infer that $vu \in \Pi_2(E, l_2)$ for each $v \in B(l_\infty(A), l_2)$. Thus (ii) implies that

$$(1) \qquad u^*v^* \in \Pi_2(l_2, E^*) \quad \text{for every } v \in B(l_\infty(A), l_2).$$

Now pick for $i = 1, 2, \ldots, x_i^* \in (l_\infty(A))^*$ so that $\sum |x^{**}(x_2{}^*)|^2 < \|x^{**}\|^2$ for every x^{**} in the second dual $(l_\infty(A))^*$. Define $v \in B(l_\infty(A), l_2)$ by $vf = (x_i^*(f))$ for $f \in l_\infty(A)$ and denote by d_i^* the i-th coordinate functional in l_2. Clearly, $\sum |d_i^*(d)|^2 = \|d\|^2$ for every $d \in l_2 = (l_2)^{**}$. Thus (1) implies that

$$\sum \|u^*v^*d_i^*\|^2 < C(u^*v^*).$$

Hence $\Sigma \|u^*x_i^*\|^2 < C(u^*v^*)$, because $v^*d_i^* = x_i^*$ for $i = 1, 2, \ldots$ Therefore $u^* \in \Pi_2((l_\infty(A))^*, E^*)$. Hence, by the Pietsch Factorization Theorem (cf., [3], p. 285), u^* is a hilbertian operator. Thus, by [3], Proposition 5.1, u is hilbertian. Since u is an isometrically isomorphic embedding of E and u is hilbertian, E is isomorphic to an inner product space (because the Banach space $u(E)$ is the range of a bounded linear operator from a Hilbert space). This completes the proof.

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STUDIA MATHEMATICA, T. XXXVIII. (1970)

Colloquium on
Nuclear Spaces and Ideals in Operator Algebras

On a class of operators in Hilbert space*

by

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0. Introduction. One version of the spectral theorem for Hermitian and normal operators in Hilbert space is a consequence of the Gelfand representation of the uniform closure R_T of the algebra generated by the normal operator T, its adjoint and the identity. The essential fact is that R_T is commutative and isometrically $*$-isomorphic to the algebra $C(X)$ of all complex-valued continuous functions on a compact Hausdorff space ([4]-[6]).

On the other hand, If T is *any* operator on *any* Hilbert space H, then R_T is isometrically $*$-isomorphic to *some* algebra $C(X, A)$ of all continuous A-valued functions on X, where X is a compact Hausdorff space and A is a C^*-algebra. Indeed, we may use for A the algebra R_T and for X any one-point space $\{x\}$. Clearly, what is desirable for any extension of spectral theory is the choice of a canonical or minimal algebra A and of a usefully simple topological space X so that the isometric $*$-isomorphism $R_T \overset{*}{\cong} C(X, A)$ permits some analysis of T.

To pursue these ideas the author has discussed various aspects of a natural and fruitful generalization of the notion of commutative Banach algebra ([4]-[6]). Indeed, since a commutative Banach algebra A is one such that all its quotients by regular maximal ideals are isomorphic (to C, the field of complex numbers), the generalization in question is a so-called Q-uniform Banach algebra defined as follows:

An algebra A is a *Q-uniform algebra* if:

a. Q is a simple Banach algebra with identity;

b. A is a Q-bimodule such that for $a_1, a_2 \in A$, $q_1, q_2 \in Q$,

$$(q_1a_1)q_2 = q_1(a_1q_2), \quad (q_1a_1)a_2 = q_1(a_1a_2), \quad (a_1a_2)q_1 = a_1(a_2q_1),$$
$$(q_1q_2)a_1 = q_1(q_2a_1), \quad a_1(q_1q_2) = (a_1q_1)q_2, \quad |a_1q_1|, |q_1a_1| \leqslant |a_1|\,|q_1|,$$

and where the left and right actions of Q on A are unitary;

* This research was supported in part by NSF Grant GP-13288 for which the author is grateful.

c. The quotient of A by any regular maximal ideal is isomorphic to Q.

A C^*-algebra A is *Q-*uniform* if it is Q-uniform, where Q is a C^*-algebra and if:

c′. The quotient of A by any regular maximal ideal is *-isomorphic to Q.

One aspect of a Q-uniform algebra A is that it is canonically at least homomorphic and sometimes isomorphic to a subset of $C(\mathrm{Epi}_C(A, Q), Q)$, where $\mathrm{Epi}_C(A, Q)$ is the set of continuous C-epimorphisms $\eta: A \to Q$ [3]. (When we speak of *-Q-uniform algebras we use $\mathrm{Epi}_C^*(A, Q)$, the set of continuous C-*-epimorphisms $\eta: A \to Q$.) The circumstances in which the canonical homomorphism $A \to C(\mathrm{Epi}_C(A, Q), Q)$ is an isomorphism or an isometric isomorphism or an isometric *-isomorphism require detailed study. Alternatively one can seek a topological space X such that A and $C_b(X, Q)$ are isometrically *-isomorphic.

For any simple Banach algebra Q and any compact Hausdorff space X the algebra $C(X, Q)$ is Q-uniform [3]. If Q is a C^*-algebra then $C(X, Q)$ is *-Q-uniform. With this in mind we consider as the natural generalization of a normal operator in a Hilbert space an operator T such that R_T is *-Q-uniform for some simple C^*-algebra Q and more particularly is isometrically *-isomorphic to $C_b(X, Q)$, with a suitable involution * and where X is a compact Hausdorff space. If $R_T \overset{*}{\cong} C_b(X, A)$, where X is completely regular and Q is a simple C^*-algebra with identity, then the following facts concerning X and Q are easily verified:

(i) X is separable and in $C_b(X, Q)$ there is a separating function γ_T that is never zero and such that $C_b(X, Q) = R_{\gamma_T}$. (For any C^*-algebra A and element a of A, R_a denotes the closed ring generated in the uniform topology by the identity, a and a^*. We say that A is *singly generated* if for some a, $A = R_a$.)

(ii) Q is a separable singly generated C^*-algebra.

A kind of converse situation obtains in that if there are given a topological space X and a simple C^*-algebra, then for some suitably chosen Hilbert space H and some suitably chosen operator T in $L(H)$ the relationship $R_T \overset{*}{\cong} C_b(X, Q)$ obtains. Furthermore, it can be established rather easily that when $R_T \overset{*}{\cong} C(X, Q)$ and X is compact Hausdorff, the operator T determines the algebra Q uniquely. Thus, the study of operators T of the kind described is intimately related to the study of algebras Q as described in (ii) above.

Simple examples of algebras Q satisfying (ii) above are the algebras $\mathrm{End}_C(C^n)$ consisting of all linear endomorphisms of complex n-dimensional space. In this case $C(X, Q)$, as P. Halmos has remarked, is an instance of what might be termed an n-normal ring of Arlen Brown [1].

1. Singly generated C^*-algebras. Let A be a Q-uniform algebra with continuous involution * and let M be a regular maximal ideal of A. Then

(a) *M^* is a regular maximal ideal.*

Proof. M^* is certainly a regular ideal. If M^* is not maximal, let $M^* \subsetneq N$, a maximal ideal. Then $M^{**} \subsetneq N^*$, a contradiction.

*If A is *-Q-uniform, then $M = M^*$.*

Proof. Let $M = \ker(\eta)$, $\eta \in \mathrm{Epi}_C^*(A, Q)$. Then $x \in M$ iff $\eta(x) = 0$, iff $\eta(x)^* = \eta(x^*) = 0$ iff $x^* \in M$ iff $x \in M^*$.

(b) *Q has a continuous involution.*

Proof. Let $\eta \in \mathrm{Epi}_C(A, Q)$. For $q \in Q$, let $\eta(a) = q$, $\ker(\eta) = M$. Fix η^* so that $\ker(\eta^*) = M^*$ and let $q^* = \eta^*(a^*)$. Then if $\eta(a_1) = q$, $a_1 - a \in M$, $a_1^* - a^* \in M^*$ and $\eta^*(a_1^*) = \eta^*(a^*)$, whence q^* is uniquely defined. For fixed η and η^*, let $q_n \to q$. By the open mapping theorem there is a sequence $a_n \to a$, where $\eta(a_n) = q_n$, $\eta(a) = q$. Then $a_n^* \to a^*$ and thus $\eta^*(a_n^*) = q_n^* \to q^* = \eta^*(a^*)$. Thus * as defined for Q is continuous.

(c) *Let A be singly generated, i.e., $A = R_a$ for some a. Then Q is singly generated.*

Proof. Let $\eta(a) = q$ and let $q_1 \in Q$. Let polynomials $p_n(e, a, a^*) \to a_1$, where $\eta(a_1) = q_1$. Then $p_n(e, q, q^*) \to q_1$, as the following shows. The map $\theta: p_n(e, a, a^*) \to p_n(e, q, q^*)$ is well-defined on the dense subset consisting of polynomials. If $|\eta|, |\eta^*| \leqslant K$, then $|\theta| \leqslant K_0^2$, where $K_0 = \max(K, 1)$. Thus θ is uniformly continuous and uniquely extendable to all A as a continuous homomorphism.

If for some Hilbert space H and $T \in L(H)$ and some Banach algebra A with identity there obtains the formula $R_T \overset{*}{\cong} C_b(X, A)$, where X is a completely regular topological space, then

(d) *X and A are separable.*

Proof. Since R_T is separable, so is $C_b(X, A)$, which contains the subset consisting of constant functions, a subset isometrically *-isomorphic to A. Thus A is separable. If $\{U_n\}$ is a countable base for C and if $\{f_n\}$ is a countable dense subset of the subset $C_b(X, A)$, then $\{W_{m,n} \equiv f_n^{-1}(U_m)\}$ is a countable base for X. Indeed, if V is open in X, let $x \in V$, $f(x) = 1$, $f = 0$ off V, $f \in C_b(x)$, and let $|f - f_{n_0}| < \frac{1}{3}$. Let $U_{m_0} \ni f_{n_0}(x)$, $\operatorname{diam} U_{m_0} < \frac{1}{3}$. Then for $f_{n_0}(y) \in U_{m_0}$, i.e., $y \in W_{m_0,n_0}$ we find

$$\begin{aligned} |f(y)-1| &= |f(y)-f(x)| \\ &= |f(y)-f_{n_0}(y)| + |f_{n_0}(y)-f_{n_0}(x)| + |f_{n_0}(x)-f(x)| \\ &< \tfrac{1}{3}+\tfrac{1}{3}+\tfrac{1}{3} = 1. \end{aligned}$$

Thus $f(y) \neq 0$, $y \in V$ and $W_{m_0,n_0} \subset V$.

(e) *$C_b(X, A)$ contains a separating function γ that is never 0 and $C_b(X, A) \doteq R_\gamma$.*

Proof. Let $T \to \gamma_T \in C_b(X, A)$. Then if $\gamma_T(x_1) = \gamma_T(x_2)$, we find that, since $R_\gamma \overset{*}{\cong} C_b(X, A)$, for all $f \in C_b(X, A)$, $f(x_1) = f(x_2)$, whereas $C_b(X)$ is a separating family in $C_b(X, A)$. Thus γ_T is separating in $C_b(X, A)$ and since γ_T is bounded, for some $q_0 \in q$, $q_0 e + \gamma_T$ is never 0 and separating. Clearly, $C_b(X, A) = R_{\gamma_T}$.

(f) *A is a C^*-algebra.*

Proof. A may be regarded as isometrically and $*$-isomorphically embedded as the set of constant functions in $C_b(X, A)$. Thus A may be regarded as a closed $*$-subalgebra of $C_b(X, A)$, hence as a closed $*$-subalgebra of R_T, hence as a C^*-algebra.

(g) *If X is compact Hausdorff and if Q is simple, then Q is unique.*

Proof. If $R_T \overset{*}{\cong} C(X_1, Q_1) \overset{*}{\cong} C(X_2, Q_2)$, where X_1 and X_2 are compact Hausdorff and Q_1 and Q_2 are simple, then R_T is $*$-Q_i-uniform, $i = 1, 2$, whence Q_1 and Q_2 are $*$-isomorphic.

Note that in consequence of (g) the study of operators T for which $R_T \overset{*}{\cong} C(X, Q)$, X compact, Q simple may be implemented by the study of separable, (d) singly generated, (c) C^*-algebras. This situation is discussed further in [3].

2. $C_b(X, A) \overset{*}{\cong} R_T$. Let X be a topological space. On X define discrete measure μ by $\mu(S) = \mathrm{card}(S)$ if the S is finite; otherwise $\mu(S) = \infty$. Let $C_b(X, A) \equiv B$ be the set of bounded continuous A-valued functions on A, where A is a C^*-algebra over a Hilbert space H_A. Then H_A allows the construction of a new Hilbert space $L^2(X, H_A, \mu) \equiv H$.

(a) H is a B-module (via multiplication) as follows: if $f \in H$, $\gamma \in B$, then $(\gamma \cdot f)(x) = \gamma(x) f(x) \equiv T_\gamma f$. The map $\gamma \to T_\gamma$ is a $*$-homomorphism.

(b) Clearly, $|T_\gamma| \leqslant |\gamma|$. In fact, $|T_\gamma| = |\gamma|$.

Proof. If $|T_\gamma| < |\gamma|$ for some γ, let $|\gamma(x_0)| > (1-\delta)|\gamma|$, $0 \neq f = \chi_{x_0} \cdot \varphi$, where φ is such that $|\gamma(x_0)\varphi| > (1-\delta)|\gamma|\ |\varphi|$ and $|T_\gamma| < (1-\varepsilon)|\gamma|$, where $0 < \varepsilon, \delta < 1$. Then

$$|T_\gamma f| = |\gamma(x_0)\varphi| > (1-\delta)|\gamma|\ |\varphi|,$$
$$|T_\gamma f| \leqslant |T_\gamma|\ |f| = |T_\gamma|\ |\varphi| < (1-\varepsilon)|\gamma|\ |\varphi|.$$

Thus $(1-\varepsilon)|\gamma|\ |\varphi| > (1-\delta)|\gamma|\ |\varphi|$ or $\varepsilon < \delta$.

Since $\delta > 0$ is arbitrary and $\varepsilon > 0$ is given, we have a contradiction. Hence $|T_\gamma| = |\gamma|$ and so the map $\gamma \to T_\gamma$ is an isometric $*$-isomorphism.

(c) B is a C^*-algebra: $|\gamma\gamma^*| = \sup_x |\gamma(x)\gamma^*(x)| = \sup_x |\gamma(x)|^2 = |\gamma|^2$.

(d) If there is a function $\gamma \in C_b(X, A)$ such that $C_b(X, A) = R_\gamma$, then $C_b(X,A) \overset{*}{\cong} R_{T_\gamma}$.

In summary:

If X is a topological space and if A is a C^*-algebra such that for some $\gamma \in C_b(X, A) \equiv B$ there obtains $B = R_\gamma$, then there is a Hilbert space H and an operator T_γ such that $C_b(X, A) \overset{*}{\cong} R_{T_\gamma}$.

3. Examples and problems. It is of interest to exemplify situations described in the summary of the preceding section.

If X is a topological space admitting a real continuous function $p(x)$ such that $C_b(X)$ is singly generated by $p(x)$ and if M_n is the C^*-algebra $\mathrm{End}_{\boldsymbol{C}}(\boldsymbol{C}^n)$ of $\boldsymbol{C}$-endomorphisms of $\boldsymbol{C}^n$, then $C_b(X, M_n)$ is singly generated by γ_p defined as follows:

$$\gamma_p(x) = \begin{pmatrix} 0 & p(x) & 0 & \dots & 0 \\ 0 & 0 & p(x) & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & p(x) \\ 0 & 0 & & \dots & 0 \end{pmatrix}.$$

Indeed, let $T_x = (t_{ij})$, where $t_{i,i+1} = x$ $(i = 1, 2, \dots, n-1)$, $t_{ij} = 0$ otherwise. Then $T_x^k T_x^{l*} = (r_{ij}^{kl})$, where $r_{i,i+m}^{kl} = x^{k+l}$ $(i = 1, 2, \dots, n-m)$, $r_{ij}^{kl} = 0$ otherwise if $k = l+m$ and $r_{i+m,i}^{kl} = x^{k+l}$ $(i = 1, 2, \dots, n-m)$, $r_{ij}^{kl} = 0$ otherwise if $l = k+m$. Similarly, $T_x^{l*} T_x^k = (s_{ij}^{lk})$, where $s_{i+l,\,i+k}^{lk} = x^{k+l}$ $(i = 1, 2, \dots, \min(n-k, n-l))$, $s_{ij}^{lk} = 0$ otherwise. Thus for any set $S \subset \boldsymbol{R}$ for which the Müntz theorem applies, e.g., S compact, we can by use of linear combinations, adjoints and uniform approximation techniques conclude that $R_{T_x} = C(S, M_n)$. In consequence, $C_b(X, M_n) = R_{\gamma_p}$.

A corollary to the results above is that any $C(X)$ may be viewed as a family of commuting normal operators on some Hilbert space H. Thus any $C(X, M_n)$ may be viewed as an algebra of n-normal operators on some Hilbert space H [1]. If $C(X)$ is singly generated, then, as shown above, $C(X, M_n)$ is also singly generated. Thus if for some operator T on a Hilbert space H the ring R_T is isometrically $*$-isomorphic to some $C(X, M_n)$, T may be regarded as an n-normal operator.

The author is not aware of any singly generated simple C^*-algebras other than M_n for all n. If in fact, there are no others then the only operators T such that $R_T \overset{*}{\cong} C(X, Q)$ for some simple C^*-algebra Q are the n-normal operators.

The theory is at present incomplete for a number of reasons. These are summarized in the following questions:

1. If X is completely regular and if Q is a simple Banach algebra is $C_b(X, Q)$ Q-uniform? (If X is compact, the answer is affirmative [3].)

2. In what circumstances is a Q-uniform Banach algebra A of the form $C(X, Q)$? Clearly, some restrictions are needed, e.g., when $Q = \boldsymbol{C}$. If R_T is Q-uniform, where T is an operator on some Hilbert space, is

(a) Q a C^*-algebra; (b) the map $R_T \to C(\mathrm{Epi}_C(R_T, Q), Q)$ one-to-one; (c) $R_T \stackrel{*}{\simeq} C_b(X, A)$ for some (compact) topological space X?

3. If Q is a simple singly-generated C^*-algebra, is there an n such that $Q = M_n$?

4. If Q is a C^*-algebra and $C(X_1, Q) \stackrel{*}{\simeq} C(X_2, Q)$, where X_1 and X_2 are compact Hausdorff, are X_1 and X_2 homeomorphic?

5. Are $C_b(X, A)$ and $C(\beta(X), A)$ isomorphic? (Here $\beta(X)$ is the Čech compactification of the completely regular space X.)

If the answers to 2 and 3 are affirmative, then the suggested generalization of a normal operator may be studied in the context of [1].

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STUDIA MATHEMATICA, T. XXXVIII. (1970)
Colloquium on
Nuclear Spaces and Ideals in Operator Algebras

Nukleare Funktionenräume und singuläre elliptische Differentialoperatoren

von

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Grothendieck hat die Frage aufgeworfen, ob jeder nukleare (F)-Raum eine Basis besitzt [6]. Dieses Problem ist zur Zeit ungelöst. Damit ist es von Interesse, für spezielle nukleare (F)-Räume die Existenz einer Basis nachzuweisen. Wie Mitjagin zeigen konnte [13], ist jede Basis eines nuklearen (F)-Raumes absolut. Neben der Frage nach der Existenz absoluter Basen in nuklearen (F)-Räumen ist die Isomorphie spezieller nuklearer (F)-Räume untereinander von Interesse. Wie T. und Y. Komura beweisen konnten [11] ist jeder nukleare Raum isomorph zu einem Teilraum des Tychonovproduktes $(s)^A$. Dabei ist A eine passende Indexmenge. s ist der Raum der schnell fallenden Folgen, also

$$s = \{\xi = (\xi_j)_{j=1,2,\ldots},\ \xi_j \text{ komplex}, \sup_{j=1,2,\ldots} |\xi_j| j^k < \infty \text{ für } k = 0, 1, 2, \ldots\}$$

mit der üblichen Topologie. Ist der nukleare Raum ein (F)-Raum, so kann man $A = \{1, 2, 3, \ldots\}$ setzen. Das folgt unmittelbar aus den Beweisen der Arbeit von T. und Y. Komura [11]. Für konkrete Räume dieser Art ist somit die explizite Bestimmung eines isomorphen Teilraumes von $(s)^A$ von Interesse. Sämtliche in dieser Arbeit untersuchten Räume sind isomorph zu s. Damit ist zugleich die Frage nach der Existenz absoluter Basen in den hier betrachteten Räumen positiv beantwortet.

Im Mittelpunkt der Arbeit stehen nukleare Funktionenräume und ihre Beziehungen zu singulären elliptischen Differentialoperatoren in Hilberträumen. Die Verwendung von Hilberträumen scheint zumindest plausibel zu sein, wenn man berücksichtigt, daß die Topologie eines nuklearen Raumes durch Hilberthalbnormen erzeugt werden kann [18], S. 71. Die Benutzung eines selbstadjungierten Operators A in einem Hilbertraum zur Konstruktion des lokalkonvexen Raumes

$$D(A^\infty) = \bigcap_{n=1}^{\infty} D(A^n)$$