in B(L^\varrho, \mathscr{X})$ is compact iff, in its representation (of Theorem 3.2), (a) $\{G_1(E)\colon \varrho(\chi_E) \leqslant 1\} \subset \mathscr{X}$ is conditionally compact, and (b) G_2 takes its values in a conditionally compact subset of $\mathscr{X}$.*

Proof. By the representation, for $f \in L^\varrho$,

$$Tf = \int_\Omega f dG_1 + \int_\Omega F(f)\, dG_2 = T_1 f + T_2 f \quad \text{(say)}. \tag{2}$$

T is compact iff T_i are, $i = 1, 2$. Since Theorem 1.7 (and a known result of Grothendieck) implies N^ϱ is an (AM)-space (without a unit) so that it is isometrically isomorphic to a closed subspace of $C(S)$, on some compact Hausdorff space S. Then regarding T as a map on that subspace to $\mathscr{X}$, it follows from a result of Bartle-Dunford-Schwartz ([9], p. 496), with trivial modifications, that G_2 has a conditionally compact range in $\mathscr{X}$ iff T_2 is compact. It remains to consider T_1.

If T_1 is compact, then $A = \overline{T_1 U} \subset \mathscr{X}$ is a compact set, where U is the unit ball in M^ϱ. Taking $f \in U, f = \chi_E$ yields $G_1(E) \in A$. Thus the set in (a) is conditionally compact in $\mathscr{X}$. Conversely, if (a) holds, let $K = \{G_1(E)\colon \varrho(\chi_E) \leqslant 1\}$, so that $\bar{K}$ is compact. Then $T_1 U \subset$ convex hull of $\{(iK) \cup \cup (-iK)\}$, and so is precompact. This implies T_1 is compact, and the result follows.

Remarks. 1. If ϱ is continuous at zero, then $G_2 = 0$ and G_1 is σ-additive, when μ is localizable, G_1 has a density g, and a corresponding condition on g is more involved here than in Theorem 4.1.

2. Theorem 4.2 can be specialized to get Theorems 7.1 and 7.3 of [20]. In fact, let $\mathscr{X} = L^{\varrho_1}(S, \mathscr{B}, \lambda)$, $T\colon L^\varrho(\Sigma) \mapsto L_a^{\varrho_1}(\mathscr{B})$. Then the compactness of T implies the set $E = \{T\chi_A\colon \varrho(\chi_A) \leqslant 1\} \subset \mathscr{X}$ has a uniform a.c.n., (i.e., $\varrho(\chi_{A_n} g) \downarrow 0$ as $A_n \downarrow \emptyset$ uniformly in $g \in E$). The converse that T is compact holds if E has uniform a.c.n. and that $U_\pi E$ is conditionally compact for each π, where $\{U_\pi\}$ is a generalized base in $L_a^{\varrho_1}(\mathscr{B})$, i.e. U_π are degenerate and $U_\pi g \to g$ in norm as π is refined, $\|U_\pi\| \leqslant \alpha < \infty$.

For, let $A_n \in \Sigma$, $A_n \downarrow \emptyset$ be arbitrary. Let $U_n = \chi_{A_n^c}\colon L_a^{\varrho_1} \mapsto L_a^{\varrho_1}$. Then $U_n g \to g$ for each $g \in L_a^{\varrho_1}$ by a.c.n. of g. Since $L_a^{\varrho_1}$ is a B-space, by Corollary 2.5, $U_n g \to g$ uniformly for g in any compact subset of $L_a^{\varrho_1}$, by [9], IV. 5. 4. In particular, if T is compact, then $\bar{E}$ is compact by theorem and $U_n(E) \to E$, or E has a uniform a.c.n. by what is just proved. The converse under the above hypothesis is a consequence of the same result of [9]. That such $\{U_\pi\}$ exists is shown in the next theorem.

Another condition for the converse part is to demand that $T^*\colon L^{\varrho'_1} \mapsto L_a^{\varrho'}$, and that T be defined by a "kernel". This was shown in [20].

Definition 4.3. A B-space $\mathscr{X}$ is said to have the *metric approximation property* (m.a.p.) if the identity map on $\mathscr{X}$ can be uniformly approximated on precompacts of $\mathscr{X}$, by degenerate operators (i.e., those of finite dimensional range) of bound 1.

The purpose of the next two results is to prove that L^ϱ and $(L^\varrho)^*$ have m.a.p.'s. This is done by first proving it for M^ϱ and then extending it by use of Theorems 4.1 and 4.2 above. It appears also possible to prove this same result using the procedure of [25], Thms, 6.1 and 6.4, after using certain isomorphism theorems of the next chapter (particularly, Thm. 1.1 there). The following approach has independent interest.

THEOREM 4.4. *The space M^ϱ has the m.a.p.*

Proof. Let $f \in M^\varrho$ and $\pi = \{E_1, \ldots, E_n\}$, $E_i \in \Sigma_0$, disjoint and $\mu(E_i) > 0$. Let $0 \leqslant G^0 \in \mathscr{A}_{\varrho'}(\mu)$ with $G_{E_i}^0 = G^0(\cap \cdot)$, $i = 1, \ldots, n$ (non-zero), and define f_π as

$$f_\pi = \sum_{i=1}^n \frac{1}{\varrho(\chi_{E_i})\, \varrho'(G_{E_i}^0)} \Big(\int_{E_i} f dG^0\Big) \chi_{E_i}. \tag{3}$$

Then f_π $(\in M^\varrho)$ is a simple function relative to Σ_0. The G^0 thus far is not required to verify any conditions. However, it will be chosen to satisfy also

$$\varrho'(G_{E_i}^0) = G^0(E_i) \cdot [\varrho(\chi_{E_i})]^{-1}. \tag{4}$$

This is possible. In fact by the Hahn-Banach theorem, there is an $x_{E_i}^* \in (L^\varrho)^*$ and a $0 \leqslant G_{E_i} \in \mathscr{A}_{\varrho'}(\mu)$, vanishing outside E_i, by Theorem 1.13, such that $\varrho(\chi_{E_i}) = x_{E_i}^*(\chi_{E_i}) = G_{E_i}(E_i)$, and $\|x_{E_i}^*\| = 1 = \varrho'(G_{E_i})$. So let $G_{E_i}^0 = \varrho'(G_{E_i}^0) G_{E_i}$, and G^0 be their sum. So (4) holds. Thus G^0 is determined by π. Now if

$$f = \sum_{i=1}^n a_i \chi_{E_i} \in M^\varrho,$$

then clearly

$$f_\pi = \sum_{i=1}^n \frac{a_i}{\varrho(\chi_{E_i})} \frac{G^0(E_i)}{\varrho'(G_{E_i}^0)} \chi_{E_i} = f$$

by (4).

It may be noted that if μ has FSP, so that ϱ' exists on point functions nontrivially, then (4) was directly proved in [11], p. 580, under the hypothesis that ϱ is a leveling norm and in [13], p. 8, that ϱ verifies (J). Then it becomes $\varrho'(\chi_{E_i})\, \varrho(\chi_{E_i}) = \mu(E_i)$. These additional hypotheses are not now available, and the above form suffices in this general situation and subsumes the quoted results.

Now define the degenerate maps $U_\pi: f \mapsto f_\pi$. To see that $\{U_\pi\}$ are uniformly bounded, consider

$$\varrho(U_\pi f) = \varrho(f_\pi) = \sup\{|x^*(f_\pi)|: \|x^*\| \leqslant 1\}$$
$$= \sup\left\{\left|\int_\Omega f_\pi dG\right| : \varrho'(G) \leqslant 1\right\} \quad \text{(by Theorem 1.13)}$$
$$\leqslant \sup\left\{\sum_{i=1}^n \frac{\left|\int_{E_i} f dG^0\right| \|G(E_i)\|}{\varrho(\chi_{E_i})\varrho'(G^0_{E_i})} : \varrho'(G) \leqslant 1\right\} \quad \text{(by (3))}$$
$$\leqslant \varrho(f) \sup\left\{\sum_{i=1}^n \int_\Omega \frac{\chi_{E_i}}{\varrho(\chi_{E_i})} d|G| : \varrho'(G) \leqslant 1\right\}$$
$$\leqslant \varrho(f) \sup\left\{\sum_{i=1}^n \left[\sup \int_{E_i} |g| d|G| : \varrho(g) \leqslant 1\right] : \varrho'(G) \leqslant 1\right\}$$
$$\leqslant \varrho(f) \sup\left\{\sup\left[\int_\Omega |g| d|G| : \varrho(g) \leqslant 1\right] : \varrho'(G) \leqslant 1\right\}$$
$$\leqslant \varrho(f) \sup\{\varrho'(G): \varrho'(G) \leqslant 1\} \leqslant \varrho(f),$$

since $\varrho'(G) = \varrho'(|G|)$. Thus U_π is a contraction for each π.

Since $U_\pi f = f$ for step functions $f \in M^\varrho$, and appropriate π, by the earlier paragraph, it follows that $\varrho(U_\pi f - f) \to 0$ for all step functions, as π is refined. Since step functions are dense in M^ϱ by definition, this holds for all $f \in M^\varrho$. This implies, by [9], IV. 5.4, the statement of the theorem, as asserted.

Note that since for $L^\varrho = L^p$, $M^\varrho = L^p$, $1 \leqslant p \leqslant \infty$, for any μ, this case already slightly improves [9], IV.8.18.

THEOREM 4.5. *Every compact $T \in B(L^\varrho, \mathscr{X})$ can be uniformly approximated by degenerate operators of norm, at most $\|T\|$.*

Proof. If $T \in B(L^\varrho, \mathscr{X})$, then let $T = T_1 + T_2$, where $T_1 \in B(M^\varrho, \mathscr{X})$ and $T_2 \in B(N^\varrho, \mathscr{X})$ as before. If T is compact, then so are T_1 and T_2. Since M^ϱ has m.a.p., by [14], p. 179 (A_2), it can be approximated uniformly by $T_{1\pi_0} \in B(M^\varrho, \mathscr{X})$, degenerate and bounded by $\|T_1\|$. $(N^\varrho)^*$ is an (AL)-space, and so has the m.a.p. (by [14], or Theorem 4.4 above). Hence by [14], p. 167, N^ϱ also has the m.a.p. Consequently, T_2 can be uniformly approximated by $T_{2\pi_1} \in B(N^\varrho, \mathscr{X})$, degenerate and bounded by $\|T_2\|$. If π is the common refinement of π_0 and π_1. Then $T_\pi = T_{1\pi_0} + T_{1\pi_1}$ is degenerate, $T_\pi \in B(L^\varrho, \mathscr{X})$ bounded by $\|T\|$, and $\|T - T_\pi\| \to 0$ as π is refined, and the Moore-Smith limit is taken. This gives the desired result.

The main result on the m.a.p. is given by the following

THEOREM 4.6. *Both L^ϱ and $(L^\varrho)^*$ have the m.a.p.*

Proof. By the preceding result $T \in B(L^\varrho, \mathscr{X})$, compact, implies the existence of a degenerate $T_\pi \in B(L^\varrho, \mathscr{X})$ with $\|T - T_\pi\| \to 0$. So T_π is of the form (if π has n elements)

$$T_\pi = \sum_{i=1}^n y_i^*(\cdot)x_i, \quad y_i^* \in (L^\varrho)^*,\ x_i \in \mathscr{X},\ i = 1, \ldots, n.$$

This means $T_\pi \in (L^\varrho)^* \otimes \mathscr{X}$, the formal tensor product. Since $T_\pi \to T$ uniformly, it follows that T lies in the norm closure of $(L^\varrho)^* \otimes \mathscr{X}$ in $B(L^\varrho, \mathscr{X})$. Hence by [14], p. 167, Prop. 36 (in which only the completeness of $\mathscr{X}$ is needed), $(L^\varrho)^*$ has the m.a.p. But this implies, by another part of the same result, L^ϱ itself has the m.a.p. This completes the proof.

Remarks. 1. If $L^\varrho = L^p$, $1 \leqslant p \leqslant \infty$, the above result was proved in [25], and in [14]. The above theorem also implies that every Orlicz space, on any measure space, has the m.a.p.

2. Actually, Grothendieck's propositions 36 on p. 167, [14], and 39 on p. 179 (cf. conditions (A′) on 167 and conditions (A) $\Leftrightarrow$ (A_3) on p. 179) imply that a B-space $\mathscr{X}$ has the m.a.p. iff any one, and hence all, of its $n (n \geqslant 1)$ conjugates has the m.a.p. This then yields a more general result that the n-th $(n \geqslant 1)$ conjugate of L^ϱ-space has the m.a.ep. also

1.5. Tensor products. The importance of tensor products in function spaces L^ϱ is due, in large part, to the fact that some of the key results of the scalar case can be transfered to the vector valued spaces $L^\varrho_{\mathscr{X}}$. Since this is indeed the case, for some work in Chapter 2 below, certain results on the greatest and least cross-norms, for L^ϱ and $\mathscr{X}$, will be given here. They also extend and complement certain results in [14] and [12].

Definition 5.1. (a) If $\mathscr{X}$ is a normed linear space and $\mathscr{Y}$ is a B-space, then $\mathscr{A}_{\varrho'}(B(\mathscr{X}, \mathscr{Y}), \mu)$ stands for the class of additive set functions $G: \Sigma_0 \mapsto B(\mathscr{X}, \mathscr{Y})$, vanishing on μ-null sets such that $\mathscr{V}_{\varrho'}(G) < \infty$ where the norm $\mathscr{V}_{\varrho'}(\cdot)$ is defined by

(1) $$\mathscr{V}_{\varrho'}(G) = \sup\{n_{\varrho'}(Gz): \|z\| \leqslant 1, z \in \mathscr{X}\},$$

then ${}_{\varrho'}(\cdot)$ and other symbols being as before (cf. Definition 3.1).

(b) The class $\mathscr{B}_{\varrho'}(B(\mathscr{X}, \mathscr{Y}), \mu)$ stands for the additive set functions $\nu: \Sigma \mapsto B(\mathscr{X}, \mathscr{Y})$, vanishing on μ-null sets and for each $y^* \in \mathscr{Y}^*$, $z \in \mathscr{X}$, $y^* \nu z$ is pfa with its support contained in that of an element of $L^\varrho - M^\varrho$, and $|||\nu|||(\Omega) < \infty$, where

(2) $$|||\nu|||(\Omega) = \sup\{\|\nu z\|(\Omega): \|z\| \leqslant 1, z \in \mathscr{X}\}.$$

(c) Let $\mathscr{U}_{\varrho'}(B(\mathscr{X},\mathscr{Y}),\mu) = \mathscr{A}_{\varrho'}(B(\mathscr{X},\mathscr{Y}),\mu)\oplus\mathscr{B}_{\varrho'}(B(\mathscr{X},\mathscr{Y}),\mu)$, and $G\in\mathscr{U}_{\varrho'}$, $G = G_1+G_2$ implies $R_{\varrho'}(G)<\infty$, where

$$R_{\varrho'}(G) = \sup\{\|Gz\|_{\varrho'}\colon \|z\|\leqslant 1,\ z\in\mathscr{X}\}. \tag{3}$$

This definition is necessary to restate a version of Theorem 3.2 when $\mathscr{X} = B(\mathscr{X},\mathscr{Y})$, since it is very useful in characterizing some tensor product spaces $L^\varrho\otimes_\gamma\mathscr{X}$ and $L^\varrho\otimes_\lambda\mathscr{X}$, where γ,λ are the greatest and least cross-norms. Recall, from [33], that if

$$t = \sum_{i=1}^n x_i\otimes y_i\in\mathscr{X}\otimes\mathscr{Y},$$

a formal tensor product, then

$$\gamma(t) = \inf\Big\{\sum_{i=1}^n \|x_i\|\ \|y_i\|\colon \text{all representations of } t\Big\}, \tag{4}$$

and $\mathscr{X}\otimes_\gamma\mathscr{Y}$ is the completion, in this norm, of $\mathscr{X}\otimes\mathscr{Y}$. Similarly, if t is a given element of $\mathscr{X}\otimes\mathscr{Y}$ as above, then

$$\lambda(t) = \sup\Big\{\Big|\sum_{i=1}^n x^*(x_i)y^*(y_i)\Big|\colon \|x^*\|\leqslant 1,\ \|y^*\|\leqslant 1\Big\}, \tag{5}$$

and $\mathscr{X}\otimes_\lambda\mathscr{Y}$ is the completion, in this norm, of $\mathscr{X}\otimes\mathscr{Y}$.

THEOREM 5.2. *In the notation introduced above, $B(L^\varrho, B(\mathscr{X},\mathscr{Y}))$ is isometrically isomorphic to $\mathscr{U}_{\varrho'}(B(\mathscr{X},\mathscr{Y}),\mu)$. More explicitly, every $T\in B(L^\varrho, B(\mathscr{X},\mathscr{Y}))$ can be uniquely represented as*

$$Tf = \int_\Omega f\,dG = \int_\Omega f\,dG_1 + \int_\Omega F(f)\,dG_2,\quad f\in L^\varrho, \tag{6}$$

with

$$\|T\| = R_{\varrho'}(G),\quad G = G_1+G_2\in\mathscr{U}_{\varrho'}(B(\mathscr{X},\mathscr{Y}),\mu).$$

The proof is identical with that of Theorem 3.2. Note that by definition, $\|T\| = \sup\{\sup[\|T(fz)\|_{\mathscr{Y}}\colon \|z\|\leqslant 1]\colon \varrho(f)\leqslant 1\}$. From this theorem, it is possible to characterize the subspace $B(\mathscr{M}^\varrho_{\mathscr{X}},\mathscr{Y})$ of $B(L^\varrho_{\mathscr{X}},\mathscr{Y})$, where $\mathscr{M}^\varrho_{\mathscr{X}} = \overline{sp}\{fx\colon f\in L^\varrho,\ x\in\mathscr{X}\}\subset L^\varrho_{\mathscr{X}}$.

It is again convenient to introduce the following

Definition 5.3. For any $T\colon L^\varrho\mapsto B(\mathscr{X},\mathscr{Y})$ let the two norms be given by

$$|||T||| = \sup\{\|y^*(Tf)\|_{\mathscr{X}^*}\colon \varrho(f)\leqslant 1,\ \|y^*\|\leqslant 1\} \tag{7}$$

and

$$|||T|||^{\sim} = \sup\Big\{\sum_{i=1}^n \|y^*T(f_i)\|_{\mathscr{X}^*}\colon \varrho\Big(\sum_{i=1}^n f_i\Big)\leqslant 1,\ \|y^*\|\leqslant 1,\ f_i\wedge f_j = 0\Big\}, \tag{8}$$

and if $\tilde{T}\colon \mathscr{M}^\varrho_{\mathscr{X}}\mapsto\mathscr{Y}$, then similarly

$$|||\tilde{T}|||^{\sim} = \sup\Big\{\sum_{i=1}^n |y^*(\tilde{T}(f_iz_i))|\colon \varrho\Big(\sum_{i=1}^n f_iz_i\Big)\leqslant 1,\ \|y^*\|\leqslant 1\Big\}. \tag{8'}$$

If $G\in\mathscr{U}_{\varrho'}(B(\mathscr{X},\mathscr{Y}),\mu)$, also let

$$R^{\sim}_{\varrho'}(G) = \sup\{|||y^*G|||_{\varrho'}\colon \|y^*\|\leqslant 1\}\quad (\geqslant R_{\varrho'}(G) \text{ of (3) above}). \tag{9}$$

(As usual, here $f\wedge g = 0$ means f,g have disjoint supports.)

THEOREM 5.4. *If $T\in B(\mathscr{M}^\varrho_{\mathscr{X}},\mathscr{Y})$, where $\mathscr{Y}$ is a B-space and $\mathscr{X}$ a normed linear space, then there exists a unique $T'\in B(L^\varrho, B(\mathscr{X},\mathscr{Y}))$ given by the correspondence $T(fz) = (T'f)z$ for all $f\in L^\varrho$, $z\in\mathscr{X}$. Moreover, for all T with $|||T|||<\infty$, there is a unique $G\in\mathscr{U}_{\varrho'}(B(\mathscr{X},\mathscr{Y}),\mu)$, such that*

$$R_\varrho(G) = \|T'\|\leqslant\|T\|\leqslant|||T'||| = |||T|||^{\sim} = R^{\sim}_{\varrho'}(G), \tag{10}$$

and thus (7) and (8) are equal in this case. Furthermore, if $\mathscr{Y}$ is the scalars, then $\|T\| = |||T'|||$, and if $L^\varrho = L^1$, then $\|T'\| = \|T\| = |||T'|||$.

Remark. If $\mathscr{X}$ is the scalars, this reduces to Theorem 3.2 since $\mathscr{M}^\varrho_{\mathscr{X}} = L^\varrho$, and if $L^\varrho = L^p$, $p\geqslant 1$, so that $\mathscr{M}^\varrho_{\mathscr{X}} = L^p_{\mathscr{X}}$, this result is essentially given in [5], p. 196. The norm equations (8) and (9) were considered in the L^p and respectively in the Orlicz space cases in [5] and in [37]. The proof follows the ideas in [5] and will be sketched.

Proof. For any given $T\in B(\mathscr{M}^\varrho_{\mathscr{X}},\mathscr{Y})$, let T' be as in the statement. The boundedness of T' and the first part of (10) follow from

$$\|T'(f)z\|_{\mathscr{Y}} = \|T(fz)\|_{\mathscr{Y}}\leqslant\|T\|\,\varrho(f)\,\|z\|.$$

Hence by Theorem 5.2, there is a unique $G\in\mathscr{U}_{\varrho'}(B(\mathscr{X},\mathscr{Y}),\mu)$ with

$$T'f = \int_\Omega f\,dG,\ \text{or}\ T(fz) = \int_\Omega \langle fz, dG\rangle,\quad \|T'\| = R_{\varrho'}(G). \tag{11}$$

Thus if $T' = 0$, then $G = 0$ so that $T = 0$ since linear combinations of $\{fz\}$ are dense in $\mathscr{M}^\varrho_{\mathscr{X}}$. So $T\leftrightarrow T'$ are in one-to-one correspondence.

To prove (10) completely, first note that $|||T'||| = R^{\sim}_{\varrho'}(G)$, which is a consequence of (6), (7), and (9) and Definition 3.1(c). Let

$$g = \sum_{i=1}^n f_iz_i\in\mathscr{M}^\varrho_{\mathscr{X}},\quad z_i\in\mathscr{X},\ f_i\wedge f_j = 0.$$

Such elements are dense in the latter, and using (8), since $T\colon \mathscr{M}^{\varrho}_{\mathscr{X}} \mapsto \mathscr{Y}$, one has

(12)
$$|||T|||^{\sim} = \sup\Big\{\sum_{i=1}^{n} |y^* T(f_i z_i)|\colon \ \varrho(g) \leqslant 1,\ \|y^*\| \leqslant 1\Big\}$$
$$\leqslant \sup\Big\{\sum_{i=1}^{n} \|y^* T'(f_i)\|_{\mathscr{X}^*}\|z_i\|\colon \ \varrho(g) \leqslant 1,\ \|y^*\| \leqslant 1\Big\}$$
$$= \sup\Big\{\sum_{i=1}^{n} \Big\|\int_{\Omega} \|f_i z_i\|\, d(y^* G)\Big\|_{\mathscr{X}^*}\colon \ \varrho(g) \leqslant 1,\ \|y^*\| \leqslant 1\Big\} \text{ (by (11))}$$
$$\leqslant \sup\Big\{\Big\|\int_{\Omega} |h|\, d(y^* G)\Big\|_{\mathscr{X}^*}\colon \ \varrho(h) \leqslant 1,\ \|y^*\| \leqslant 1\Big\} = R^{\sim}_{\varrho'}(G) = |||T'|||.$$

For the opposite inequality, one can proceed as in [5]. Thus for any $\varepsilon > 0$, and $g = \sum_{i=1}^{n} f_i$, $\varrho(g) \leqslant 1$ with $f_i \wedge f_j = 0$, choose a $z_i \in \mathscr{X}$, $\|z_i\| = 1$ (Hahn-Banach theorem) such that

$$\|y^*(T' f_i)\|_{\mathscr{X}^*} < |y^*(T' f_i) z_i| + \varepsilon/n, \qquad i = 1, \ldots, n.$$

Then

$$\tilde{g} = \sum_{i=1}^{n} f_i z_i \in \mathscr{M}^{\varrho}_{\mathscr{X}}, \qquad \varrho(\|\tilde{g}\|) = \varrho(|g|) \leqslant 1,$$

and, by (8′),

$$\sum_{i=1}^{n} \Big\|\int_{\Omega} f_i\, d(y^* G)\Big\|_{\mathscr{X}^*} - \varepsilon = \sum_{i=1}^{n} \Big\|y^* \int_{\Omega} f_i\, dG\Big\|_{\mathscr{X}^*} - \varepsilon < \sum_{i=1}^{n} |y^* T(f_i z_i)| \leqslant |||T|||^{\sim}.$$

Hence

(13)
$$|||T'||| \leqslant |||T|||^{\sim}.$$

Now (12) and (13) yield (10), and the main part of the theorem follows.

The assertion about $L^{\varrho} = L^1$, and that when $\mathscr{Y}$ is scalars, is similar to a result in [5], p. 197. Thus the result follows.

The above theorem together with the fact that for any B-spaces $\mathscr{X}_1, \mathscr{X}_2$, $(\mathscr{X}_1 \otimes_{\gamma} \mathscr{X}_2)^* \cong B(\mathscr{X}_1, \mathscr{X}_2^*)$, by [33], p. 47, where γ is the greatest cross-norm (cf. (4)), one obtains the following result (proved differently in [14], cf. also [15]):

COROLLARY 5.5. *$B(L^1_{\mathscr{X}}, C) \cong B\big(L^1, B(\mathscr{X}, C)\big)$, where C = scalars. More explicitly, $(L^1_{\mathscr{X}})^* \cong B(L^1, \mathscr{X}^*) \cong (L^1 \otimes_{\gamma} \mathscr{X})^*$, and that $L^1_{\mathscr{X}} \cong L^1 \otimes_{\gamma} \mathscr{X}$, where $\mathscr{X}$ is any B-space.*

Also since the theorem implies the inclusions

(14)
$$(\mathscr{M}^{\varrho}_{\mathscr{X}})^* = B(\mathscr{M}^{\varrho}_{\mathscr{X}}, C) \subset B\big(L^{\varrho}, B(\mathscr{X}, C)\big) \cong (L^{\varrho} \otimes_{\gamma} \mathscr{X})^*,$$

one has the following important consequence:

COROLLARY 5.6. *For any B-space $\mathscr{X}$, and the BFS L^{ϱ}, one has*

(15)
$$L^{\varrho} \otimes_{\gamma} \mathscr{X} \subset \mathscr{M}^{\varrho}_{\mathscr{X}} \subset L^{\varrho}_{\mathscr{X}},$$

where the first space (with the strongest topology) is densely embedded in the second (in the norm topology of the latter) and the second space is isometrically embedded in the third. There is equality throughout if either $L^{\varrho} = L^1$ or $\mathscr{X} = C$, and between the last two if ϱ is an a.c.n.

Another consequence of the theorem is given in the following

COROLLARY 5.7. *Let S be a compact Hausdorff space and $C(S)$ be the space of real continuous functions on S. Then*

(16)
$$C^*(S) \otimes_{\gamma} \mathscr{X} \cong L^1_{\mathscr{X}}(\tilde{S}, \nu) \subset \mathrm{ca}(\tilde{S}, \nu, \mathscr{X}),$$

where $\tilde{S}$ is the Stone-Gelfand-Kakutani space, ν is a finite regular Borel measure on the clopen-sets of $\tilde{S}$, and S is homeomorphic to a closed subset of $\tilde{S}$. The last symbol stands for the space of $\mathscr{X}$-valued ν-continuous σ-additive set functions on the Borel field of $\tilde{S}$, of finite total variation. The isometric embedding on the right is onto iff the B-space $\mathscr{X}$ has the R-N property relative to ν.

Proof. By the classical result of Kakutani ([13], p. 1020), the (AL)-space $C^*(S) \cong L^1(\tilde{S}, \nu)$, where the right space is as described. Then using the preceding corollary the first half of (16) obtains, and the last inclusion is obvious. If $\mathscr{X}$ has the R-N property, then the last space can be expressed as $L^1_{\mathscr{X}}(\tilde{S}, \nu)$ and only then, as is well-known. (See also [37], p. 33, Thm. 5.)

It may be of interest to note that $\mathscr{X}$ has R-N property with respect to a (finite) measure ν which is not purely atomic, then it has the property relative to every (finite) measure. (Cf. [3], p. 26, about this and related results.) Since, as noted in the proof of Theorem 3.3, S can be homeomorphically embedded as a closed subset of $\tilde{S}$, the above result slightly extends [12], Thms. 6.2-6.4. As pointed out, for instance, in [12], the result can be stated for S locally compact, by reducing it to the one considered here by a standard compactification argument.

Now a general result on least cross-norms will be given. It illuminates the duality theory of [33], p. 138-143, and extends also [12], Thm. 6.1, to the L^{ϱ}-spaces. Recall that $f\colon \Omega \mapsto \mathscr{X}$ is weakly measurable iff the numerical function $x^* f$ is measurable for each $x^* \in \mathscr{X}$. Let $W^{\varrho}_{\mathscr{X}} = \overline{sp}\{(f\colon \Omega \mapsto \mathscr{X})\colon f$ weakly measurable and $w_{\varrho}(f) < \infty\}$, where the norm $w_{\varrho}(\cdot)$, defined by Pettis [24] for the L^p-case, is given by

(17) $$w_\varrho(f) = \sup\{\varrho(x^*f): \|x^*\| \leqslant 1\}.$$

The main result here is given by the following

THEOREM 5.8. *With the above notation, $L^\varrho \otimes_\lambda \mathscr{X} \cong W^\varrho_{\mathscr{X}}$. Thus, $W^\varrho_{\mathscr{X}}$ can be identified isometrically as the space of all compact linear operators of $(L^\varrho)^*$ into $\mathscr{X}$.*

An immediate consequence of this result, and of (15), is the

COROLLARY 5.9. *If L^ϱ is a BFS, then*

(18) $$L^\varrho \otimes_\gamma \mathscr{X} \subset \mathscr{M}^\varrho_{\mathscr{X}} \subset L^\varrho_{\mathscr{X}} \subset W^\varrho_{\mathscr{X}} \cong L^\varrho \otimes_\lambda \mathscr{X},$$

where the inclusions are both algebraic and topological. If ϱ is a.c.n., then there is equality between the second and third spaces.

Proof of Theorem 5.8. Consider the formal tensor product $L^\varrho \otimes \mathscr{X}$. If $t \in L^\varrho \otimes \mathscr{X}$, then

$$t = \sum_{i=1}^n f_i \otimes x_i \quad \left(= \sum_{i=1}^n f_i x_i\right) \quad \text{for some } n \geqslant 1.$$

If $f^t = \sum_{i=1}^n f_i x_i$, then f^t is weakly (even strongly) measurable and

$$w_\varrho(f^t) \leqslant \sum_{i=1}^n \varrho(f_i)\|x_i\| < \infty.$$

Let T^t: $(L^\varrho)^* \mapsto \mathscr{X}$ be defined by

(19) $$x^*(T^t G) = \int_\Omega x^*(f^t)\,dG, \quad G \in \mathscr{A}_{\varrho'}(\mu),\ x^* \in \mathscr{X}^*$$

(by Theorem 1.13). Then by definition of $\lambda(\cdot)$ in (5), and with (19), one has

(20) $$\lambda(t) = \|T^t\| = \sup\{\sup|x^*(T^t G)|: \varrho'(G) \leqslant 1, \|x^*\| \leqslant 1\}$$
$$= \sup\{\varrho(x^*(f^t)): \|x^*\| \leqslant 1\} = w_\varrho(f^t).$$

It follows that $L^\varrho \otimes_\lambda \mathscr{X} \subset W^\varrho_{\mathscr{X}}$, and the inclusion is an isometric embedding. Thus the correspondence $t \to T^t \to f^t$ is an isometry.

In order to prove the opposite inclusion, let f: $\Omega \mapsto \mathscr{X}$ be a weakly measurable function with $w_\varrho(f) < \infty$, and let T^f: $(L^\varrho)^* \mapsto \mathscr{X}$ be defined by

(21) $$x^*(T^f G) = \int_\Omega x^*(f)\,dG, \quad G \in \mathscr{A}_{\varrho'}(\mu),\ x^* \in \mathscr{X}^*.$$

Then, as above, $\|T^f\| = w_\varrho(f) < \infty$ and, moreover, T^f is a compact operator. The latter fact follows from an argument of Pettis' [24], Lemma 6.11, since the present case can be proved along the same lines using the properties of integration ([9], III. 2). Alternately, the present case can be reduced to that of [24] (in which $\Omega = [0,1]$ and $dG = g\,dx$ are not crucial) as follows. Consider the restriction σ-field $\Sigma(E)$, where $E \in \Sigma_0$, and note that G: $\Sigma(E) \mapsto$ scalars, is a bounded additive set function. Then by the important isomorphism theorem ([9], IV. 9. 11), $G \mapsto \hat{G}$, a regular σ-additive set function on a compact (Stone) space, where this correspondence is an isometric isomorphism, and the result on the latter space is now true by Pettis' Theorem [24]. Consequently, the same holds true by considering the preimage under the isometry on $\Sigma(E)$, in exactly the same way as in [9], IV. 9. 12, and since $E \in \Sigma_0$ is arbitrary the complete statement obtains.

Now by Theorem 4.6, both L^ϱ and $(L^\varrho)^*$ have the m.a.p. So the compact T^f can be uniformly approximated by degenerate operators T^f_π. But, by [14], p. 168, $L^\varrho \otimes_\lambda \mathscr{X}$ can be isometrically identified with the subspace of compact operators, in $B((L^\varrho)^*, \mathscr{X})$, which are uniform limits of degenerate operators iff L^ϱ or $\mathscr{X}$ has the m.a.p. Since T^f is shown to be one such, it follows that there exist $t_n \in L^\varrho \otimes \mathscr{X}$, and T^{t_n}, as in the first paragraph, with $\|T^f - T^{t_m}\| \to 0$. Hence $\lambda(t_n - t_m) = \|T^{t_n} - T^{t_m}\| \to 0$ as $n, m \to \infty$. Hence $t_n \to t$ in the $\lambda(\cdot)$ norm, and one has the correspondence $t \to T^t = T^f$. Thus $t = f$ and $f \in L^\varrho \otimes_\lambda \mathscr{X}$, since the latter is closed. So $W^\varrho_{\mathscr{X}} \subset L^\varrho \otimes_\lambda \mathscr{X}$ and $w_\varrho(f) = \lambda(t)$. This completes the proof.

If $L^\varrho = L^1$, this result was given in [12], as noted above, and if $L^\varrho = L^\infty$, it was noted in [14], p. 90. The above and Theorem 4.6 have the following consequence:

COROLLARY 5.10. *For any B-space $\mathscr{X}$, one has*

(22) $$(L^\varrho \otimes_\lambda \mathscr{X}) \cong (L^\varrho)^* \otimes_\gamma \mathscr{X}^* \quad \text{and} \quad (L^\varrho \otimes_\gamma \mathscr{X})^* \supset (L^\varrho)^* \otimes_\lambda \mathscr{X}^*.$$

The first equality is immediate from the preceding proof, since L^ϱ has the m.a.p., and of [33], p. 51, Thm. 3.6. The last inclusion is proved in [33], p. 52, Thm. 3.7. with $\alpha = \gamma$ here. There is equality in the last inclusion iff every bounded operator on L^ϱ to $\mathscr{X}^*$ is compact. Thus by [33], p. 141, this fact together with the reflexivity of both L^ϱ and $\mathscr{X}$ (when this is assumed) imply, and is implied by, the reflexivity of $L^\varrho \otimes_\lambda \mathscr{X}$ and of $L^\varrho \otimes_\gamma \mathscr{X}$. (The first condition holds only if either L^ϱ or $\mathscr{X}$ is finite dimensional!)

The next result is important for a projection problem in the next chapter.

PROPOSITION 5.11. *Let L^ϱ be a BFS, $\mathscr{S}^\varrho \subset L^\varrho$ be a closed subspace, and $\mathscr{S}^\varrho_{\mathscr{X}} = \overline{sp}\{fx: f \in \mathscr{S}^\varrho, x \in \mathscr{X}\}$. Let $\mathscr{Y} \subset \mathscr{X}$ be a closed subspace. Then the following two statements are equivalent:*

(I) *There exists a contractive projection P on $\mathscr{M}^\varrho_{\mathscr{X}}$ onto its subspace $\mathscr{S}^\varrho_{\mathscr{Y}}$.*

(II) *There exist contractive projections P_1 on L^ϱ onto $\mathscr{S}^\varrho$, and P_2 on $\mathscr{X}$ onto $\mathscr{Y}$.*

Proof. (I) ⇒ (II). From the fact that $\mathscr{S}^{\varrho}_{\mathscr{Y}}$ is a closed subspace of the B-space $\mathscr{M}^{\varrho}_{\mathscr{X}}$, L^{ϱ} can be identified isometrically with a subspace of $\mathscr{M}^{\varrho}_{\mathscr{X}}$ and $\mathscr{S}^{\varrho}$ with the corresponding subspace of $\mathscr{S}^{\varrho}_{\mathscr{Y}}$. In fact, if $f \in L^{\varrho}$ with $\varrho(f) = 1$, and $x \in \mathscr{X}$ with $\|x\| = 1$, then let P_1 be defined by $P_1(g)\cdot x = P(gx) \in \mathscr{S}^{\varrho}_{\mathscr{Y}}$. Then the subspace $\tilde{\mathscr{S}} = \{P(gx)\colon g \in L^{\varrho}\}$ of $\mathscr{S}^{\varrho}_{\mathscr{Y}}$ is linearly isometric to $\mathscr{S}^{\varrho}$ and the mapping $P_1\colon L^{\varrho} \mapsto \mathscr{S}^{\varrho}$ given by $P_1\colon g \mapsto P_1(g)$, above, is the required contractive projection. Similarly, the mapping, $(P_2x)f = P(fx)$ defines $P_2\colon x \mapsto P_2x \in \mathscr{Y}$, a contractive projection on $\mathscr{X}$ onto $\mathscr{Y}$. Thus (II) holds.

(II) ⇒ (I). If $P_1\colon L^{\varrho} \mapsto \mathscr{S}^{\varrho}$, and $P_2\colon \mathscr{X} \mapsto \mathscr{Y}$ are contractive projections. Then by [33], p. 58, it follows that there exists a contractive projection on $L^{\varrho} \otimes_{\gamma} \mathscr{X}$ onto $\mathscr{S}^{\varrho} \otimes_{\gamma} \mathscr{Y}$ and the latter is a closed subspace of the former. But by Corollary 5.9, these two spaces are respectively dense subspaces of $\mathscr{M}^{\varrho}_{\mathscr{X}}$ and $\mathscr{S}^{\varrho}_{\mathscr{Y}}$ in the topology of the latter. Also $\mathscr{S}^{\varrho}_{\mathscr{Y}}$ is a closed subspace of $\mathscr{M}^{\varrho}_{\mathscr{X}}$, from definition. It can now be checked directly, by considering adjoint mapping of P, that (I) obtains. It is also a simple consequence of [32] which asserts that there is a contractive projection $\tilde{P}\colon \mathscr{M}^{\varrho}_{\mathscr{X}} \mapsto \mathscr{S}^{\varrho}_{\mathscr{Y}}$ such that $\tilde{P} | L^{\varrho} \otimes_{\gamma} \mathscr{X} = P$. This completes the proof.

1.6. Remarks and open problems. The L^{ϱ}-spaces have certain "closure" properties among function spaces in the following sense. It is known that, when $\mu(\Omega) < \infty$,

$$L^1(\mu) = \cup \left\{ L^{\Phi}(\mu)\colon \frac{\Phi(t)}{t} \uparrow \infty \right\},$$

where L^{Φ} is an Orlicz space corresponding to the Young's function Φ, and that every Orlicz space is isomorphic to a strictly convex (= rotund) Orlicz space. ($L^1(\mu)$, for any μ, is isomorphic to an Orlicz, but not a Lebesgue space!) In this way the L^{Φ}-spaces form a "closure" of the L^p, $p \geqslant 1$, spaces. On the other hand, if $M^{\Phi} \subset L^{\Phi}$ corresponds to M^{ϱ} of this paper, then M^{Φ} is not necessarily an Orlicz space in that it is not an L^{Q}, for some Young's function Q. However, it is an L^{ϱ}-space on the same measure space if $\varrho(\cdot)$ is defined by $\varrho(f) = N_{\Phi}(f)$ for $f \in M^{\Phi}$, and $= +\infty$ for $f \in L^{\Phi} - M^{\Phi}$, where $N_{\Phi}(\cdot)$ is a norm on the L^{Φ}-spaces. Thus the L^{ϱ}-spaces form a "cover" to the L^{Φ}-spaces. It is remarkable that, as seen from Section 1.1 and [26], the L^{Φ}- theory already reflects the intricacies of the L^{ϱ}-spaces, at least for the duality theory.

The function norm $\varrho(\cdot)$ can be interpreted, in conjunction with the lifting property (cf. [7]), as the uniform norm also, as was noted in the preceding sections. Thus the representation theory of $C(S)$ can be formally obtained from that of the L^{ϱ}-spaces. This will be briefly illustrated by deducing the result of [35], and this clarifies a discussion in [12] on this theorem. It illuminates the structure of the problem.

Let $C(S)$ be the real continuous function spaces on a compact Hausdorff S. Then, as before, it can be identified with a closed subspace of $L^{\infty}(\tilde{S}, \nu)$ with S being homeomorphically embedded in the compact $\tilde{S}$, through the use of [18]. Then $C_{\mathscr{X}}(S)$, of $\mathscr{X}$-valued continuous functions ($\mathscr{X}$ a B-space) can be isometrically identified as a subspace of $L^{\infty}_{\mathscr{X}}(\tilde{S}, \nu)$. Since $L^{\infty}(\tilde{S}, \nu) \otimes_{\gamma} \mathscr{X}$ is a dense subspace of $L^{\infty}_{\mathscr{X}}(\tilde{S}, \nu)$, it follows from Section 1.5, that

$$\big(C_{\mathscr{X}}(S)\big)^* \subset \big(L^{\infty}_{\mathscr{X}}(\tilde{S}, \nu)\big)^* \subset \big(L^{\infty}(\tilde{S}, \nu) \otimes_{\gamma} \mathscr{X}\big)^* \cong B\big(L^{\infty}(\tilde{S}, \nu), \mathscr{X}^*\big)$$
$$\cong \mathscr{A}_{\varrho'}(\mathscr{X}^*, \nu),$$

where $\varrho'(\cdot) = \|\cdot\|_{1,\nu}$, the $L^1(\nu)$-norm. Here the first is an isometric embedding and the second topological ($M^{\varrho} = L^{\infty}$ here!). If $T \in B\big(L^{\infty}(\tilde{S}, \nu), \mathscr{X}^*\big)$, and $\tilde{T} = T | C(S)$, $\tilde{T}'(fx) = \tilde{T}(f)\cdot x$, $x \in \mathscr{X}$, then $\tilde{T}' \in \big(C_{\mathscr{X}}(S)\big)^*$ and by Theorem 5.4, $\|\tilde{T}'\| = |||\tilde{T}'|||$, so that, arguing and using Theorem 3.3, one gets (with $S \subset \tilde{S}$)

$$\tilde{T}'(fx) = \int_S (fx)\, d\tilde{G}, \quad x \in \mathscr{X}, \ \|\tilde{T}\| = |||\tilde{T}'||| = R_{\tilde{\varrho}'}(\tilde{G}),$$

where $\tilde{G} = G | S$, $G \in \mathscr{A}_{\varrho'}(\mathscr{X}^*, \nu)$. But in the present case, $R_{\tilde{\varrho}'}(\tilde{G})$ = total variation of $\tilde{G}$. Thus

$$\tilde{T}'(f) = \int_S f\, d\tilde{G}, \quad f \in C_{\mathscr{X}}(S), \ \|\tilde{T}\| = \operatorname{var}(\tilde{G}).$$

Since S is compact $|\tilde{T}'f_n| \to 0$ as $\|f_n\| \to 0$ one sees that $\tilde{G}$ is weakly and hence strongly σ-additive, and if $M(S, \mathscr{X}^*)$ is the space of $\mathscr{X}^*$-valued measures of finite variation, then, by the above, $\big(C_{\mathscr{X}}(S)\big)^* \cong M(S, \mathscr{X}^*)$. Note that by [7], p. 269, $\mathscr{X}^*$ has the R-N property so there is equality here (compare with Corollary 5.7). This result was originally given in [35], and discussed differently in [12], without the L^{ϱ}-space theory.

PROBLEM 1. If $\mathscr{Y}$ is not scalars and $\mathscr{X}$ is not scalars, $B(L^{\varrho}_{\mathscr{X}}, \mathscr{Y})$ when $L^{\varrho}_{\mathscr{X}} \neq \mathscr{M}^{\varrho}_{\mathscr{X}}$, has not been represented. This can be done as in Theorem 5.4, if $(L^{\varrho}_{\mathscr{X}})^*$ is determined. The solution of this will also be useful for the projection problem of the next chapter.

PROBLEM 2. It will be of interest to obtain an intrinsic characterization of $L^{\varrho} \otimes_{\gamma} \mathscr{X}$. This seems to be closely related to the study of absolutely summing operators, related to $\varrho(\cdot)$, generalizing the theory of [19], p. 304. For $L^{\varrho} = L^p$, $p \geqslant 1$, the work of [4] is relevant in this context, but the general case has been open.

II. CONTRACTIVE PROJECTIONS AND APPLICATION

2.1. Auxiliary notions; first reduction. In this section a few concepts on the L^ϱ-spaces and certain isomorphisms will be recalled. Then a useful isomorphism theorem, necessary for a first reduction of the projection problem, will be proved. It has some independent interest.

Recall that for each $\varrho(\cdot)$, an *associate norm* $\varrho'(\cdot)$ *on point-functions* is given by

(1) $$\varrho'(f) = \sup\Big\{\Big|\int_\Omega fg\,d\mu\Big| : \varrho(g) \leqslant 1\Big\},$$

and the higher associates are defined as $\varrho'' = (\varrho')'$ etc., all of which are function norms. Moreover, $\varrho'(\cdot)$ (and thus all higher associates), has the *Fatou property* (i.e., $0 \leqslant f_n \uparrow f$, a.e. implies $\varrho(f_n) \uparrow \varrho(f)$). A result of Halperin and Luxemburg shows, as noted earlier, that ϱ' is non-trivial (on point-functions) iff μ has the FSP, and that $\varrho = \varrho''$ iff ϱ also has the Fatou property, and they may only be equivalent otherwise. $\varrho(\cdot)$ has the *weak Fatou property* if $0 \leqslant f_n \uparrow f$ and $\sup \varrho(f_n) < \infty$ implies $\varrho(f) < \infty$. Then there is a (unique) constant $0 < \gamma \leqslant 1$, such that $\gamma\varrho \leqslant \varrho'' \leqslant \varrho$ (when μ has FSP). The Fatou properties imply the Riesz-Fischer property and thus L^ϱ is a BFS. (For details, see [20], [21], or [39].) Also in what follows all measures and σ-fields are assumed complete, since they can be replaced by completions otherwise.

The work of the following sections depends on several isomorphism theorems on the equivalence of measure and function spaces, in the sense of [18] and [34]. The first reduction, based on [27], with [18] and [34] (cf. also [10]), is contained in the following

THEOREM 1.1. *Let $L^\varrho(\Sigma)$ be a real BFS on (Ω, Σ, μ). Then there exists a measure space $(S, \mathscr{B}, \nu)$, where S is a locally compact Hausdorff space, $\mathscr{B}$ is the σ-field generated by the compact subsets of S, and ν is finite on each compact set, in terms of which $L^\varrho(S, \mathscr{B}, \nu)$, or $L^\varrho(\mathscr{B})$, is isometrically (and lattice) isomorphic to $L^\varrho(\Sigma)$. Moreover, every element of $L^\varrho(\mathscr{B})$ has a σ-compact support.*

Proof. First suppose that there is an $f_0 \in L^\varrho(\Sigma)$ with $f_0 > 0$ a.e. Let $\mathscr{C} \subset L^\varrho(\Sigma)$ be the algebra of all essentially bounded functions. It is clear that, if Σ_1 is the σ-field generated by $\mathscr{C}$, then $L^\varrho(\Sigma_1) = L^\varrho(\Sigma)$ in the sense that any function in the one space differs only on a null set from a function in the other and vice versa, where the functions (in $\mathscr{C}$) are selected with the lifting map (cf. [7]). Since $\mathscr{C}$ is a vector lattice, the closed subspace $\overline{\mathscr{C}}$ determined by $\mathscr{C}$ in $L^\infty(\Sigma)$, is an (AM)-space, [18]. Since $f_{0,n} = \min(f, n) \in \mathscr{C}$, and $f_{0,n} > 0$, a.e., it follows by an argument used in the first proof of [27], Thm. 2.1, that there exists a compact (Stone) space S_1 such that $\overline{\mathscr{C}} \simeq C(S_1)$, of real continuous functions. So its adjoint space $(C(S_1))^*$,

being an (AL)-space, is (by another result of [18]) again isometrically equivalent to $L^1(S, \mathscr{B}, \nu)$, where ν is a finite regular measure on the Borel sets $\mathscr{B}$ of S which is also a compact (Stone) space. As noted earlier (cf., Thm. I.3.2), S_1 can be (homeomorphically) identified with a closed subset of S and thus, with the lifting map, $\mathscr{C}(S_1) \subset L^\infty(S, \mathscr{B}, \nu)$, so that $\overline{\mathscr{C}} \cong C(S_1) = C$, say.

Let $\varphi: f \mapsto \hat{f} \in C$ be the mapping on $\overline{\mathscr{C}}$ onto C, then $\hat{f} = 0$ a.e. (ν) iff $f = 0$, a.e. (μ), and $\|f\|_{\infty,\mu} = \|\hat{f}\|_{\infty,\nu}$. A characteristic function of $\mathscr{C}$ goes into a characteristic function of C. Let ϱ on $\mathscr{C}$ be defined as follows. For $f \in \mathscr{C}$, let $\hat{\varrho}(\hat{f}) = \varrho(f)$, where $\hat{f} = \varphi(f)$. Clearly, $\hat{\varrho}$ is a function norm on C. If $0 \leqslant f \in L^\varrho(\Sigma)$, let $f_n = \min(f, n) \in \mathscr{C}$, and $\hat{f}_n = \varphi(f_n) \in \mathscr{C}$, so that $\hat{\varrho}(\hat{f}_n) = \varrho(f_n)$ for all n. Since $f_n \uparrow f$ a.e. (μ), and $\hat{f}_n \uparrow \hat{f}$ a.e. (ν), where φ is extended to have $\hat{f} = \varphi(f)$ unambiguously, one can define $\hat{\varrho}(\hat{f}) = \varrho(f)$. The definition is correct and does not depend on the particular sequence used. In fact, this is immediate if ϱ has the Fatou property. Otherwise, since $L^\varrho(\Sigma)$ is BFS, there exists another norm $\bar{\varrho}$ on $L^\varrho(\Sigma)$, which has Fatou's property (cf. [39], p. 450, where this latter norm is denoted ϱ_L, and the assumption of σ-finiteness of μ in that proof is not essential and the result holds for general μ), and $\bar{\varrho} \leqslant \varrho$. Hence the extension is obtained for this norm. Since $L^{\bar{\varrho}} \supset L^\varrho$, the elements of L^ϱ and $L^{\hat{\varrho}}$ have simultaneously finite or infinite ϱ and $\hat{\varrho}$ values, and from this the above definition for ϱ itself follows at once. Let $L^{\hat{\varrho}}(S, \mathscr{B}, \nu)$, or $L^{\hat{\varrho}}(\mathscr{B})$, be the corresponding BFS. Then the extended φ on $L^\varrho(\Sigma)$ onto $L^{\hat{\varrho}}(\mathscr{B})$ is an isometric isomorphism and preserves lattice operations. Identifying ϱ and $\hat{\varrho}$, the result follows in this case, with the desired measure space as $(S, \mathscr{B}, \nu)$.

Now consider the general case that there need not be an $f_0 > 0$ a.e. in $L^\varrho(\Sigma)$. Let $\mathscr{S} = \{f_\alpha, \alpha \in I\} \subset L^\varrho(\Sigma)$, where $f_\alpha \wedge f_{\alpha'} = 0$ for $\alpha \neq \alpha'$ in I, i.e., f_α's have disjoint supports, and are non-null. Since $L^\varrho(\Sigma)$ is a vector lattice, it may be assumed that $f_\alpha \geqslant 0$, a.e. for all $\alpha \in I$. Since every linearly ordered (by inclusion) subcollection of $\mathscr{S}$ has an upperbound (their union), there exists a maximal element $\mathscr{S}_0 \subset \mathscr{S}$, by Zorn's lemma. Let $\mathscr{S}_0 = \{f_\beta, \beta \in I_0 \subset I, f_\beta \wedge f_{\beta'} = 0\}$. If $S_\beta = \operatorname{supp}(f_\beta)$, then $S_0 = \bigcup\{S_\beta, \beta \in I_0\}$ contains and hence equals Ω except for a null set. For, if this is false, there exists $E \subset \Omega - S_0$, with $\mu(E) > 0$. That is, even if $A = \Omega - S_0$ is not μ-measurable (since μ is complete), there exists a μ-measurable $B \supset A$ with $\mu^*(A) = \mu(B)$ by the classical Carathéodory theory, (μ^* is the outer measure of μ). From this the above assertion immediately holds except for a trivial case, of no interest. Then there exists an $f_1 \in L^\varrho(\Sigma)$, $f_1 > 0$ a.e. on E, so that $f_1\chi_E \wedge f_\beta = 0$, all $\beta \in I_0$. Thus $\mathscr{S}_0 \subsetneqq \{\mathscr{S}_0, f\chi_E\} \subset \mathscr{S}$, and contradicts the maximality of $\mathscr{S}$. This proves the assertion.

Now consider $L^\varrho(\Sigma(S_\beta)) \subset L^\varrho(\Sigma)$, where $\Sigma(S_\beta)$ is the restriction σ-field of Σ to S_β. Then by the special case above $L^\varrho(\Sigma(S_\beta)) \cong L^{\hat{\varrho}}(\hat{S}_\beta, \mathscr{B}_\beta, \nu_\beta)$, where $\hat{\mathcal{S}}_\beta$ is a compact (Stone) space etc. Let $S = \bigcup\{\hat{S}_\beta,\ \ \beta\in I_0\}$, $\mathscr{B} = \sigma(\bigcup\hat{\mathscr{B}}_\beta: \beta\in I_0)$. More precisely, $S = \bigcup\{\{\beta\}\times\hat{S}_\beta: \beta\in I_0\}$ and $\mathscr{B} = \sigma\{\psi_\beta(E): E\in\Sigma$, and $\psi_\beta: x\mapsto(\beta,\hat{x})$ is an isomorphism of $\Omega\mapsto\hat{S}_\beta\}$, the σ-field generated by the sets shown. Then $A\subset S$ is measurable iff $A\cap\hat{S}_\beta\in\hat{\mathscr{B}}_\beta$ for at most countably many β's. Let ν on $\mathscr{B}$ be defined by

$$\nu(A) = \sup\Big\{\sum\nu_\beta(A\cap\hat{S}_\beta): \beta\in I_0\Big\},\qquad A\in\mathscr{B}.$$

Then S becomes a locally compact (extremally disconnected) space with the topology of the so-called "topological set sum" of the topologies on $\hat{S}_\beta$ of Bourbaki (cf. [34], p. 288, and further discussion in [10]). From this definition it now follows that $L^\varrho(\Sigma)\cong L^\varrho(\mathscr{B})$ and the correspondence is both an isometric (and lattice) isomorphism. Moreover, if $f\in L^\varrho(\Sigma)$, $A_n = \{|f|>1/n\}\in\Sigma_0$, since $\varrho(\chi_{A_n})<\infty$, and $\chi_{A_n}\in\mathscr{C}$ so that $\varphi(\chi_{A_n}) = \chi_{\hat{A}_n}\in C$, $\hat{A}_n\subset S$ being compact. Since

$$\operatorname{supp}(f) = \bigcup_{n=1}^{\infty}A_n,$$

t follows that

$$\operatorname{supp}(\hat{f}) = \bigcup_{n=1}^{\infty}\hat{A}_n,$$

which is σ-compact in S. Also $\nu_\beta(\hat{S}_\beta)<\infty$ implies $\nu(\cdot)$ is finite on compacts of S. Thus $L^\varrho(\Sigma)\cong L^\varrho(\mathscr{B})$, with $(S,\mathscr{B},\nu)$ thus constructed satisfying all the requirements. This completes the proof of the theorem. (This extends to the complex case also, with trivial changes of proof.)

Some useful consequences will now be noted.

COROLLARY 1.2. *If (Ω,Σ,μ) is σ-finite, then $L^\varrho(\Sigma)$ is isometrically (and lattice) isomorphic to $L^\varrho(S,\mathscr{B},\nu)$, where S is a compact (extremally disconnected) Hausdorff space and $\nu(S)<\infty$, $\nu(\cdot)$ being a regular Borel measure on $\mathscr{B}$, the σ-field of clopen sets of S. More generally, if (Ω,Σ,μ) is such that there exists an $f_0\in L^\varrho(\Sigma)$, $f_0>0$, a.e., then the same conclusion holds (clopen means closed-open).*

The last statement was proved in the first paragraph of the above proof. As for the first part, since μ is σ-finite, there is a weak unit $0<f_0\in L^\varrho(\Sigma)$ which is determined by a sequence $\{\chi_{A_n}\}\subset L^\varrho(\Sigma)$, $\bigcup_{n=1}^{\infty}A_n = \Omega$, $\mu(A_n)<\infty$, $A_n\uparrow$. (This is the ϱ-admissible sequence, cf. [20], p. 153.) Thus the result follows from the preceding.

The above corollary and [34], Thm. 4.1, yield the following complementing the result in [34], p. 306;

COROLLARY 1.3. *A measure ring of a σ-finite measure space (which is therefore complete, see [34] for the terminology) is strongly equivalent to the measure ring of a finite measure space.*

Another consequence is the following result which strengthens and elaborates the last part of [27], Thm. 2.2:

COROLLARY 1.4. *If $M^\varrho\subset L^\varrho$ and $\mathscr{A}\subset M^\varrho$ be a subalgebra (or a vector lattice) of bounded functions satisfying the conditions:*

(i) *there is an $f_0\in\mathscr{A}$,*

(ii) *$f\in\mathscr{A}$ implies its complex conjugate $\bar{f}\in\mathscr{A}$, and*

(iii) *$A_1, A_2\in\Sigma$, $A_1\cap A_2 = \emptyset$, $\mu(A_i)>0$, implies the existence of $f_i\in\mathscr{A}$ with $f_1\geqslant 0$ a.e. on A_1 and $f_2<0$ a.e. on A_2.*

Then $\overline{sp}(\mathscr{A}) = M^\varrho$.

Proof. By definition of M^ϱ, given $\varepsilon>0$, and $f\in M^\varrho$, there exists $f_\varepsilon\in M^\varrho$, bounded, and $\varrho(f-f_\varepsilon)<\varepsilon/2$. It suffices to show that there is $f'_\varepsilon\in\mathrm{sp}(\mathscr{A})$ with $\varrho(f_\varepsilon-f'_\varepsilon)<\varepsilon/2$. By Corollary 1.2, using (i), for this proof one may assume μ to be a finite measure, by going into the isometrically isomorphic image in $L^\varrho(\mathscr{B})$. But then this follows from the fact that every bounded function in $L^\varrho(\Sigma)$ (μ finite) can be uniformly approximated by step functions in $\mathrm{sp}(\mathscr{A})$ (if the σ-algebra generated by $\mathscr{A}$ is Σ_2, then $L^\varrho(\Sigma)\cong L^\varrho(\Sigma_2)$ by [27], Thm. 2.1) and thus in $\varrho(\cdot)$ norm. This completes the proof.

Remark. Such a result, in a special case, was used in the proof of [20], Thm. 1.4, and an alternate proof of it was given in [39], Ch. 15. As noted in the context of Orlicz spaces filling L^1 if $\mu(\Omega)<\infty$ (cf. Section I. 1. 6), it may be remarked that among the function norms, for $L^\varrho(\Sigma)$ with $\mu(\Omega)<\infty$, the topology of uniform convergence of elements in L^ϱ is stronger than that induced by the $\varrho(\cdot)$ norm.

The next result gives the existence and structure, in an important special case, of contractive projections which will be useful later. The Fatou property (FP) will be needed.

THEOREM 1.5. *Let $L^\varrho(\Sigma)$ be a BFS on (Ω,Σ,μ) such that ϱ, with FP, has the localizable property (cf. Thm. I.2.6). If $\mathscr{B}\subset\Sigma$ is a σ-field and $L^\varrho(\mathscr{B})\subset L^\varrho(\Sigma)$ is the corresponding subspace of $\mathscr{B}$-measurable functions, then there exists a contractive projection $P: L^\varrho(\Sigma)\mapsto L^\varrho(\mathscr{B})$. One such operator is $P = E^{\mathscr{B}}$, which is uniquely defined by*

$$\int_A f\,d\mu = \int_A E^{\mathscr{B}}(f)\,d\mu,\qquad f\in L^\varrho(\Sigma),\ A\in\mathscr{B}_0\subset\mathscr{B},\tag{2}$$

where $\mathscr{B}_0$ is the ring of sets of finite μ-measure. $E^{\mathscr{B}}$ satisfying (2) is termed the generalized conditional expectation relative to $\mathscr{B}$.

Proof. If $L^\varrho(\mathscr{B}) = \{0\}$ so that $\mathscr{B}_0$ has only μ-null sets, define $E^{\mathscr{B}} = 0$. If $L^\varrho(\mathscr{B})\neq\{0\}$, then $\mathscr{B}_0$ is non-trivial, and let $\mathscr{B}_1$ be the tribe ($=\sigma$-ring) generated by $\mathscr{B}_0$. Then $L^\varrho(\mathscr{B}_1) = L^\varrho(\mathscr{B})$. To see this, since clearly $L^\varrho(\mathscr{B}_1) \subset L^\varrho(\mathscr{B})$, let $f \in L^\varrho(\mathscr{B})$. Then there exist step functions $f_n \in L^\varrho(\mathscr{B})$ such that $f_n \to f$ a.e. Since ϱ has the localizable property, and each f_n takes only a finite number of values, it follows that $\operatorname{supp}(f_n) \in \mathscr{B}_0$. So f_n is $\mathscr{B}_1$-measurable and hence f is $\mathscr{B}_1$-measurable, or a.e. equal to one that is. Thus $f \in L^\varrho(\mathscr{B}_1)$ and the opposite inequality holds.

Let $0 \leqslant f \in L^\varrho(\mathscr{B})$, and define

$$\nu_f(A) = \int_A f d\mu, \qquad A \in \Sigma.$$

Then $\nu_f(\cdot)$ on Σ is a measure and let $\tilde{\nu}_f = \nu_f | \mathscr{B}$, so that $\tilde{\nu}_f$ is $\mu_{\mathscr{B}}$ $(= \mu|\mathscr{B})$-continuous. If $S_0 = \operatorname{supp}(\nu_f)$, then $S_0 \in \mathscr{B}_1$. To see this, note that the localizable property of ϱ implies the FSP of μ and hence ϱ' exists as a non-trivial associate norm on point functions. Then

$$(3) \qquad \infty > \varrho''(\nu_f) \geqslant \varrho''(\tilde{\nu}_f) = \sup\Big\{\Big|\int_\Omega g d\tilde{\nu}_f\Big| : \varrho'(g) \leqslant 1,\ g \in L^{\varrho'}(\mathscr{B}_1)\Big\}$$

$$= \lim_{n\to\infty}\Big\{\int_\Omega g_n d\tilde{\nu}_f,\ 0 \leqslant g_n \in L^{\varrho'}(\mathscr{B}_1),\ \varrho'(g_n) \leqslant 1\Big\}.$$

Let $S_n = \operatorname{supp}(g_n) \in \mathscr{B}_1$, and $\bar{S} = \bigcup_{n=1}^{\infty} S_n \in \mathscr{B}_1$. Then $S_0 = \bar{S}$ except possibly for a null set (which can and will be ignored). If this equality is false, then there exists an $A \in \mathscr{B}_0$, $\mu(A) > 0$, $\bar{S} \cap A = \varnothing$ and $\tilde{\nu}_f(A) > 0$. Then there exists a $\tilde{g} \in L^\varrho(\mathscr{B}_1)$, $\varrho'(\tilde{g}) \leqslant 1$ and $\alpha = \int_A \tilde{g} d\nu_f > 0$. Consequently,

$$\varrho''(\tilde{\nu}_f) = \sup\Big\{\int_\Omega g d\tilde{\nu}_f :\ 0 \leqslant g \in L^{\varrho'}(\mathscr{B}_1),\ \varrho'(g) \leqslant 1\Big\}$$

$$\geqslant \sup\Big\{\int_A g d\tilde{\nu}_f + \int_{\bar{S}} g d\tilde{\nu}_f :\ \varrho'(g) \leqslant 1,\ g \in L^{\varrho'}(\mathscr{B}_1)\Big\}$$

$$\geqslant \alpha + \lim_{n\to\infty} \int_{\bar{S}} g_n d\tilde{\nu}_f,\ \varrho'(g_n) \leqslant 1,\ g_n \in L^{\varrho'}(\mathscr{B}_1) \qquad \text{(as in (3))}$$

$$= \alpha + \varrho''(\tilde{\nu}_f) \qquad \text{(by (3))}.$$

This contradiction proves $S_0 = \bar{S}$ a.e., and thus $S_0 \in \mathscr{B}_1$.

By the definition of $\mathscr{B}_1$, there exist $A_n \in \mathscr{B}_0$, $A_n \uparrow S_0$ (a.e.), and by the R-N theorem (since S_0 is a σ-finite set), there is a unique $\mathscr{B}_1$-measurable $\tilde{f}$, vanishing a.e. outside S_0, and such that

$$\tilde{\nu}_f(A) = \int_A \tilde{f} d\mu, \qquad A \in \mathscr{B}_0.$$

If now the mapping $E^{\mathscr{B}}: L^\varrho(\Sigma) \mapsto L^\varrho(\mathscr{B})$ is defined by $E^{\mathscr{B}}(f) = \tilde{f}$ (or $= 0$ in the trivial case considered earlier), first for $f \geqslant 0$, and by linearity for the general case, and noting that $L^\varrho = L^{\varrho''} \subset (L^\varrho)^{**}$, and the norms are equal for set functions, then $P = E^{\mathscr{B}}(\cdot)$ is a (positive) contractive projection. This completes the proof. [FP is used only at the end.]

The operator $E^{\mathscr{B}}$ has the following properties of interest, and they can be proved with the properties of the R-N derivatives and a standard computation.

Proposition 1.6. *The mapping $E^{\mathscr{B}}: L^\varrho(\Sigma) \mapsto L^\varrho(\mathscr{B})$ of the above theorem verifies the following relations a.e. (μ):*

(i) *$E^{\mathscr{B}}$ is a positive contractive projection,*

(ii) $E^{\mathscr{B}}(fE^{\mathscr{B}}g) = (E^{\mathscr{B}}f)(E^{\mathscr{B}}g)$, $f \in L^\infty(\Sigma)$, $g \in L^\varrho(\Sigma)$,

(iii) $\mathscr{B} \subset \mathscr{B}_1 \subset \Sigma$ *implies* $E^{\mathscr{B}}E^{\mathscr{B}_1}(g) = E^{\mathscr{B}_1}E^{\mathscr{B}}(g) = E^{\mathscr{B}}(g)$,

(iv) (i)-(iii) *also hold for arbitrary function norms ϱ, with FP, if $\mu_{\mathscr{B}}$ is localizable.*

As an immediate consequence of Theorem 1.5 and of Theorem I.2.6, one has the following

Corollary 1.7. *Let $L^\varrho(\Sigma)$ be reflexive, and $\mathscr{B} \subset \Sigma$ be a sub-σ-field. Then there exists a contractive projection $P: L^\varrho(\Sigma) \mapsto L^\varrho(\mathscr{B})$.*

A simple direct proof of this result can be based on the representation of $(L^\varrho)^* \cong L^{\varrho'}$, as noted in [29] for the Orlicz spaces. Except in this special case the projections onto $L^\varrho(\mathscr{B})$ in general need not be unique. This and the characterization problem will be considered in the next three sections.

2.2. Contractive projections; scalar case. In this section a complete characterization of contractive projections on L^ϱ-spaces of scalar functions will be presented, breaking up the problem into a number of cases. It will be considered first for $\mu(\Omega) < \infty$, and the general case will be obtained through several isomorphisms, using the above Theorem 1.1 and many other results of [34]. *Hereafter, ϱ will be assumed to have the weak Fatou property.*

Proposition 2.1. *If $\mathscr{M} \subset L^\varrho(\Sigma)$, on a finite measure space (Ω, Σ, μ), is a closed subspace, then there exists a function $f_0 \in \mathscr{M}$ with support S_0 such that every $f \in \mathscr{M}$ is null outside S_0. (S_0 itself, with the latter property is called the support of $\mathscr{M}$.) If $\mathscr{M}$ is also a lattice and self-adjoint, then $f_0 > 0$ a.e. on S_0 also holds.*

Proof. If $\mathscr{S} = \{\operatorname{supp}(f): f \in \mathscr{M}\} \subset \Sigma$, then from the fact that every finite measure is localizable (cf. [34], p. 284), it follows that there is an $S_0 \in \Sigma$, which is the supremum of $\mathscr{S}$. So it is also the support of $\mathscr{M}$. This set can be approximated. In fact, there exist $\{f_n\} \subset \mathscr{M}$, such that

$$S_0 = \bigcup_{n=1}^{\infty} \operatorname{supp}(f_n).$$

(For a simple construction of such a sequence, see [8], p. 448, or [1], p. 396). If

$$f_0 = \sum_{n=1}^{\infty} \frac{f_n}{2^n} (\varrho(f_n))^{-1} \quad \text{and} \quad g_n = \sum_{i=1}^{n} f_i (\varrho(f_i))^{-1}/2^i \in \mathscr{M},$$

then $\varrho(f_0 - g_n) \to 0$ and since $\mathscr{M}$ is complete, $f_0 \in \mathscr{M}$. Clearly, $\operatorname{supp}(f_0) = S_0$. If $\mathscr{M}$ is also a lattice, and s.a., then $f_n \in \mathscr{M}$ implies $|f_n| \in \mathscr{M}$, so that replacing f_n by $|f_n|$ in the definition of f_0, one has $f_0 > 0$, a.e. on S_0, as desired.

The next result, on the structure of vector sublattices of L^ϱ, is important. Hereafter $\mathscr{M}$ is a B-lattice iff it is s.a. and its real functions form a lattice.

THEOREM 2.2. *Let $\mathscr{M} \subset L^\varrho(\Sigma)$ be a B-lattice, $\mu(\Omega) < \infty$, and $0 \leqslant f_n \uparrow f$ a.e., $f_n \in \mathscr{M}$, $f \in L^\varrho(\Sigma)$ imply $f \in \mathscr{M}$. Then there exist* (a) *a σ-field $\mathscr{B} \subset \Sigma$,* (b) *a $0 < f_0 \in L^\varrho(\mathscr{B}) \cap L^\infty(\mathscr{B})$, called a weak unit which may be taken to be in $L^{\varrho'}(\mathscr{B})$ also, and* (c) *$0 \leqslant g_0 \in \mathscr{M}$ with $g_0 > 0$, a.e. on S_0, the support of $\mathscr{M}$ such that $f_0 \mathscr{M} \subset g_0 L^\varrho(\mathscr{B}) \subset \mathscr{M}$, where the inclusions are both algebraic and topological.*

Proof. Let S_0 be the support of $\mathscr{M}$ and $0 \leqslant h_0 \in \mathscr{M}$ with support S_0, as assured by the preceding result. Let $\mathscr{B} = \{A \in \Sigma\colon h_0\chi_A \in \mathscr{M}\}$. Then $S_0 \in \mathscr{B}$ and $\mathscr{B}$ is a field. Since $A_n \in \mathscr{B}$, $A_n \uparrow A$ implies $h_0\chi_{A_n} \uparrow h_0\chi_A \in L^\varrho(\Sigma)$, the hypothesis implies $h_0\chi_A \in \mathscr{M}$ so that $A \in \mathscr{B}$. So $\mathscr{B}$ is also a σ-field. Let $L^\varrho(\mathscr{B}) \subset L^\varrho(\Sigma)$ be the set of $\mathscr{B}$-measurable functions. It results from [20], p. 163, with a trivial modification of the construction there, that a weak unit $0 < f_0 \in L^\varrho(\mathscr{B}) \cap L^{\varrho'}(\mathscr{B})$ exists, and it may be assumed bounded (by 1). The next two steps will establish the stated inclusions.

(I) There exists a $0 \leqslant g_0 \in \mathscr{M}$ such that $\operatorname{supp}(g_0) = S_0$ and $g_0 L^\varrho(\mathscr{B}) \subset \mathscr{M}$. For, if f_n is any $\mathscr{B}$-measurable simple function, then $f_n h_0 \in \mathscr{M}$. Now define the measures μ_{S_0} and ν on $\mathscr{B}$ by the equations

$$\mu_{S_0}(A) = \int_{S_0 \cap A} f_0\, d\mu, \quad \nu(A) = \int_A f_0 h_0\, d\mu, \quad A \in \mathscr{B}, \tag{1}$$

where f_0 is the weak unit noted above. Since μ_{S_0} and ν are equivalent (finite) measures, there is a unique $\mathscr{B}$-measurable $l \geqslant 0$ (by the R-N theorem) such that

$$\int_{S_0 \cap A} f_0\, d\mu = \mu_{S_0}(A) = \int_A l\, d\nu = \int_A l f_0 h_0\, d\mu, \quad A \in \mathscr{B}. \tag{2}$$

Since $d\mu$ can be replaced by the equivalent measure $f_0 d\mu$ here, it is seen that l depends only on h_0 but not on f_0. Let $g_0 = l f_0 h_0$. Then $\operatorname{supp}(g_0) = S_0$ and (even though $S_0 \in \mathscr{B}$) g_0 need not be $\mathscr{B}$-measurable. To see that $g_0 \in \mathscr{M}$, consider

$$\varrho''(g_0) = \sup\Big\{\Big|\int_\Omega g_0 g\, d\mu\Big|\colon \varrho'(g) \leqslant 1,\ g \in L^{\varrho'}(\mathscr{B})\Big\}$$
$$= \sup\Big\{\Big|\int_{S_0} f_0 g\, d\mu\Big|\colon \varrho'(g) \leqslant 1\Big\} \quad (\text{by } (2))$$
$$= \varrho''(f_0) \leqslant \varrho(f_0) < \infty.$$

Since ϱ and ϱ'' are equivalent, $g_0 \in L^\varrho(\Sigma)$. But there exist $0 \leqslant l_n \uparrow l f_0$ a.e. where l_n are $\mathscr{B}$-simple. So $l_n h_0 \in \mathscr{M}$, as noted above, and $l_n h_0 \uparrow l f_0 h_0 = g_0 \in L^\varrho(\Sigma)$. Hence $g_0 \in \mathscr{M}$ by the second part of the hypothesis. It remains to show that g_0 verifies the inclusion relation also.

First, if $\mathscr{B}_{g_0} = \{A \in \Sigma\colon g_0\chi_A \in \mathscr{M}\}$, then as earlier $\mathscr{B}_g$ is a σ-field. If $A \in \mathscr{B}_{g_0}$, then $h_n = (h_0 - n g_0 \chi_A)^+ \in \mathscr{M}$, for each n since $\mathscr{M}$ is a lattice. Since $h_n \uparrow h_0 \chi_{A^c} \in L^\varrho(\Sigma)$, one has $h_0\chi_{A^c} \in \mathscr{M}$ and so $A^c \in \mathscr{B}$. Thus $\mathscr{B}_{g_0} \subset \mathscr{B}$. Interchanging h_0 and g_0 here the opposite inclusion obtains, so that $\mathscr{B}_{g_0} = \mathscr{B}$ and the σ-field is determined by $\mathscr{M}$ alone, and does not depend on h_0 or g_0. To see that $g_0 L^\varrho(\mathscr{B}) \subset \mathscr{M}$, let $0 \leqslant f \in L^\varrho(\mathscr{B})$, so that $f_0 f \in L^\varrho(\mathscr{B})$ (f_0 being bounded). So there exist $0 \leqslant f_n \uparrow f$, f_n are $\mathscr{B}$-simple, and $f_n g_0 \in \mathscr{M}$. Since $f_n g_0 \uparrow f g_0$ a.e., it will follow that $f g_0 \in \mathscr{M}$ provided $\varrho(f g_0) < \infty$. For this, consider

$$\varrho''(f_n g_0) = \sup\Big\{\int_\Omega f_n g_0 |g|\, d\mu\colon \varrho'(g) \leqslant 1\Big\}$$
$$= \sup\Big\{\sum_{i=1}^n a_i \int_{A_i} g_0 |g|\, d\mu\colon \varrho'(g) \leqslant 1\Big\},$$
$$f_n = \sum_{i=1}^n a_i \chi_{A_i},\ A_i \in \mathscr{B},\ a_i \geqslant 0$$
$$= \sup\Big\{\sum_{i=1}^n a_i \int_{S_0 \cap A_i} f_0 |g|\, d\mu\colon \varrho'(g) \leqslant 1\Big\} \quad (\text{by } (2))$$
$$= \sup\Big\{\int_{S_0} f_0 f_n |g|\, d\mu\colon \varrho'(g) \leqslant 1\Big\}$$
$$= \varrho''(f_0 f_n) \leqslant \varrho''(f_0 f) < \infty.$$

Thus by the Fatou property of ϱ''

$$\varrho''(f g_0) = \varrho''(f_0 f). \tag{3}$$

Since ϱ and ϱ'' are equivalent norms, this yields $f g_0 \in L^\varrho(\Sigma)$, as desired. Now both $L^\varrho(\mathscr{B})$ and $\mathscr{M}$ are vector lattices. Consequently, the above proof implies the desired result for all $f \in L^\varrho(\mathscr{B})$. Note that if ϱ has the Fatou property $\varrho = \varrho''$ in (3), and the inclusion would be an isometry.

(II) It remains to prove $f_0 \mathscr{M} \subset g_0 L^\varrho(\mathscr{B})$. For this it again suffices to consider positive elements. So let $0 \leqslant f \in \mathscr{M}$, and it is to be shown that there is a $u \in L^\varrho(\mathscr{B})$ such that $f_0 f = g_0 u$. Let for any $a > 0$,

$$f_n = \min\big([n(f - a g_0)^+],\, g_0\big) \in \mathscr{M}.$$

Then $0 \leqslant f_n \uparrow g_0 \chi_A \in L^\varrho(\Sigma)$ so that $g_0 \chi_A \in \mathscr{M}$ and $A \in \mathscr{B}$, where $A = \{f/g_0 > a\}$. Thus $\tilde{u} = f/g_0$ is $\mathscr{B}$-measurable, and $f_0 f = f_0 \tilde{u} g_0$. If $u = f_0 \tilde{u}$, which is $\mathscr{B}$-measurable, it is to be shown that $\varrho(u) < \infty$. Now there exist $0 \leqslant u_n \uparrow \tilde{u}$, u_n are $\mathscr{B}$-simple. Thus $u_n g_0 \uparrow \tilde{u} g_0 = f$, a.e., and $\varrho(f) < \infty$. Consider again, with the equivalent norm ϱ'' of ϱ for a computation as in (3),

$$(4) \qquad \infty > \varrho''(f) \uparrow \varrho''(g_0 u_n) = \sup\Big\{\int_\Omega g_0 u_n |g|\, d\mu:\ \varrho'(g) \leqslant 1\Big\}$$

$$= \sup\Big\{\sum_{i=1}^n b_i \int_{S_0 \cap B_i} f_0 |g|\, d\mu;\ \varrho'(g) \leqslant 1\Big\}$$

$$u_n = \sum_{i=1}^n b_i \chi_{B_i},\ b_i \geqslant 0,\ B_i \in \mathscr{B},\ \text{and (2)},$$

$$= \sup\Big\{\int_{S_0} f_0 u_n |g|\, d\mu:\ \varrho'(g) \leqslant 1\Big\} = \varrho''(u_n f_0).$$

Thus with the Fatou property

$$(5) \qquad \varrho''(u) = \varrho''(\tilde{u} f_0) = \varrho''(f) < \infty.$$

Hence $f_0 f = g_0 u$, and $u \in L^\varrho(\mathscr{B})$, by the equivalence of ϱ and ϱ''. So the desired inclusion follows.

Finally, let $T: \mathscr{M} \mapsto f_0 \mathscr{M} = \tilde{\mathscr{M}}$. Then the boundedness of f_0 implies that of T and since $f_0 > 0$, a.e., T is one-to-one, and onto $\tilde{\mathscr{M}}$. Since $\tilde{\mathscr{M}}$ is complete, by the inverse boundedness theorem, T^{-1} is also bounded. Thus the inclusions are topological and the proof of the theorem is complete.

Remark. If $L^\varrho(\mathscr{B})$ contains constants, then $f_0 = 1$, and $\mathscr{M} = g_0 L^\varrho(\mathscr{B})$ and the relation is an isometric equivalence in the above, whenever $\varrho = \varrho''$ (i.e. ϱ verifies Fatou's property). Taking

$$\varrho(f) = \int_0^1 \frac{|f|}{x}\, dx$$

shows $L^\varrho(\mathscr{B})$ need not have non-trivial constants. Also note that if $\varrho(\cdot)$ is an a.c.n., then the second conditions on $\mathscr{M}$ ($0 \leqslant f_n \in \mathscr{M}$, $f_n \uparrow f \in L^\varrho(\Sigma)$ implies $f \in \mathscr{M}$) is *automatic*, by Corollary I.2.3, and is needed in the general case. The underlying idea of the proof of this and the next two theorems is abstracted from the L^1-case of [8] and the Orlicz space case of [29].

The above analysis admits the first characterization as follows:

THEOREM 2.3. *Let $L^\varrho(\Sigma)$ be a BFS with FP and μ finite. Then $\mathscr{M} \subset L^\varrho(\Sigma)$ is the range of a positive contractive projection on $L^\varrho(\Sigma)$ iff* (i) *$\mathscr{M}$ is a B-lattice, and* (ii) *$0 \leqslant f_n \uparrow f \in L^\varrho(\Sigma)$, $f_n \in \mathscr{M}$, $n \geqslant 1$, implies $f \in \mathscr{M}$.*

Proof. Suppose $\mathscr{M}$ verifies (i) and (ii). Then by Theorem 2.2, $f_0 \mathscr{M} \subset g_0 L^\varrho(\mathscr{B}) \subset \mathscr{M}$, with the same notations as in it. But by Theorem 1.5, $E^{\mathscr{B}}: L^\varrho(\Sigma) \mapsto L^\varrho(\mathscr{B})$, is a positive contractive projection. Let $P = g_0 E_{\mathscr{B}}(\cdot)$, so that $P: L^\varrho(\mathscr{B}) \mapsto g_0 L^\varrho(\mathscr{B})$ is a positive linear operator, and since $0 < f_0 \leqslant k_0$, a.e., $0 < k_0 \leqslant 1$ for an appropriate k_0, and keeping it fixed, it will be shown that P is a contraction. In fact, k_0 is the constant connecting ϱ and ϱ'' through the weak Fatou property of ϱ so that $k_0 \varrho \leqslant \varrho'' \leqslant \varrho$. Thus (with FP, $k_0 = 1$ can be taken)

$$k_0 \varrho(Pf) \leqslant \varrho''(Pf) = \varrho''\big(g_0 E^{\mathscr{B}}(f)\big) = \varrho''\big(f_0 E^{\mathscr{B}}(f)\big) \qquad \text{(by (3))}$$

$$= \varrho''\big(E^{\mathscr{B}}(f_0 f)\big) \leqslant \varrho''(f_0 f)$$

$$(f_0 \text{ is } \mathscr{B}\text{-measurable and Proposition 1.6})$$

$$\leqslant k_0 \varrho''(f) \leqslant k_0 \varrho(f) \qquad (\text{since } 0 < f_0 \leqslant k_0).$$

Thus P is a contraction in ϱ-norm. To see $P^2 = P$, consider

$$(6) \qquad P^2 f = g_0 E^{\mathscr{B}}\big(g_0 E^{\mathscr{B}}(f)\big) = g_0 E^{\mathscr{B}}(g_0 h),$$

where $h = E^{\mathscr{B}}(f)$ is $\mathscr{B}$-measurable. To show $P^2 f = Pf$, it suffices to establish $E^{\mathscr{B}}(f_0 g_0 h) = f_0 h \chi_{S_0}$, a.e., for all $h \in L^\varrho(\mathscr{B})$. The following equation is a consequence of (2), first for simple $h \in L^\varrho(\mathscr{B})$, and then for the general case by monotone convergence:

$$(7) \qquad \int_{S_0 \cap A} f_0 h\, d\mu = \int_A f_0 g_0 h\, d\mu = \int_A E^{\mathscr{B}}(f_0 g_0 h)\, d\mu, \qquad A \in \mathscr{B}.$$

Since $f_0 \chi_{S_0} = f_0$ is $\mathscr{B}$-measurable, (7) implies $E^{\mathscr{B}}(g_0 f_0 h) = h f_0 \chi_{S_0}$, a.e., as asserted. Hence $P = g_0 E^{\mathscr{B}}$ is a positive contractive projection. Now let $\tilde{P} = T^{-1} P T$, $T: \mathscr{M} \mapsto f_0 \mathscr{M}$, defined by $Tf = f_0 f$, as in the proof of Theorem 2.2. Then $\tilde{P}: L^\varrho(\Sigma) \mapsto \mathscr{M}$ is a positive projection. But

$$(8) \qquad \varrho(\tilde{P} f) = \varrho\big(T^{-1} g_0 E^{\mathscr{B}}(f_0 f)\big) = \varrho\Big(\frac{1}{f_0}\, g_0 E^{\mathscr{B}}(f_0 f)\Big)$$

$$= \varrho\big(g_0 E^{\mathscr{B}}(f)\big) = \varrho(Pf) \leqslant \varrho(f)$$

since f_0 is $\mathscr{B}$-measurable. Thus $\tilde{P}$ is a contraction, and the result holds in one direction.

Conversely, let P be a positive contractive projection on $L^\varrho(\Sigma)$ and $\mathscr{M} = P(L^\varrho(\Sigma))$. Then $\mathscr{M}$ is a closed subspace, and is s.a. To see that it is a lattice, let $f \in \mathscr{M}$, and to see that $|f| \in \mathscr{M}$, note that $|f| = |Pf| \leqslant P|f|$, since P is a positive projection. Since $L^{\varrho'}$ is norm determining for $L^{\varrho''}$ (cf. [20]), there exist $0 \leqslant g_n$, $\varrho'(g_n) \leqslant 1$, and $\varrho''(|f|) = \lim_n \int_\Omega |f| g_n d\mu$. Also

$$0 \leqslant \overline{\lim_n} \int_\Omega (P(|f|) - |f|) g_n d\mu \leqslant \overline{\lim_n} \Big[\varrho''(P|f|) \cdot 1 - \int_\Omega |f| g_n d\mu\Big] = 0,$$

since P is a contraction in ϱ'' ($\leqslant \varrho$) also. This implies $P(|f|) = |f| \in \mathscr{M}$, so (i) holds. To prove (ii), let $0 \leqslant f_n \in \mathscr{M}$, and $f_n \uparrow f \in L^\varrho(\Sigma)$. Again by definition $\varrho(h) = \sup\{|\int_\Omega h dG|: \ \varrho'(G) \leqslant 1\}$, one has $L^\varrho(\Sigma) = \cap\{L^1(\Sigma, G): G \in \mathscr{A}_{\varrho'}(\mu),\ \varrho'(G) \leqslant 1\}$.

So consider $0 \leqslant G_0$, $\varrho(G_0) = 1$, and G_0 is σ-additive. Such G_0 not only exist but are total on $L^\varrho (= L^{\varrho''})$ since $L^{\varrho'}$ is non-trivial. Let $\mathscr{L} = \mathrm{sp}(L^\varrho(\Sigma)) \subset L^1(\Sigma, G_0)$, since $\|\cdot\|_{1,G_0} \leqslant \varrho(\cdot)$. Then P has a norm-preserving unique extension $\tilde{P}$ to $\mathscr{L}$. Thus

$$\text{(9)} \qquad \|f - \tilde{P}f\|_{1,G_0} = \|f - Pf\|_{1,G_0} \leqslant \|f - f_n\|_{1,G_0} + \|P(f_n - f)\|_{1,G_0} \leqslant 2\|f - f_n\|_{1,G_0} \to 0, \quad n \to \infty,$$

by the monotone convergence theorem. Hence $f = Pf \in \mathscr{M}$, and (ii) holds(1). As above, ϱ can be replaced by ϱ'' here since P is also a contraction in ϱ''-norm. This completes the proof.

Thus far, $\mu(\Omega) < \infty$ was assumed. It will now be shown that the result extends to arbitrary μ, using the work of Section 1.1 above, and of [34], in the following two propositions, in which FP for ϱ is assumed.

PROPOSITION 2.4. *Let (Ω, Σ, μ) be a localizable measure space, and $L^\varrho(\Sigma)$ be the corresponding BFS. Then $\mathscr{M} \subset L^\varrho(\Sigma)$ is the range of a positive contractive projection iff conditions* (i) *and* (ii) *of Theorem* 2.3 *hold.*

Proof. By [34], p. 301 and p. 282, $L^\varrho(\Sigma) = \sum_{a \in A} \oplus_a L^\varrho(\Omega_a, \Sigma_a, \mu_a)$ where $(\Omega_a, \Sigma_a, \mu_a)$ is a finite measure space, $\Omega_a \cap \Omega_{a'} = \emptyset$, and $f \in L^\varrho(\Sigma)$ can be expressed as $f = \sum_{i=1}^\infty f_{a_i}$, $f_{a_i} \in L^\varrho(\Omega_{a_i}, \Sigma_{a_i}, \mu_{a_i}) = L^\varrho(\Sigma_{a_i})$, say, for at most a countable set of indices (depending on f) and $\mu(E) = \sum_{i=1}^\infty \mu_{a_i}(E \cap \Omega_{a_i})$. Let $\mathscr{M}_a \subset L^\varrho(\Sigma_a)$ be the corresponding decomposition $\mathscr{M} = \Sigma \oplus_a \mathscr{M}_a$,

(1) Alternately, $Pf = \int_\Omega f d\nu$, $0 \leqslant \nu \in \mathscr{A}_{\varrho'}(\mu)$ by Theorem I.3.2, and $0 \leqslant f_n = Pf_n = \int_\Omega f_n d\nu_1 + \int_\Omega f_n d\nu_2$ (ν_2 is Pfa), $f_n \uparrow f$, and ν_1 is σ-additve. So $\lim_n \int_\Omega f_n d\nu_2 = 0$ and $f = \int_\Omega f d\nu_1$, and then $\nu = \nu_1$ follows. This gives (ii) also.

$[\mathscr{M}_a = \chi_{\Omega_a} \mathscr{M}]$. Then it is seen that $\mathscr{M}_a$ satisfies, for each $a \in A$, conditions (i) and (ii) iff $\mathscr{M}$ satisfies these conditions.

To show the existence of the required projection, for each $a \in A$, by the preceding theorem, there exists a P_a: $L^\varrho(\Sigma_a) \mapsto \mathscr{M}_a$ which is a positive contractive projection. Let $P = \Sigma \oplus P_a$ by setting

$$Pf = \sum_{i=1}^\infty P_{a_i} f_{a_1} \in \mathscr{M}.$$

It is easily seen that P is well-defined, since the representation of f by $\{f_{a_i}\}$, the direct sum, is unique. Now P is clearly positive and $P^2 = P$. To show $\|P\| \leqslant 1$, consider

$$\varrho(Pf) = \varrho\Big(\sum_{i=1}^\infty P_{a_i} f_{a_i}\Big) = \sup\Big\{\Big|\int_\Omega \Big(\sum_{i=1}^\infty P_{a_i} f_{a_i}\Big) dG\Big| : \ \varrho'(G) \leqslant 1\Big\}$$
$$\leqslant \sup\Big\{\Big|\sum_{i=1}^\infty \int_{\Omega_{a_i}} f_{a_i} dG_{a_i}\Big| : \ G = \sum_{i=1}^\infty G_{a_i},\ \varrho'(G) \leqslant 1\Big\}$$

by definition of the direct sum decomposition here, and P_{a_i} is a contraction,

$$\leqslant \sup\Big\{\Big|\int_\Omega f dH\Big| : \ \varrho'(H) \leqslant 1\Big\} = \varrho(f).$$

The converse implication that if $P: L^\varrho(\Sigma) \mapsto \mathscr{M}$ is a positive contractive projection, then $\mathscr{M}$ verifies (i) and (ii) is immediate from the proof of Theorem 2.3 where the finiteness of μ was not used, and the FSP was enough. Here localizability of μ implies FSP.

The general case of μ can now be handled as follows:

PROPOSITION 2.5. *Let (Ω, Σ, μ) be an arbitrary measure space, and $L^\varrho(\Sigma)$ be the corresponding BFS. Then a closed subspace $\mathscr{M}$ of $L^\varrho(\Sigma)$ is the range of a positive contractive projection iff conditions* (i) *and* (ii) *of Theorem* 2.3 *hold.*

Proof. By Theorem 1.1, $L^\varrho(\Sigma)$ is isometrically (lattice) isomorphic to $L^\varrho(\mathscr{B})$ on $(S, \mathscr{B}, \nu)$, as described there. Then $L^\varrho(\mathscr{B})$ verifies the hypothesis of [34], Thm. 3.4, and hence there exists a localizable measure space $(\tilde{S}, \tilde{\mathscr{B}}, \tilde{\nu})$, with $L^\varrho(\tilde{\mathscr{B}})$ on it, such that there is an isometric (lattice) isomorphism into (this will be onto iff ν is localizable) and such that $\mathscr{B}$ is mapped into a sub σ-ring of $\tilde{\mathscr{B}}$. Since then $L^\varrho(\mathscr{B}) \subset L^\varrho(\tilde{\mathscr{B}})$, under the identification, and $L^\varrho(\mathscr{B})$ trivially satisfies (i) and (ii), there exists, by the preceding proposition, a positive contractive projection P of $L^\varrho(\tilde{\mathscr{B}})$ onto $L^\varrho(\mathscr{B})$.

If P_1: $L^\varrho(\mathscr{B}) \mapsto \mathscr{M}$, where the given subset of $L^\varrho(\Sigma)$ is now identified with its image in $L^\varrho(\mathscr{B})$, is a positive contractive projection, then it follows immediately that $PP_1 = P_1 P (= \tilde{P}$ say$)$ is a positive contractive projection on $L^\varrho(\tilde{\mathscr{B}}) \mapsto \mathscr{M}$, under the identifications. Hence, by the preceding proposition $\mathscr{M}$ satisfies (i) and (ii).

Conversely, if $\mathscr{M}$ satisfies (i) and (ii) and since $L^\varrho(\mathscr{B})$ always verifies them in $L^\varrho(\tilde{\mathscr{B}})$, it follows immediately that $\mathscr{M}$ satisfies (i) and (ii), in $L^\varrho(\tilde{\mathscr{B}})$ also. So by the preceding result, there is a Q on $L^\varrho(\tilde{\mathscr{B}})$ onto $\mathscr{M}$, and if $P_1 = QP\ (= PQ)$, where $P: L^\varrho(\tilde{\mathscr{B}}) \mapsto L^\varrho(\mathscr{B})$, then $P_1|\mathscr{M} = Q|\mathscr{M} = I$ since $\mathscr{M} \subset L^\varrho(\mathscr{B}) \subset L^\varrho(\tilde{\mathscr{B}})$, and $P_1: L^\varrho(\mathscr{B}) \mapsto \mathscr{M}$ is the desired projection. Transfering these isometric maps to $L^\varrho(\Sigma)$, the result obtains, and the proof is complete.

Thus far, only positive projections are considered. It will now be shown that the general case of contractive projections can be reduced to the above case. In this reduction, the following property (D), found in [8], will be useful and it is automatic for the L^p, $1 < p < \infty$, and the Orlicz spaces L^Φ, with $\Phi(t)/t \uparrow \infty$ (cf. [1] and [29]). ((D) was (*) in [8].)

Definition 2.6. (a) A projection $P: L^\varrho(\Sigma) \mapsto \mathscr{M}$ is said to have *property* (D), if $\mathscr{N} = \{f \in L^\varrho(\Sigma): f\cdot \mathscr{M} = 0\}$, then $P(\mathscr{N}) = 0$, i.e., P annihilates all functions in $L^\varrho(\Sigma)$ whose supports are disjoint from the supports of all functions of $\mathscr{M}$.

(b) If $\mathscr{M} \subset L^\varrho(\Sigma)$, and $(\tilde{S}, \tilde{\mathscr{B}}, \tilde{\nu})$ is the localizable measure space into which (Ω, Σ, μ) is embedded under the algebraic isomorphism of [34] (cf. proof of Proposition 2.5) let $\mathscr{M}$ be identified as a closed subspace $\tilde{\mathscr{M}}$ of $L^\varrho(\tilde{\mathscr{B}}) = \Sigma \oplus_a L^\varrho(\tilde{\mathscr{B}}_a)$. If S_a^0 is the support of $\mathscr{M}_a \subset L^\varrho(\tilde{\mathscr{B}}_a)$, guaranteed by Proposition 2.1, let S^0 be the supremum of $\{S_a^0\} \subset \tilde{S}$. Thus (cf. [34], p. 282) S^0 is a $\tilde{\mathscr{B}}$-measurable set, and any function in $\tilde{\mathscr{M}}$ whose support is disjoint with S^0 is null. Then the inverse image of S^0, say S_0, in Σ, will be called the *support* of $\mathscr{M}$.

Remark. If $\mathscr{M} = P(L^\varrho(\Sigma))$, where P is a contractive projection then clearly $Qf = P(\chi_{S_0} f)$ is also a contractive projection onto $\mathscr{M}$, where S_0 is the support of $\mathscr{M}$. Moreover, Q hasth e property (D). If $Nf = P(\chi_{S_0^c} f)$, then $N^2 = 0$, and $P = Q + N$. (Hereafter FP for ϱ will be assumed.)

In view of the above remark, in the analysis of the range of contractive projections, it is no restriction to assume (D). Thus the general case, extending Proposition 2.5, can be given as follows:

THEOREM 2.7. *Let $\mathscr{M} \subset L^\varrho(\Sigma)$. Then $\mathscr{M}$ is the range of a contractive projection P, on $L^\varrho(\Sigma)$, with property* (D), *iff there is a multiplication map U_η $(U_\eta f = \eta f)$, where η is measurable $|\eta| = 1$ a.e., such that $Q = U_{\bar{\eta}} P U_\eta$ is a positive contractive projection, onto $U_\eta(\mathscr{M})$, with property* (D).

Remark. This result extends the L^1-case, of [8] and also contains the results of [1], [29], and [36]. The proof utilizes some computations of [8], and will be outlined here.

Proof. The direct part is easy since then there exists, by Proposition 2.5, a positive contractive projection $Q: L^\varrho(\Sigma) \mapsto U_\eta(\mathscr{M})$, which by the earlier remark, may be taken to have (D). So $P = U_{\bar{\eta}} Q U_\eta: L^\varrho(\Sigma) \mapsto \mathscr{M}$ is a contractive projection, with (D), onto $\mathscr{M}$.

For the converse, let $P: L^\varrho(\Sigma) \to \mathscr{M}$ be a contractive projection. The map U_η will now be constructed in the following four steps. In view of Propositions 2.4 and 2.5, in which the positivity of P was not needed, it suffices to prove the result for $\mu(\Omega) < \infty$.

(i) For any $f \in L^\varrho$, let $\theta_f = f/|f|$ if $f \neq 0$, and $= 1$ if $f = 0$, so that θ_f is measurable and $|\theta_f| = 1$, a.e. If now $f \in \mathscr{M}$, and $0 \leqslant h \leqslant |f|$, then $|f - \theta_f h| = |f| - h$ and $Pf = f$. It is claimed that if $Q = \theta_{\bar{f}} P \theta_f$, then (a) $\operatorname{supp}(Q) = \operatorname{supp}(Ph) \subset \operatorname{supp}(h) \subset \operatorname{supp}(f)$, and (b) $Qh \geqslant 0$. To see this, since $L^\varrho(\Sigma) = \bigcap \{L^1(\Sigma, G): 0 \leqslant G \in \mathscr{A}_{\varrho'}(\mu),\ \varrho'(G) = 1\}$, as noted earlier, consider a $0 \leqslant G \in \mathscr{A}_{\varrho'}(\mu)$, $\varrho'(G) = 1$, and G is σ-additive. Then $\|\cdot\|_{1,G} \leqslant \varrho(\cdot)$, and let $\mathscr{L} = \operatorname{sp}(L^\varrho(\Sigma)) \subset L^1(\Sigma, G)$. So P on $L^\varrho(\Sigma)$ has a unique (norm-preserving) extension $\tilde{P}$ to $\mathscr{L}$, and $\tilde{P}f = f$ still. Thus $\|\tilde{P}(f - \theta_f h)\|_{1,G} \leqslant \|f - \theta_f h\|_{1,G}$. Hence the L^1-theory applies to $\mathscr{L}$, and the inequalities

$$(10)\quad 0 \leqslant \int_\Omega (|f| - h)\,dG = \|f - \theta_f h\|_{1,G} \geqslant \|\tilde{P}(f - \theta_f h)\|_{1,G}$$

$$= \|\theta_{\bar{f}} f - Qh\|_{1,G} \geqslant \|f\|_{1,G} - \|Qh\|_{1,G} \geqslant \int_\Omega (|f| - h)\,dG\ (^2)$$

yield both (a) and (b) at once.

(ii) If $0 \leqslant h \in L^\varrho(\Sigma)$, $\operatorname{supp}(h) \subset \operatorname{supp}(f)$, then also (a) and (b) hold for this h and f $(\in \mathscr{M})$. This is again obtained by reducing the case to (i). Thus, if $A = \{h \leqslant |f|\}$, then $h\chi_A$ verifies (i) and $h\chi_{A^c} = h_0 \in L^\varrho(\Sigma)$, can be approximated by $0 \leqslant h_n \uparrow h_0$, $h_n \leqslant n|f|$ $(\in \mathscr{M})$, and if $g_n = Qh_n$, then by (i) $0 \leqslant g_n \uparrow g_0 \in L^\varrho(\Sigma)$. However,

$$\|g_0 - Qh_0\|_{1,G} \leqslant \|g_0 - g_n\|_{1,G} + \|Q(h_n - h_0)\|_{1,G} \to 0,$$

so that $g_0 = Qh_0$, and thus the statements hold. (Actually the above two steps can also be proved, replacing monotone sequences by sequences converging in G-measure, and using the proof of [9], III.3.8, without using Propositions 2.4 and 2.5 so that $0 \leqslant G \in \mathscr{A}_{\varrho'}(\mu)$ can be arbitrary.)

(iii) If $f \in \mathscr{M}$ and $h \in L^\varrho(\Sigma)$, then $f\chi_S, f\chi_{S^c} \in \mathscr{M}$ where $S = \operatorname{supp}(h)$. This follows from (i) and (ii) since $\operatorname{supp}(f\chi_S) \subset S$, so that

$$f\chi_{S^c} = (P(f\chi_S) + Pf\chi_{S^c})\chi_{S^c} = P(f\chi_{S^c})\cdot\chi_{S^c},$$

and P is a contraction so that $\|f\chi_{S^c}\|_{1,G} = \|Pf\chi_{S^c}\|_{1,G}$ (²) and $P(f\chi_S^c) = f\chi_{S^c}$.

(iv) To construct the U_η, let S_0 be the support of $\mathscr{M}$. Then, as noted earlier, $S_0 = \bigcup S_a$, $S_a \in \Sigma$, disjoint, and there is an $f_a \in \mathscr{M}$ with support S_a.

(²) Such a G can be chosen by Theorem I.3.2, again, since Q or P is a projection (cf. (¹)).

Let η be defined as

$$\eta = \begin{cases} f_a/|f_a| & \text{on } S_a, \\ 1 & \text{on } S_0^c. \end{cases} \tag{11}$$

Then η is measurable, $|\eta| = 1$ a.e., and $U_\eta(f) = \eta f$ is the desired operator, i.e. $Q = U_{\bar\eta} P U_\eta$ is a contractive projection with (D) (since P has (D)). Only positivity of Q need be checked here. So let $0 \leqslant h \in L^\varrho(\Sigma)$. Then for a countable set of indices $\{a_i\}$, one has $h = \sum_{i=1}^{\infty} h\chi_{S_{a_i}} + h\chi_{S_0^c}$, $h\chi_{S_0^c} \in \mathcal{N}$. Thus,

$$Qh = \sum_{i=1}^{\infty} U_{\bar\eta} P U_\eta (h\chi_{S_{a_i}}) + 0 = \sum_{i=1}^{\infty} \theta_{\bar f_{a_i}} P \theta_{f_{a_i}} (h\chi_{S_i}) \geqslant 0$$

by steps (i)-(iii).

It follows that $Q\colon L^\varrho(\Sigma) \mapsto U_\eta(\mathcal{M})$ is a positive contractive projection. Note that U_η is isometric and even preserves lattice operations when all functions are real. This finishes the proof [3].

Remark. The preceding two theorems imply that, when $\mu(\Omega) < \infty$, $P\colon L^\varrho \mapsto \mathcal{M}$, a contractive projection is of the form

$$P = U_{\bar\eta} \frac{1}{f_0} g_0 E^{\mathcal{B}}(\cdot) f_0 U_\eta = U_{\bar\eta} g_0 E^{\mathcal{B}}(\cdot) U_\eta, \tag{12}$$

where $g_0 \in \mathcal{M}$ has the support of $\mathcal{M}$, and U_η is an isometric multiplicative operator. Here f_0 is a weak unit in $L^\varrho(\mathcal{B})$, $\mathcal{B} \subset \Sigma$, is determined by $\mathcal{M}$, and thus f_0 is a fix point of $E^{\mathcal{B}}$.

Theorem 2.2 and the above show that $\mathcal{M}$ is topologically equivalent to $g_0 L^\varrho(\mathcal{B})$. This, however, is not a convenient enough description of $\mathcal{M}$. The desired result is given by

THEOREM 2.8. *Let $L^\varrho(\Sigma)$ be a BFS, $\mu(\Omega) < \infty$. Then $\mathcal{M} \subset L^\varrho(\Sigma)$ is the range of a contractive projection implies $\mathcal{M}$ is topologically equivalent to a BFS $L^\varrho(\mathcal{B})$ on some measure space $(S, \mathcal{B}, \nu)$ ($S = \Omega$, $\mathcal{B} \subset \Sigma$ can be arranged). Conversely, if $\mathcal{M}$ is isometrically isomorphic to some $L^\varrho(\mathcal{B})$ on a $(S, \mathcal{B}, \nu)$, then it is the range of a contractive projection. More generally, $\mathcal{M}$ is the range of a contractive projection iff it is isometrically isomorphic to some BFS, $L^{\tilde\varrho}(\mathcal{B})$, on a measure space $(S, \mathcal{B}, \nu)$, and $\tilde\varrho$ is equivalent to $\tilde\varrho$, i.e., $k_0 \varrho \leqslant \tilde\varrho \leqslant \varrho$ for some $k_0 > 0$.*

Proof. The main ideas are already contained in the preceding computations. The result is completed on reducing it to the L^1-case.

[3] A shorter proof is obtained by noting, $Pf = \int_\Omega f d\nu$ (Theorem I.3.2), and $\nu = \nu^+ - \nu^-$, $\Omega = \Omega^+ \cup \Omega^-$ the Jordan and Hahn decompositions, and let $\tilde\nu = \nu \circ \theta$ with $\theta = 1$ on Ω^+, $= -1$ on Ω^- and $Qf = \int_\Omega f d\tilde\nu$ (cf. Wright, TAMS, 139).

Direct Part. Since by Theorems 2.2 and 2.7, if $\mathcal{M} = P(L^\varrho(\Sigma))$, where P is a contractive projection, $U_\eta(\mathcal{M})$ is topologically equivalent to $g_0 L^\varrho(\mathcal{B})$, where $0 \leqslant g_0 \in U_\eta(\mathcal{M})$, and has the support of $\mathcal{M}$, it is only necessary to show that $g_0 L^\varrho(\mathcal{B}) \cong L^\varrho(\Omega, \mathcal{B}, \mu_1)$ for some measure μ_1 on $\mathcal{B}$. To prove this, consider the equation (2) of Theorem 2.2, i.e.,

$$\int_{S_0 \cap A} f_0 d\mu = \int_A g_0 d\mu = \mu_1(A) \quad \text{(say)}, \tag{13}$$

where f_0 (a weak unit in $L^\varrho(\mathcal{B}) \cap L^\infty(\mathcal{B})$) and g_0 are defined there. Then μ_1 has support S_0, and by equations (3)-(5) there,

$$k_0 \varrho(fg_0) \leqslant \varrho(ff_0) \leqslant k' \varrho(fg_0), \quad f \in L^\varrho(\mathcal{B}, \mu), \tag{14}$$

for some $0 < k_0, k' < \infty$, and $k_0 = k' = 1$ if ϱ has Fatou's property and L^ϱ has constants in it. Write ϱ_μ for ϱ with measure μ. Note that the mapping $f \mapsto f_0 f$ is a linear homeomorphism of $L^\varrho(\mathcal{B})$ onto itself, i.e. $\varrho(f_0 \cdot)$ and $\varrho(\cdot)$ are equivalent. If $\varrho_{\mu_1}(f) = \varrho_\mu(g_0 f)$, then (14) and the preceding observation imply that ϱ_{μ_1} and ϱ_μ are equivalent norms, μ and μ_1 being measures on $(\Omega, \mathcal{B})$. Consequently, $g_0 L^\varrho(\mathcal{B}, \mu)$ and $L^\varrho(\mathcal{B}, \mu_1)$ are equivalent. Thus $\mathcal{M}$ is topologically equivalent to $L^\varrho(\Omega, \mathcal{B}, \mu_1)$.

Converse. Suppose there is a $T\colon L^\varrho(S, \mathcal{B}, \nu) \mapsto \mathcal{M}$, a topological equivalence. If $g_0 \in \mathcal{M}$ is an element with support that of $\mathcal{M}$, (by Proposition 2.1), then $h = T^{-1}(g_0) \in L^\varrho(S, \mathcal{B}, \nu)$ and $|h| > 0$, a.e. Then by Corollary 1.2 of the preceding section $L^\varrho(S, \mathcal{B}, \nu)$ is isometrically equivalent to an $L^\varrho(\tilde S, \tilde{\mathcal{B}}, \tilde\nu)$, with $\tilde\nu(\tilde S) < \infty$. Hence, for this proof it may be assumed $\nu(S) < \infty$, to start with. Then there is a weak unit $0 < f_0 \leqslant k_0$, a.e., $f_0 \in L^\varrho(\mathcal{B}, \nu)$ and if $h_0 = T(f_0) \in \mathcal{M}$, $\operatorname{supp}(h_0) = S_0$. If $\eta = h_0/|h_0|$ on S_0, and $= 1$ elsewhere, then $U_\eta(h_0) \geqslant 0$, and U_η is an isometry on $L^\varrho(\Sigma)$. If $V = U_\eta T\colon L^\varrho(\mathcal{B}) \mapsto U_\eta(\mathcal{M})$, then V is a topological equivalence and is an isometry if T is. To complete the proof, it is only necessary (by Theorem 2.7) to show that $U_\eta(\mathcal{M})$ is a lattice and is s.a., since it is clearly complete.

Since T is, moreover, given to be an isometry for this part, so is V. To show that V is positive (so that $U_\eta(\mathcal{M})$ is a lattice), the trick is to reduce the result to an L^1-case, using the representation of Theorem I.1.13. Thus if $0 \leqslant G_1 \in \mathcal{A}_{\varrho'}(\mu)$, $\varrho'(G_1) = 1$, $L^\varrho(\Sigma) \subset L^1(\Sigma, G_1)$, let $0 \leqslant G \in \mathcal{A}_{\varrho'}(\nu)$, $\varrho'(G) = 1$, $L^\varrho(\mathcal{B}) \subset L^1(\mathcal{B}, G)$. Let $\mathcal{L}_1$ and $\mathcal{L}$ be as considered before. The isometric map V extends to these subspaces as a contraction, which have weaker topologies. Using the same symbol, let $0 \leqslant f_0$, $f \in L^\varrho(\mathcal{B})$, f_0 being the weak unit noted above. Then $\|V(f_0+f)\|_{1,G_1} = \|(f_0+f)\|_{1,G} = \|f_0\|_{1,G} + \|f\|_{1,G} \geqslant \|Vf_0\|_{1,G_1} + \|Vf\|_{1,G_1}$ and since $Vf_0 \geqslant 0$ because $f \geqslant 0$, it follows that $Vf \geqslant 0$, and $U_\eta \mathcal{M}$ is thus s.a. The same equation also shows, on replacing f_0, f by f_1, f_2 with $f_1 \wedge f_2 = 0$, that $(Vf_1) \wedge (Vf_2) = 0$. With this one deduces if $f \in U_\eta(\mathcal{M})$, then $|f| \in U_\eta(\mathcal{M})$. In fact, $V^{-1}(f) \in L^\varrho(\mathcal{B})$, so that $(V^{-1}(f))^{\pm}\in L^{\varrho}(\mathscr{B})$, the latter being a lattice. So $V((V^{-1}(f))^{\pm})\in U_{\eta}(\mathscr{M})$, and have disjoint supports. Since V^{-1} is also an isometry on $U_{\eta}(\mathscr{M})\mapsto L^{\varrho}(\mathscr{B})$, and $f = V((V^{-1}f)^{+}-(V^{-1}f)^{-})$, one can now choose the $\bar{G}$, $\bar{G}_1$ (with similar notations) such that V^{-1} is the (extension of the) isometry on the corresponding subspaces of $L^1(\Sigma,\bar{G}_1)$ and $L^1(\mathscr{B},\bar{G})$. This is possible since the various operations are taking place only on pairs of elements of $L^{\varrho}(\mathscr{B})$-and $L^{\varrho}(\Sigma)$-spaces themselves and these are contained in all the subspaces $\mathscr{L}$ for all G's considered. Thus, using the isometry in the original L^{ϱ}-spaces,

$$\|f\|_{1,\bar{G}_1} = \|V(V^{-1}f)^{+}\|_{1,\bar{G}_1}+\|V(V^{-1}f)^{-}\|_{1,\bar{G}_1}\leqslant \|(V^{-1}f)^{+}\|_{1,\bar{G}}+\|(V^{-1}f)^{-}\|_{1,\bar{G}}$$
$$= \|(V^{-1}f)^{+}\pm(V^{-1}f)^{-}\|_{1,\bar{G}} = \|V^{-1}(f)\|_{1,\bar{G}}\leqslant \|f\|_{1,\bar{G}_1}.$$

For the last part, if $\mathscr{M}$ is topologically equivalent to an $L^{\varrho}(\mathscr{B})$, let $T\colon \mathscr{M}\mapsto L^{\varrho}(\mathscr{B})$ be the map effecting this equivalence. Then $f\in\mathscr{M}$ implies $g = Tf\in L^{\varrho}(\mathscr{B})$, and let $\tilde{\varrho}(g) = \varrho(Tf)\leqslant k\varrho(f)$. Then there is k' $(0\leqslant k, k' < \infty)$ with $k'\varrho(f)\leqslant\varrho(Tf)\leqslant k\varrho(f)$, and it may be assumed that $k = 1$ here. So $\tilde{\varrho}$ is equivalent to ϱ and $L^{\tilde{\varrho}}(\mathscr{B}) = L^{\varrho}(\mathscr{B})$. Thus $T\colon \mathscr{M}\mapsto L^{\varrho}(\mathscr{B})$ is an isometry, and the above computations can be given to $\tilde{\varrho}$. This completes the proof of the theorem ([4]).

The following special case is stated for a reference and is a consequence of the above proof together with that of Theorems 2.2 and 2.4:

THEOREM 2.9. *Let $L^{\varrho}(\Sigma)$ be a BFS, $\mu(\Omega)<\infty$, ϱ has the Fatou property and L^{ϱ} has constants. Then $\mathscr{M}\subset L^{\varrho}(\Sigma)$ is the range of a contractive projection iff it is isometrically isomorphic to some $L^{\varrho}(\mathscr{B})$ on some measure space $(S,\mathscr{B},\nu)$.*

Using the results of [34] and of Theorem 1.1 of the preceding section, the above result can be extended, for arbitrary measures, in exactly the same way as Propositions 2.4 and 2.5. In the decomposition there, $L^{\varrho}(\Sigma) = \Sigma\oplus_{\alpha}L^{\varrho}(\Sigma_{\alpha})$, for the localizable case, one notes that the maps of the topological equivalence depend only on the bound of the weak unit in each $L^{\varrho}(\mathscr{B}_{\alpha})$, on a finite measure space. Since the latter can be chosen to be uniformly bounded (by 1), the proofs proceed without any major modifications. The main result of this section can thus be presented, as a summary of the preceding work, as follows:

THEOREM 2.10. *Let $L^{\varrho}(\Sigma)$ be a BFS on a measure space (Ω,Σ,μ). Then the following statements are equivalent (ϱ has FP):*

(a) *$\mathscr{M}$ is the range of a (positive) contractive projection on L^{ϱ}.*

(b) *There exists an isometric isomorphism $\psi\colon L^{\varrho}(\Sigma)\mapsto L^{\varrho}(\Sigma)$ $(\psi = I)$ such that (i) $\psi(\mathscr{M})$ is a B-lattice, and (ii) $0\leqslant f_n\uparrow f\in L^{\varrho}(\Sigma)$, $f_n\in\psi(\mathscr{M})$, all n, implies $f\in\psi(\mathscr{M})$.*

(c) *$\mathscr{M}\subset L^{\varrho}(\Sigma)$ is (positively) isometrically isomorphic to an $L^{\tilde{\varrho}}(\mathscr{B})$ on a measure space $(S,\mathscr{B},\nu)$, where ϱ and $\tilde{\varrho}$ are equivalent function norms: $k_0\varrho\leqslant\tilde{\varrho}\leqslant\varrho$, $0<k_0\leqslant 1$.*

If ϱ has also the Fatou property, and $\varrho(\chi_A)<\infty$ for each $\mu(A)<\infty$, then (c) can be strengthened to $\varrho = \tilde{\varrho}$, where $\tilde{\varrho}$ also has the Fatou property.

Remark. If $L^{\varrho} = L^{p}$, $1\leqslant p<\infty$, the above result was proved, for various cases, in [8], [1], [19] and [36], and if $L^{\varrho} = L^{\Phi}$, Φ satisfies a growth condition, μ is σ-finite, it was obtained in [29]. Note that if $L^{\varrho} = l^{\infty}$, and $\mathscr{M} = c_0$, of convergent sequences to zero, then b(ii) is not satisfied and the well-known result that there is no contractive projection on l^{∞} onto c_0 is a consequence of the above theorem.

2.3. Contractive projections; vector case. In this section, the above results will be extended to a class of $L^{\varrho}_{\mathscr{X}}$-spaces of $\mathscr{X}$-valued strongly measurable functions, where $\mathscr{X}$ is a B-space. This will generalize the main result on $L^{1}_{\mathscr{X}}$ in [15], and also includes the $L^{p}_{\mathscr{X}}$-spaces for $1\leqslant p\leqslant\infty$. Unfortunately it is not so comprehensive as Theorem 2.10, but seems to be the best that the present methods yield.

Let $\mathscr{M}^{\varrho}_{\mathscr{X}} = \overline{\mathrm{sp}}\{fx\colon f\in L^{\varrho}(\Sigma), x\in\mathscr{X}\}\subset L^{\varrho}_{\mathscr{X}}$, as in Chapter I. If ϱ is a. c. n., so $M^{\varrho} = L^{\varrho}$, then $\mathscr{M}^{\varrho}_{\mathscr{X}} = L^{\varrho}_{\mathscr{X}}$, and may be a proper subset otherwise. Also let ϱ have FP.

THEOREM 3.1. *Let $L^{\varrho}(\Sigma)$ be BFS on (Ω,Σ,μ) and $\mathscr{X}$ be a B-space. If $\mathscr{Y}\subset\mathscr{X}$ and $\mathscr{S}^{\varrho}\subset L^{\varrho}(\Sigma)$ are closed subspaces, and $\mathscr{M}^{\varrho}_{\mathscr{X}}$ as above, let $\mathscr{S}^{\varrho}_{\mathscr{Y}} = \overline{\mathrm{sp}}\{gy\colon g\in\mathscr{S}^{\varrho}, y\in\mathscr{Y}\}$. Then the following statements are equivalent:*

(a) *There exists a contractive projection on $\mathscr{M}^{\varrho}_{\mathscr{X}}$ onto the subspace $\mathscr{S}^{\varrho}_{\mathscr{Y}}$.*

(b) *There exist contractive projections on $L^{\varrho}(\Sigma)$ onto $\mathscr{S}^{\varrho}$ and on $\mathscr{X}$ onto $\mathscr{Y}$.*

(c) *$\mathscr{S}^{\varrho}_{\mathscr{Y}}$ is isometrically isomorphic to an $\mathscr{M}^{\tilde{\varrho}}_{\mathscr{Y}}\subset L^{\tilde{\varrho}}_{\mathscr{Y}}(\mathscr{B})$ on some measure space $(S,\mathscr{B},\nu)$, $\tilde{\varrho}$ is equivalent to ϱ, $k_0\varrho\leqslant\tilde{\varrho}\leqslant\varrho$ for some $0<k_0\leqslant 1$, and there is a contractive projection on $\mathscr{X}$ onto $\mathscr{Y}$.*

If, moreover, ϱ has the Fatou property, and $\varrho(\chi_A)<\infty$ for all $\mu(A)<\infty$, then $\varrho = \tilde{\varrho}$ in (c).

Proof. That (a) $\Rightarrow$ (b) was proved in Proposition I.5.11, (b) $\Rightarrow$ (c) is a consequence of Theorem 2.10 above and Proposition I.5.11. The last comment is a direct consequence of Theorem 2.9.

Since the last additional hypothesis always holds for the Orlicz spaces, the following is a special case of the above result:

COROLLARY 3.2. *Let $L^{\Phi}(\Sigma)$ be an Orlicz space on (Ω,Σ,μ). If $\mathscr{M}^{\Phi}_{\mathscr{X}}$ and $\mathscr{S}^{\Phi}_{\mathscr{Y}}$ are the corresponding spaces, then all the statements of the theorem*

([4]) A shorter proof is again obtained, with Theorem 1.3.2 for P and the fact that a positive projection T on L^{ϱ} can be expressed as $T(f\cdot Tg) = T(Tf\cdot Tg)$ for $f,g\in L^{\infty}\cap L^{\varrho}$ (cf. Lloyd, TAMS, 125). A multiplication can be introduced in Rang(T), which contains an algebra. Then the result follows from [27], Theorem 2.1.

are equivalent with the same function norm $\varrho(\cdot)$, which can be either of the two known norms in the theory of Orlicz spaces.

Any mention of $\mathscr{M}^\varrho$ can be dropped if $\varrho(\cdot) = \|\cdot\|_p$, $1 \leqslant p \leqslant \infty$. Thus the above result takes a particularly simple form for the L^p-spaces:

COROLLARY 3.3. *If $\mathscr{S}^p \subset L^p$, $1 \leqslant p \leqslant \infty$, is a closed subspace (for any measure space (Ω, Σ, μ)) and $\mathscr{Y} \subset \mathscr{X}$, then the following are equivalent:*

(a) *There is a contractive projection on the subspace $\mathscr{S}^p_{\mathscr{Y}}$ of $L^p_{\mathscr{X}}$.*

(b) *There exist contractive projections on L^p onto $\mathscr{S}^p$ and on $\mathscr{X}$ onto $\mathscr{Y}$.*

(c) *There exists an $L^p_{\mathscr{Y}}(\mathscr{B})$ on some $(S, \mathscr{B}, \nu)$ such that $\mathscr{S}^p_{\mathscr{Y}}$ is isometrically equal to it, and there is a contractive projection on $\mathscr{X}$ onto $\mathscr{Y}$.*

Remark. If $p = 1$, this was given in [15], as a consequence of certain results on a metric characterization of the $L^1_{\mathscr{X}}$-spaces among all B-spaces. On the other hand, the above result, and particularly Proposition I.5.11, is based on a characterization of $B(L^p_{\mathscr{X}}, \mathscr{Y})$, holding for other cases as well.

2.4. More on projections. Thus far only the existence of contractive projections on the L^ϱ-spaces was determined, but the uniqueness and structure of such projections was not considered. This is again non-trivial, and two cases will be treated in this section to elucidate the problem.

PROPOSITION 4.1. *Let $P\colon L^\varrho(\Sigma) \mapsto L^\varrho(\mathscr{B})$ be a contractive projection where $\mathscr{B} \subset \Sigma$ is a σ-field and $\mu_{\mathscr{B}}$ has the FSP. Then there exists a locally integrable quasi-function g^* (i.e., measurable on every set of μ-finite measure) such that $Pf = E^{\mathscr{B}}(g^* f)$, $f \in L^\varrho(\Sigma)$. If $0 < f_0 \in L^\varrho(\mathscr{B})$ and μ is localizable, then g^* is measurable and $E^{\mathscr{B}}(g) = 1$, a.e.*

(About quasi-functions, cf. [22], or [39], p. 257 ff., where they are called cross-sections.)

Proof. By the FSP of μ, $L^{\varrho'}(\Sigma)$ is non-trivial (and the same holds of $L^{\varrho'}(\mathscr{B})$), and in fact is norm-determining for $L^{\varrho''}(\Sigma)$ where ϱ and ϱ'' are equivalent. Let $A \in \Sigma$ be a set such that $0 < \varrho(\chi_A) < \infty$, $0 < \varrho'(\chi_A) < \infty$. In fact, for every set $A \in \Sigma$ of finite measure there exist both "ϱ and ϱ'-admissible" sequences $\{A_n\}$, $A_n \uparrow A$ and $0 < \varrho(\chi_{A_n}) < \infty$, $0 < \varrho'(\chi_{A_n}) < \infty$ for all n (cf. [20]). So A may be replaced by A_n for large enough n. Consider now $x^*_A \in (L^\varrho(\Sigma))^*$ defined by

$$(1) \qquad x^*_A(f) = \int_\Omega \chi_A P(f)\, d\mu, \quad f \in L^\varrho(\Sigma).$$

It is continuous and can be regarded as $x^*_A \in (L^\varrho(A, \Sigma(A), \mu_A))^*$, where $\Sigma(A)$ is the trace of Σ on A, and $\mu_A(\cdot) = \mu(A \cap \cdot)$.

Since μ_A is a finite measure, by Corollary I.1.14, x^*_A can be uniquely expressed as

$$(2) \qquad \int_A Pf\, d\mu_A = x^*_A(f) = \int_A f g_A\, d\mu_A + \int_A F(f)\, dG_A, \quad f \in L^\varrho(\Sigma),$$

where $g_A \in L^{\varrho'}(\Sigma)$, and G_A is a pfa, with $|G_A|(A) < \infty$ and vanishes on μ-null subsets on A. Replacing f by any $f_0 \chi_B \in L^\varrho(\mathscr{B})$, where $f_0 > 0$ a.e., on A (which exists) and $B \in \mathscr{B}$, (2) becomes, on recalling that P is identity on $L^\varrho(\mathscr{B})$,

$$(3) \qquad \int_{A \cap B} f_0\, d\mu_A = \int_{A \cap B} f_0 g_A\, d\mu_A + \int_{A \cap B} F(f_0)\, dG_A, \quad B \in \mathscr{B}.$$

The first two integrals are σ-additive on $\mathscr{B}(A)$, and the last one is a pfa on it and hence it must vanish on A. Thus (4) becomes with Proposition 1.5 above,

$$(4) \qquad \int_B Pf\, d\mu = \int_B f g_A\, d\mu = \int_B E^{\mathscr{B}}(f g_A)\, d\mu, \quad B \in \mathscr{B}(A).$$

Since the integrals on the extreme are $\mathscr{B}$ (and even $\mathscr{B}(A)$)-measurable, they can be identified. As A varies on μ-finite sets $\{g_A\}$ defines a quasi-function g^*, by definition. Thus the relation can be expressed as $Pf = E^{\mathscr{B}}(f g^*)$, $f \in L^\varrho(\Sigma)$. (In passing from the ϱ-admissible sequences to those of A of finite μ-measure to using FSP, the result of [13], p. 19, can be consulted.)

Finally, if there is a weak unit $f_0 \in L^\varrho(\mathscr{B})$, $f_0 > 0$, a.e. and μ is localizable, then g^* can be chosen μ-measurable, so that

$$(5) \qquad f_0 = Pf_0 = E^{\mathscr{B}}(f_0 g) = f_0 E^{\mathscr{B}}(g) \quad \text{a.e.},$$

by Proposition 1.6 (ii) above. Since $f_0 > 0$ a.e., it follows that $E^{\mathscr{B}}(g) = 1$, a.e., and the proof is complete.

Remark. It can be shown that in the last part above (as was done in [28] and [29] for the Orlicz space cases) if also ϱ is an a.c.n., then $g = 1$ a.e. In particular, this shows that the generalized conditional expectation is the unique contractive projection on $L^\varrho(\Sigma) \mapsto L^\varrho(\mathscr{B})$ if $L^\varrho(\Sigma)$ is reflexive.

For other subspaces, the following result gives a sufficient condition for uniqueness:

PROPOSITION 4.2. *Let $L^\varrho(\Sigma)$ be a rotund reflexive space such that $L^{\varrho'}(\Sigma)$ is also rotund (or equivalently, L^ϱ is a rotund, reflexive and smooth BFS, where "smooth" means the norm is Gâteaux differentiable). Then a closed subspace $\mathscr{M}$ of $L^\varrho(\Sigma)$ can be the range of at most one contractive projection.*

Proof. If $\mathscr{M}$ is the range of two contractive projections P_1 and P_2, then it will be shown that their null spaces are also identical. This proves the result. But this is equivalent to proving the ranges $\mathscr{M}^*_i$ of the adjoints P^*_i are identical. (Recall that $L^\varrho = \mathscr{M} \oplus \mathscr{N}_i$; then $P^*_i\colon (L^\varrho)^* \mapsto \mathscr{N}_i^\perp \cong \mathscr{M}^*_i$.) This will be established now.

First a preliminary simplification is needed. Let $\mathscr{M} = P(L^\varrho)$ and $\mathscr{M}^* = P^*((L^\varrho)^*)$. Let $S \subset \mathscr{M}^*$, $S^* \subset \mathscr{M}^*$ be the elements of unit norms.

It will be shown that each $f \in S$ determines a unique $g_f \in S^*$ and conversely. To this end, let $f \in S$. Then there is an $x_i^* \in (L^\varrho)^*$, $\|x_f^*\| = 1$ and $1 = \varrho(f) = x_f^*(f) = \int_\Omega f g_f d\mu$, by the representation theorem. $\varrho'(g_f) = \|x_f^*\| = 1$. Since $Pf = f$ one has

$$1 = \int_\Omega (Pf) g_f d\mu = \int_\Omega f P^*(g_f) d\mu \leqslant \varrho(f)\varrho'(P^* g_f) \leqslant 1. \tag{6}$$

Thus there is equality throughout, and $\varrho'(P^* g_f) = \varrho'(g_f) = 1$. The rotundity of ϱ' implies, since $x^*(f) = \hat{f}(x^*)$, $\hat{f}(\cdot)$ attains the value (of its norm) at only one point on the unit ball of $L^{\varrho'}$. So $P^* g_f = g_f$. (The parenthetical statement is just a consequence of a classical result of V. L. Šmulian, stating that rotundity and smoothness are truely dual in reflexive spaces.) Hence $g_f \in S^*$. The mapping $\boldsymbol{i}: f \mapsto g_f$, called the *spherical image map*, is one-to-one and onto since the hypotheses on ϱ and ϱ' are symmetrical. Thus $\boldsymbol{i}: S \mapsto S^*$ and $\boldsymbol{i}^*: S^* \mapsto S$ are both onto and one-to-one.

Now to complete the proof, let $\mathscr{M}_i^* = P_i^*((L^\varrho)^*)$, $\mathscr{M} = P_i(L^\varrho)$, the latter given to be the same, $i = 1, 2$. If $S_i^* \subset \mathscr{M}_i^*$ are the corresponding sets, then $\boldsymbol{i}(S) = S_j^*, j = 1, 2$, are identical by the above paragraph. Since $\mathscr{M}_j = \overline{\mathrm{sp}}(S_j^*)$, $i = 1, 2$, $\mathscr{M}_1^* = \mathscr{M}_2^*$. This completes the proof.

Remark. The hypothesis here is not necessary as seen from the particular space $\mathscr{M} = L^\varrho(\mathscr{B}) \subset L^\varrho(\Sigma)$ when the latter is reflexive but not necessarily rotund or smooth, since then ϱ (and ϱ') is an a.c.n. Thus further work is needed on this question. If $L^\varrho = L^p$, $1 < p < \infty$, $\mu(\Omega) < \infty$, the above result was noted in [1].

2.5. Prediction operators; an application. The results of the preceding sections yield a simple characterization of the linearity of prediction operators in the general theory of non-linear prediction (and approximation) in function spaces. This is, in fact, one of the motivations for the study of the projection problem above.

Let $\mathscr{N} \subset L^\varrho$ be a subspace. It is said to be a *Tshebyshev subspace* if for each $x \in L^\varrho$ there is a unique $x_0 \in \mathscr{N}$ such that $\varrho(x - x_0) = \min\{\varrho(x-y): y \in \mathscr{N}\}$. The mapping $P_{\mathscr{N}}: L^\varrho \mapsto \mathscr{N}$ is well-defined, and has the properties: (i) $P_{\mathscr{N}}^2 = P_{\mathscr{N}}$, (ii) $P_{\mathscr{N}}(ax) = a\, P_{\mathscr{N}}(x)$, $a \geqslant 0$, (iii) $P_{\mathscr{N}}(x+y) = P_{\mathscr{N}}(x) + y$ for all $y \in \mathscr{N}$, and (iv) $x \geqslant 0$ implies $P_{\mathscr{N}}(x) \geqslant 0$ (cf. [28]). However, $P_{\mathscr{N}}$ is not necessarily linear on L^ϱ. If $Q = I - P_{\mathscr{N}}$, then Q is linear iff $P_{\mathscr{N}}$ is, and when this happens, Q is a contractive projection on L^ϱ with $\mathscr{N}$ as its null space. The subspaces $\mathscr{N}$ admitting linear prediction operators can thus be characterized as follows (as usual ϱ has the *FP*):

THEOREM 5.1. *Let L^ϱ be a real BFS on (Ω, Σ, μ) and $\mathscr{M} \subset L^\varrho(\Sigma)$ be a Tshebyshev subspace. Then the prediction operator $P_{\mathscr{M}}$ on $L^\varrho(\Sigma)$ onto $\mathscr{M}$ is linear iff the quotient space $L^\varrho/\mathscr{M}$ is isometrically isomorphic to an $L^{\tilde\varrho}(\mathscr{B})$ on some $(S, \mathscr{B}, \nu)$, where $\tilde\varrho$ is equivalent to ϱ $(k_0\varrho \leqslant \tilde\varrho \leqslant \varrho$, $0 < k_0 \leqslant 1)$.*

Proof. If $P_{\mathscr{M}}$ is linear, then $Q = I - P_{\mathscr{M}}$ is a projection with $\mathscr{M}$ as its null space. Moreover

$$\varrho(Qf) = \varrho(f - P_{\mathscr{M}} f) \leqslant \varrho(f), \quad f \in L^\varrho(\Sigma), \tag{1}$$

since $P_{\mathscr{M}} f$ is the closest to f. Thus Q is a (not necessarily positive) contractive projection and $\mathscr{N} = Q(L^\varrho(\Sigma))$ is, by Theorem 2.10 above, isometrically isomorphic to $L^{\tilde\varrho}(\mathscr{B})$ of the given description. Also $L^\varrho(\Sigma) = \mathscr{M} \oplus \mathscr{N}$ so that $\mathscr{N}$ is isometrically isomorphic to $L^\varrho(\Sigma)/\mathscr{M}$ and the latter thus is isometrically equivalent to $L^{\tilde\varrho}(\mathscr{B})$. This gives the direct part.

Conversely, let $\mathscr{N} \cong L^\varrho(\Sigma)/\mathscr{M}$, so by hypothesis $\mathscr{N} \cong L^{\tilde\varrho}(\mathscr{B})$, and $\mathscr{M}$ is a closed subspace of $L^\varrho(\Sigma)$ since $\mathscr{N}$ is complete. On the other hand, $L^\varrho(\Sigma) \overset{F}{\mapsto} \mathscr{N} \overset{T}{\mapsto} L^{\tilde\varrho}(\mathscr{B})$, where F is the canonical map and T is the isometry. Since these are onto maps and open, $F_1 = T \circ F: L^\varrho(\Sigma) \mapsto L^{\tilde\varrho}(\mathscr{B})$ is also onto and open. Then pick one element each (by the axiom of choice) in $L^\varrho(\Sigma)$ from the sets $F_1^{-1}(g)$, $g \in L^{\tilde\varrho}(\mathscr{B})$, and let this set be denoted by N. Since F_1 is linear (and open), and $L^{\tilde\varrho}(\mathscr{B})$ is complete, it follows that $F_1^{-1}: L^{\tilde\varrho}(\mathscr{B}) \mapsto N$ is a continuous one-to-one and onto operator. It is also easily seen that $N_1 = \overline{\mathrm{sp}}(N)$ is a subspace of $L^\varrho(\Sigma)$, and then is topologically equivalent to $L^{\tilde\varrho}(\mathscr{B})$. Thus it is, by Theorem 2.8, the range of a contractive projection Q on $L^\varrho(\Sigma)$. If $\tilde{\mathscr{M}} = (I - Q)(L^\varrho(\Sigma))$, then it follows that the spaces $L^\varrho(\Sigma)/\tilde{\mathscr{M}}$, N_1, $L^{\tilde\varrho}(\mathscr{B})$, $\mathscr{N}$ and $L^\varrho(E)/\mathscr{M}$ are topologically equivalent. Hence $\mathscr{M}$ and $\tilde{\mathscr{M}}$ can be identified. Thus if $P_{\mathscr{M}} = I - Q: L^\varrho(\Sigma) \mapsto \mathscr{M}$, then $P_{\mathscr{M}}(\mathscr{M}) = \mathscr{M}$, and one has, since $Q(\mathscr{M}) = 0$,

$$\varrho(f - P_{\mathscr{M}} f) = \varrho(Qf) = \varrho(Q(f-g)) \leqslant \varrho(f-g), \quad g \in \mathscr{M}. \tag{2}$$

Hence $P_{\mathscr{M}} f$ is a minimal element in $\mathscr{M}$ and since the latter is a Tshebyshev subspace $P_{\mathscr{M}} f$ is the unique element in $\mathscr{M}$ closest to f. Thus $P_{\mathscr{M}}$ is the (linear) prediction operator onto $\mathscr{M}$. This completes the proof.

Discussion. 1. The converse proof is simpler if L^ϱ is reflexive. Then $(L^\varrho)^* = L^{\varrho'}$ and $(\mathscr{N})^* \cong \mathscr{M}^\perp \cong L^{\tilde\varrho'}(\mathscr{B})$, and so there is a contractive projection onto $\mathscr{M}^\perp$. Since $(L^\varrho)^{**} = L^\varrho$ and P^{**} on $(L^\varrho)^{**}$ is a contractive projection, then $I - P^{**}: (L^\varrho)^{**} \mapsto \mathscr{M}^{**} = \mathscr{M}$ is the projection operator as in (2) above.

2. The condition given on $\mathscr{M}$ by the above theorem is not very easy to check in practical problems. For instance, if $\mathscr{M}$ is a measurable subspace, (i.e. $= L^\varrho(\mathscr{B})$ for some σ-field $\mathscr{B} \subset \Sigma$), then it was shown in [29], in the context of Orlicz spaces, that the prediction operator $P_{\mathscr{M}}$ (when it exists) will be linear iff the space is $L^2(\Sigma)$. Thus direct (and usable) conditions appear somewhat intricate.

Theorem 5.1, for $L^\varrho = L^p$, $1 < p < \infty$, using reflexivity and rotundity of L^p, was given in [1]. Also this indicates, since always $\mathcal{N} \cong \mathcal{M}^\perp$ $(\subset (L^\varrho)^*)$ above, a motivation for an investigation of the projection problem in $(L^\varrho)^*$-spaces.

2.6. Remarks and open problems. The uniqueness problem for general L^ϱ-spaces, seems difficult. Condition (D) of Definition 2.6 above, was shown to lead to uniqueness in the L^1-case in [8] (where it was introduced) and for the L^Φ-spaces with a.c.n. in [29]. The argument of the latter can be used to show that uniqueness obtains in the presence of condition (D) when ϱ is an a.c.n. On the other hand, the result of Proposition 4.2 implies that condition (D) is automatic for the reflexive rotund and smooth L^ϱ-spaces. The general case is still open.

It was noted in the preceding section, on prediction operators, that a characterization of subspaces of $(L^\varrho)^*$ admitting contractive projections is a natural problem. In general, $(L^\varrho)^*$ is only a space of (finitely additive) set functions, and such a study is not even available in the L^p-context. Even though the structure theory of Lebesgue and Orlicz spaces of additive set functions is now available (cf. [37]), the corresponding study for $\mathcal{A}_\varrho$-spaces has not been touched. This and the projection problem on spaces of set functions are completely open. In the other direction, of vector valued functions $L^\varrho_{\mathcal{X}}$, a characterization of arbitrary subspaces admitting contractive projections (analogous to Theorem 2.10) seems to need new techniques.

Finally, it is possible to define the $\mathcal{L}_{\varrho,\lambda}$-spaces for each $\lambda \geqslant 1$ with $\varrho(\cdot)$ a function norm, similar to the $\mathcal{L}_{p,\lambda}$- spaces in [19]. It will be very interesting to find conditions, with the work of this paper, for a B-space $\mathcal{X}$ to be isomorphic to an L^ϱ-space, or to be an $\mathcal{L}_{\varrho,\lambda}$-space. This is very similar to the treatment of [19], p. 309, and the work here appears particularly relevant to this problem.

References

[1] T. Andô, *Contractive projections on L^p-spaces*, Pacific J. Math. 17 (1966), p. 391-405.

[2] R. G. Bartle, *A general bilinear vector integral*, Studia Math. 15 (1956), p. 337-352.

[3] S. D. Chatterji, *Martingale convergence and the Radon-Nikodým theorem in Banach spaces*, Math. Scand., 22 (1968/69), p. 21-44.

[4] S. Chevet, *Sur certains produits tensoriels topologique d'espaces de Banach*, Z. Wahrscheinlichkeitstheorie verw. Geb. 11 (1969), p. 120-138.

[5] N. Dinculeanu, *Integral representation of vector measures and linear operations*, Studia Math. 25 (1965), p. 181-205.

[6] — *Integral representation of dominated operators on spaces of continuous vector fields*, Math. Ann. 173 (1967), p. 147-180.

[7] — *Vector measures*, London 1967.

[8] R. G. Douglas, *Contractive projections in an L^1-space*, Pacific J. Math. 15 (1965), p. 443-462.

[9] N. Dunford and J. T. Schwartz, *Linear operators, Part I: General Theory*, New York 1958.

[10] H. W. Ellis, *On the dual of L^λ*, Canadian Math. Bull. 8 (1965), p. 809-818.

[11] H. W. Ellis and I. Halperin, *Function spaces determined by a levelling length function*, Canadian J. Math. 5 (1953), p. 576-592.

[12] J. Gil de Lamadrid, *Measures and tensors, II*, ibidem 18 (1966), p. 762-793.

[13] N. Gretsky, *Representation theorems on Banach function spaces*, Bull. Amer. Math. Soc. 74 (1968), p. 705-709. (Memoirs of Amer. Math. Soc. 84 (1968), 56 pp.)

[14] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Memoirs of Amer. Math. Soc. 16 (1955), p. 336 pp.

[15] — *Une caractérisation vectorielle-métrique des espaces L^1*, Canadian J. Math. 7 (1955), p. 552-561.

[16] I. Halperin, *Reflexivity in L^λ-function spaces*, Duke Math. J. 21 (1954), p. 205-208.

[17] — *Function spaces*, Proc. International Symp. on Linear Spaces, Jerusalem (Academic Press, 1961), p. 242-250.

[18] S. Kakutani, *Concrete representation of abstract (M)-spaces*, Ann. Math. (2), 42 (1941), p. 994-1024.

[19] J. Lindenstrauss and A. Pełczyński, *Absolutely summing operators in $\mathcal{L}_p$-spaces and their applications*, Studia Math. 29 (1968), p. 275-326.

[20] W. A. J. Luxemburg and A. C. Zaanen, *Compactness of integral operators in Banach function spaces*, Math. Ann. 149 (1963), p. 150-180.

[21] — *Notes on Banach function spaces I-IV*, Nederl. Akad. Wetensch., Proc. Ser. A (= Indag. Math.), Vols. 66-68 (1963-1965).

[22] E. J. McShane, *Families of measures and representations of algebras of operators*, Trans. Amer. Math. Soc. 102 (1962), p. 328-345.

[23] — *Integrals devised for special purposes*, Bull. Amer. Math. Soc. 69 (1963), p. 597-627.

[24] B. J. Pettis, *On integration in vector spaces*, Trans. Amer. Math. Soc. 44 (1938), p. 277-304.

[25] R. S. Phillips, *On linear transformations*, ibidem 48 (1940), p. 516-541.

[26] M. M. Rao, *Linear functionals on Orlicz spaces: general theory*, Pacific J. Math. 25 (1968), p. 553-585.

[27] — *Stone-Weierstrass theorems for function spaces*, J. Math. Anal. Applic. 25 (1969), p. 362-371.

[28] — *Conditional expectations and closed projections*, Nederl. Acad. Wetensch. Proc. Ser. A (= Indag. Math.) 68 (1965), p. 100-112.

[29] — *Inference in stochastic processes, IV* (to appear).

[30] — *Produits tensoriels et espaces de fonctions*, C. R. Acad. Sci. Paris (Ser. A) 268 (1969), p. 1599-1601.

[31] — *Contractive projections and prediction operators*, Bull. Amer. Math. Soc. 75 (1969), p. 1369-1373.

[32] — *Remarks on continuity of some transformations* (to appear).

[33] R. Schatten, *A theory of cross-spaces*, Ann. Math. Studies 26, Princeton 1950.

[34] I. E. Segal, *Equivalences of measure spaces*, Amer. J. Math., 73 (1951), p. 275-313.

[35] I. Singer, *Linear functionals on spaces of continuous mappings of a compact Hausdorff space*, Rev. Math. Pures et Appl. 2 (1957), p. 309-314 (in Russian).
[36] L. Tzafriri, *Remarks on contractive projections in L^p-spaces*, Israel J. Math. 7 (1969), p. 9-15.
[37] J. J. Uhl, Jr., *Orlicz spaces of finitely additive set functions*, Studia Math. 29 (1967), p. 19-58.
[38] A. C. Zaanen, *The Radon-Nikodým theorem*, Nederl. Akad. Wetensch. Proc. Ser. A (= Indag. Math.) 64 (1961), p. 157-187.
[39] — *Integration* (2nd edition), Amsterdam 1967.

CARNEGIE-MELLON UNIVERSITY
PITTSBURGH, PENNSYLVANIA

STUDIA MATHEMATICA, T. XXXVIII. (1970)
Colloquium on
Nuclear Spaces and Ideals in Operator Algebras

Biprojective tensor products and convolutions of vector-valued measures on a compact group

by

M. DUCHOŇ (Bratislava)

Introduction. If μ and ν are regular complex-valued Borel measures on a compact Hausdorff group G, then the convolution of μ and ν can be defined by appealing to the Riesz representation theorem and letting $\mu*\nu$ be that unique regular Borel measure on G for which

$$\int_G f(z)\,d\mu*\nu(z) = \int_G \Big\{\int_G f(xy)\,d\mu(x)\Big\}\,d\nu(y)$$

holds for all continuous functions f on G. Moreover, if $\mathscr{M}(G)$ denotes the set of all complex-valued countably additive, regular Borel measures on G, then $\mathscr{M}(G)$ may be made a Banach space if we define linear operations pointwise and the norm as $\|\mu\| = |\mu|(G)$ (total variation of μ). Further $\mathscr{M}(G)$ with convolution multiplication is a Banach algebra (cf. [16], [10], [13]).

In this paper similar questions are dealt with for vector-valued measures. In the Banach space $\mathrm{rca}(\mathscr{B}(G), X)$ of all regular countably additive Borel measures with values in a Banach space X a convolution multiplication is introduced which is a bounded bilinear mapping from $\mathrm{rca}(\mathscr{B}(G), X)\times\mathrm{rca}(\mathscr{B}(G), X)$ into $\mathrm{rca}(\mathscr{B}(G\times G), X\check{\otimes} X)$, where $X\check{\otimes} X$ is the biprojective tensor product of X by X. Some properties of the convolution are given, further results will be given elsewhere.

1. Preliminaries. We need a generalization, for vector-valued measures, of the classical theorem asserting the existence of the product of measures defined on two measurable spaces. This is established in such a way that usual product of two scalars is replaced by the tensor product of two vectors. Namely, let measurable spaces $(S,\mathscr{S})$ and $(T,\mathscr{T})$, complete locally convex topological vector spaces X and Y, and (countably additive) vector-valued measures $\mu\colon\mathscr{S}\to X$ and $\nu\colon\mathscr{T}\to Y$ be given. We denote by $\mathscr{S}\otimes_\sigma\mathscr{T}$ the σ-ring generated by the sets of the form $E\times F$, $E\in\mathscr{S}$,