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Colloquium on
Nuclear Spaces and Ideals in Operator Algebras

On topological algebra sheaves of a nuclear type

by

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Introduction. We are concerned in the sequel with topological sheaves, whose fibers are topological algebras and in particular topological tensor product algebras, our main objective herein being the sheaf-theoretic version of the basic relation connecting the spectrum of a topological tensor product algebra to the spectra of the factor algebras (cf. Theorem 3.1 below). An analogous result into the same context, referring in particular to topological sheaves of Stein algebras was previously considered in [12]. Similar considerations in the recent literature have been developed in [2], [3] and [15].

Detailed proofs and further developments as well as applications to certain analytic function algebra sheaves will be given in subsequent publications.

1. Preliminaries and definitions. Let X be a (Hausdorff) topological space and let $\mathscr{E}$ be a *sheaf of* (complex linear associative) *algebras* on X.

We say that $\mathscr{E}$ is a *topological algebra sheaf* if the following conditions are satisfied:

(1.1) For every open set $U \subseteq X$ the set $\Gamma(U, \mathscr{E})$ of (continuous local) sections of $\mathscr{E}$ over U is a topological algebra ([13], p. 6).

(1.2) For any open subsets U, V of X with $U \subseteq V$, the restriction map $\varphi_{UV}\colon \Gamma(V, \mathscr{E}) \to \Gamma(U, \mathscr{E})$ is a continuous (algebra) homomorphism between the respective topological algebras.

(1.3) For every open set $U \subseteq X$ and for every open covering $(U_a)_{a\in I}$ of U, the algebra $\Gamma(U, \mathscr{E})$ has the corresponding projective limit (initial) topology with respect to the topological algebras $\Gamma(U_a, \mathscr{E})$ and the algebra homomorphisms $\varphi_{U_a U}$, $a \in I$, making it a topological algebra.

We remark that for Fréchet sheaves based on a second countable topological space, the preceding condition (1.3) is implied by the first two (cf., for instance, [1] p. 324, § 6). We have this case below by the consideration of Fréchet (locally convex and, in particular, locally m-convex topological) algebra sheaves.

Now, let $\mathscr{E}$ be a topological algebra sheaf on X. The stalk $\mathscr{E}_x$ of $\mathscr{E}$ at $x \in X$ is given, as a topological algebra, by

$$\mathscr{E}_x = \varinjlim \Gamma(U, \mathscr{E}), \tag{1.4}$$

where U ranges over a fundamental system of open neighborhoods of x, and $\mathscr{E}_x$ is equipped with the inductive limit vector space topology, such that $\mathscr{E}_x$ is a topological algebra.

In case the topological algebras involved in the sheaf $\mathscr{E}$ are locally convex with separately, respectively jointly, continuous multiplication, the corresponding inductive limit locally convex vector space topology on $\mathscr{E}_x$, $x \in X$, makes it a locally convex algebra of the same kind with those of the sheaf $\mathscr{E}$. This is not in general the case if the topological algebras considered are locally m-convex [13] in which case one considers on $\mathscr{E}_x$ the inductive limit locally m-convex topology with respect to the locally m-convex algebras $\Gamma(U, \mathscr{E})$ with U varying as in relation (1.4) above.

If the algebras involved have identity elements, then regarding the spectra $\mathfrak{M}(\Gamma(U, \mathscr{E}))$ of these algebras [8], one gets a projective system of topological spaces ([11], p. 128, Prop. 2.2), so that the spectrum of the algebra (1.4) is given by

$$\mathfrak{M}(\mathscr{E}_x) = \varprojlim \mathfrak{M}(\Gamma(U, \mathscr{E})), \tag{1.5}$$

within a homeomorphism, where U varies as in (1.4).

More generally, if K is an arbitrary subset of X and U ranges over an open neighborhood basis of K, one defines the algebra of germs of $\mathscr{E}$ over K as a topological inductive limit algebra by

$$\mathscr{E}_K = \varinjlim \Gamma(U, \mathscr{E}), \tag{1.6}$$

the spectrum of which is given, within a homeomorphism, by

$$\mathfrak{M}(\mathscr{E}_K) = \varprojlim \mathfrak{M}(\Gamma(U, \mathscr{E})). \tag{1.7}$$

Example. Suppose that X is a Stein manifold and let S be an arbitrary subset of X. Then, the "envelope of holomorphy" of S is the spectrum of the locally m-convex algebra $\mathscr{O}(S)$, the algebra of germs of holomorphic functions on S, which is obtained by the specialization of (1.7) above to the case under consideration (cf. also [6], p. 510, rel. (2.1)).

Now, given the topological algebra sheaf $\mathscr{E}$ on a Hausdorff topological space X, we define the *spectrum of* $\mathscr{E}$ by the relation

$$\mathfrak{M}(\mathscr{E}) = \bigcup_{x \in X} \mathfrak{M}(\mathscr{E}_x), \tag{1.8}$$

disjoint union of the respective topological spaces (defined by relation (1.5) above) equipped with the direct sum topology.

On the other hand, a topological algebra sheaf $\mathscr{E}$ is said to be *of a nuclear type* (or a *nuclear topological algebra sheaf*) if, for every open set $U \subseteq X$, the algebra $\Gamma(U, \mathscr{E})$ is a locally convex nuclear topological algebra (cf. also [9], p. 49).

We remark that it is equivalent to suppose in the preceding definition that the algebra $\Gamma(U, \mathscr{E})$ is nuclear, for every set U of an open basis of X: This is an easy consequence of the definition given above and condition (1.3) in the definition of a topological algebra sheaf.

Example. Suppose that $(X, \mathscr{O})$ is a reduced (second countable) complex space. Then, its structure sheaf $\mathscr{O}$ is a topological algebra sheaf of a Fréchet nuclear type, in the sence that $\Gamma(U, \mathscr{O})$ is a Fréchet locally m-convex algebra, which is also a nuclear space (cf. also [1], p. 327, Prop. 8.1).

Now, let $\mathscr{E}$ be a topological vector space sheaf on a Hausdorff topological space X ([5], p. 3) and let $\mathscr{E}'$ be the *topologically dual vector space precosheaf* on X, corresponding to $\mathscr{E}$: That is, we are given a covariant functor from the category of the open subsets of X belonging to a basis of its topology and inclusions to the category of vector spaces and linear maps, given by

$$U \to \mathscr{E}'(U) = (\Gamma(U, \mathscr{E}))', \tag{1.9}$$

U being an open subset of X as above, the last space being the topological dual of the topological vector space $\Gamma(U, \mathscr{E})$ in such a way that, for any open sets U, V in X as above with $U \subseteq V$, one has

$$\theta_V^U = {}^t r_U^V : (\Gamma(U, \mathscr{E}))' \to (\Gamma(V, \mathscr{E}))', \tag{1.10}$$

that is the transpose to the respective restriction map r_U^V(cf. also [7], p. 166).

Under the preceding circumstances, we say that $\mathscr{E}'$ is *weakly flabby* in case

$$(\Gamma(U, \mathscr{E}))' \xrightarrow{\theta_U = {}^t r_U} (\Gamma(X, \mathscr{E}))' \to 0 \tag{1.11}$$

is an exact sequence for every open set $U \subseteq X$.

Example. Let $(X, \mathscr{O})$ be a Stein space and let $\mathscr{F}$ be a coherent analytic sheaf on X. Then, the canonical Fréchet space structure on $\mathscr{F}$ defines a topologically dual weakly flabby precosheaf $\mathscr{F}'$, the corresponding by the foregoing dual precosheaf.

In connection with the preceding, we also remark that it amounts to the same of being a sheaf flabby or weakly flabby.

2. Tensor products. Let $\mathscr{E}$ and $\mathscr{F}$ be sheaves of locally convex algebras with continuous multiplication on a Hausdorff topological space X. Then,

$$\Gamma(U, \mathscr{E}) \hat{\otimes}_\pi \Gamma(U, \mathscr{F}) \tag{2.1}$$

is a complete locally convex algebra with continuous multiplication, for every open set U in X, where π denotes the projective tensorial topology ([8], p. 176, Prop. 3.2).

Now, if U varies over an open basis of the topology of X, then (2.1) defines a presheaf of topological algebras on X, so that we denote by $\mathscr{E}\pi\mathscr{F}$ the corresponding topological algebra sheaf generated by the presheaf in question.

On the other hand, suppose that one of the sheaves $\mathscr{E}, \mathscr{F}$ is of a nuclear type and moreover either one determines a topologically dual weakly flabby precosheaf. Then *for every open subset U of X, one has*

$$\Gamma(U, \mathscr{E}\pi\mathscr{F}) = \Gamma(U, \mathscr{E})\hat{\otimes}_{\pi}\Gamma(U, \mathscr{F}), \tag{2.2}$$

within a bijection.

As a consequence of relation (2.2) above, we are now in the position to state the following

THEOREM 2.1. *Let $\mathscr{E}$ and $\mathscr{F}$ be topological algebra sheaves on a Hausdorff second countable topological space X such that the corresponding local sections define Fréchet locally convex (topological) algebras. Moreover, suppose that one of $\mathscr{E}, \mathscr{F}$ is a nuclear topological algebra sheaf and either one of them defines a topologically dual weakly flabby precosheaf. Then, $\mathscr{E}\pi\mathscr{F}$ is a topological algebra sheaf, if for every open set $U \subseteq X$, the bijection defined by* (2.2) *is considered as a topological (algebraic) isomorphism, such that the corresponding local sections of $\mathscr{E}\pi\mathscr{F}$ constitute Fréchet locally convex algebras.*

The sheaf $\mathscr{E}\pi\mathscr{F}$ defined by the preceding theorem will also be denoted by $\mathscr{E}\varepsilon\mathscr{F}$, where ε denotes the biprojective tensorial topology [4], which coincides with the ε-tensor product of L. Schwartz ([14], p. 47, Corol. 1; cf. also [1], p. 328, Def. 9.1).

3. The spectrum of a topological tensor product algebra sheaf. Let $\mathscr{E}$ and $\mathscr{F}$ be topological algebra sheaves on a Hausdorff topological space X, such that the conditions of Th. 2.1 above are satisfied. Moreover, let $\mathscr{E}\pi\mathscr{F} = \mathscr{E}\varepsilon\mathscr{F}$ (cf. the comments at the end of the preceding section) the topological algebra sheaf on X, given by the same theorem.

Now, the stalk of $\mathscr{E}\pi\mathscr{F}$ at $x \in X$ is given by

$$(\mathscr{E}\pi\mathscr{F})_x = \varinjlim \Gamma(U, \mathscr{E}\pi\mathscr{F}), \tag{3.1}$$

with U varying over an open neighborhood basis of x (cf. also (1.4) in the foregoing). Therefore, by Th. 2.1, $(\mathscr{E}\pi\mathscr{F})_x$ is a locally convex topological algebra with continuous multiplication. Furthermore, suppose that $\mathscr{E}$ and $\mathscr{F}$ are topological algebra sheaves with identity elements, in the sense that, for every open set $U \subset X$, the corresponding local sections of $\mathscr{E}, \mathscr{F}$ define topological algebras with identity elements. Hence, the same is true concerning the (topological) algebras defined by (2.2) and (3.1) above. Therefore, the spectrum of the algebras (3.1) is given by

$$\mathfrak{M}((\mathscr{E}\pi\mathscr{F})_x) = \varprojlim \mathfrak{M}(\Gamma(U, \mathscr{E}\pi\mathscr{F})) = \varprojlim \mathfrak{M}(\Gamma(U, \mathscr{E})\hat{\otimes}_{\pi}\Gamma(U, \mathscr{F})), \tag{3.2}$$

within a homeomorphism.

On the other hand, suppose that the topological algebras involved have locally equicontinuous spectra [12]. Then, by [10], p. 104, Th. 2.1, the preceding relation yields

$$\mathfrak{M}((\mathscr{E}\pi\mathscr{F})_x) = \mathfrak{M}(\mathscr{E}_x) \times \mathfrak{M}(\mathscr{F}_x), \tag{3.3}$$

within a homeomorphism, for every $x \in X$.

By the preceding, we now have the following

THEOREM 3.1. *Let $\mathscr{E}$ and $\mathscr{F}$ be topological algebra sheaves on a Hausdorff topological space X such that the conditions of Theorem* 2.1 *are satisfied and, for every set U of an open basis of X, the corresponding local section algebras have identity elements and locally equicontinuous spectra. Then $\mathscr{E}\pi\mathscr{F} = \mathscr{E}\varepsilon\mathscr{F}$ is a topological algebra sheaf, such that the local sections define locally convex algebras with continuous multiplication, having identity elements and locally equicontinuous spectra. Moreover, the spectrum of $\mathscr{E}\pi\mathscr{F}$* (cf. § 1) *is given by*

$$\mathfrak{M}(\mathscr{E}\pi\mathscr{F}) = \mathfrak{M}(\mathscr{E}) \times \mathfrak{M}(\mathscr{F}), \tag{3.4}$$

within a homeomorphism.

The preceding statement may be considered as the sheaf-theoretic version of previous results on the same subject (cf., for instance, [8], [10] and [12]). A similar formula to (3.4) above has also been given in [12] referring, in particular, to Fréchet sheaves of Stein algebras. A detailed proof of the preceding result as well as further considerations will be reported elsewhere.

Added in proof. A more natural concept of the spectrum of a topological algebra sheaf as well as a corresponding extended form of Theorems 2.1 and 3.1 in the preceding are given in a subsequent paper.

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Bases in sequentially retractive limit-spaces*

by

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The object of the present paper is to generalize the classical Banach-theorem for the continuity of coefficient functionals of bases in Banach-spaces to some inductive limit-spaces. A suitable class for the treatment is the class of limit-spaces $E = \operatorname{ind}_{n\to} E_n$, for which the convergence of sequences takes place already in some generating space E_n (E_n Fréchet-spaces). These spaces seem to be interesting for other questions too, because they include two of the three most important cases: strict (LF)-spaces and (LS)-spaces (i.e. (LF)-spaces with compact linking mappings), but *not* (LS_w)-spaces (i.e. (LF)-spaces with weakly compact linking mappings; in particular those with generating reflexive normed spaces). The terminology is essentially that of [3].

1. Sequentially retractive sequences.

1.1. A sequence $E_1 \subset E_2 \subset \ldots$ of (F)-spaces E_n is called *sequentially rectractive* if for every sequence (x_i) converging in the (locally convex) inductive limit $E = \operatorname{ind}_{n\to} E_n$ there is an index n such that (x_i) converges in E_n. Furthermore, the limits of (x_i) coincide.

1.2. Sequentially retractive generated (LF)-spaces are separated and sequentially complete.

If an (LF)-space E is twice generated

$$E = \operatorname{ind}_{m\to} E_m = \operatorname{ind}_{n\to} F_n, \qquad E_m, F_n \ (F)\text{-spaces},$$

then by a theorem of *Grothendieck* ([5], I, p. 17) these sequences are mutually cofinal (i.e. for every n there is an m such that $E_n \hookrightarrow F_m$ and $F_n \hookrightarrow E_m$; "$\hookrightarrow$" means continuous embedding). Thus, if one generating sequence of an (LF)-space is sequentially retractive then all are, and one can speak of *sequentially retractive (LF)-spaces*.

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