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STUDIA MATHEMATICA, T. XXXVIII. (1970)
Colloquium on
Nuclear Spaces and Ideals in Operator Algebras

The tensor product of a locally pseudo-convex and a nuclear space

by

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The talk contained a relatively simple observation about the tensor product of a nuclear and a locally pseudo-convex space. It appears that nuclear locally convex spaces should play the same remarkable role in the category of locally pseudo-convex spaces as in the category of locally convex ones. The possibility that some, non-locally convex, but locally pseudo-convex spaces might rightly be called nuclear is not excluded, though it seems unlikely. S. Rolewicz [b] for instance mentions the fact that a locally pseudo-convex space which satisfies the "approximate dimension" condition for nuclearity is locally convex and nuclear.

Integrals of functions taking their values in locally pseudoconvex spaces were also discussed, along with applications to locally pseudo-convex algebra theory. The definition of the integrals, the relation of such integrals with topological tensor products, and applications, can be found in the literature and will not be further discussed here. (The reader may consult for example references [1] to [8]).

It does not seem that the starting point of this talk, i.e. the result about topological tensor products, has been published yet. The reader will find it here, along with a few corollaries, and counter-examples showing that the result cannot be generalized essentially.

1. Let $0 < p \leqslant 1$. A *p-semi-norm* on a (real or complex) vector space is a mapping $n\colon E \to R_+$ such that $n(x+y) \leqslant n(x)+n(y)$, $n(tx) = |t|^p n(x)$.

If $n(x) \neq 0$ for all $x \neq 0$, then n is a p-norm. We do not always wish to specify the exponent p, and speak then of a pseudo-seminorm or of a pseudo-norm. If $0 < q \leqslant 1$ and if n is a p-semi-norm, then n^q is a pq-semi-norm, which can be identified with n for all our purposes. Modulo this identification, the set of p-semi-norms is an increasing set as p decreases to zero.

A vector space topology can be associated to a set of pseudo-semi-norms $\{n_a\}_{a\in A}$. The sets

$$V_{a_1\dots a_k\varepsilon} = \{x \mid \forall\, i = 1, \dots, k\colon n_{a_i}(x) < \varepsilon\}$$

constitute a fundamental system of neighbourhoods of the orgin in this topology. Not all vector space topologies can be defined by pseudo-semi-norms in this way.

Definition. A topological vector space is *locally pseudo-convex* if its topology can be defined by a system of pseudo-semi-norms. It is locally *p-convex* if its topology can be defined by a system of p-semi-norms.

A topological vector space is therefore locally p-convex if its topology can be defined by pseudo-semi-norms n_a, of exponents $p_a \geqslant p$.

2. To simplify the statements below, we call a vector space topology on the tensor product of two topological vector spaces *admissible* if the tensor product mapping $(e_1, e_2) \to e_1 \otimes e_2$ is continuous.

Let now n_1, n_2 be p-semi-norms on vector spaces E_1, E_2. It is reasonable to call $n_1 \otimes_p n_2$ the *largest* p-semi-norm on $E_1 \otimes E_2$ which satisfies the inequality

$$n_1 \otimes n_2 (x \otimes y) \leqslant n_1(x) n_2(y).$$

If n_1 is a p_1-semi-norm on E_1, and n_2 is a p_2-semi-norm on E_2, if $r \leqslant p_1, r \leqslant p_2$, then $n_1^{r/p_1} \otimes_r n_2^{r/p_2}$ can be defined.

Observe that it is not clear that $n_1 \otimes_p n_2$ is a p-norm when n_1 and n_2 are, though the classical proof shows that this is the case when one of the p-normed spaces (E_1, n_1), (E_2, n_2) is separated by its dual. If $n_1 = n_1'^p$ where n_1' is a norm, we see that $n_1 \otimes_p n_2$ is a p-norm.

Let now E, F be locally pseudo-convex spaces, whose topologies are defined by pseudo-semi-norms $\{n_a\}_{a\in A}, \{m_\beta\}_{\beta\in B}$. Call p_a the exponent of n_a, q_β the exponent of m_β. The strongest admissible locally pseudo-convex topology on $E \otimes F$ is defined by the pseudo-semi-norms

$$n_a^{r/p_a} \otimes_r m_\beta^{r/q_\beta}$$

(with $r \leqslant \min(p_a, q_\beta)$).

3. Let E be a nuclear, locally convex space, E^* its dual with the equicontinuous boundedness. Let F be a locally pseudo-convex space and $\mathscr{L}_e(E^*, F)$ the space of linear mappings of E^* into F which map equicontinuous subsets on to bounded ones, topologized by uniform convergence on equicontinuous sets. This is a locally pseudo-convex space, and a complete space when F is complete.

We map $E \otimes F$ into $\mathscr{L}_e(E^*, F)$, mapping $e \otimes f$ on to the mapping $u \to \langle e, u\rangle f$, $E^* \to F$. It is clear that this is a continuous mapping. If we pull back to $E \otimes F$ the topology of $\mathscr{L}_e(E^*, F)$, we find an admissible topology on $E \otimes F$.

PROPOSITION. *This is the strongest admissible locally pseudo-convex topology on $E \otimes F$.*

The topology of $\mathscr{L}_e(E^*, F)$ is defined by pseudo-semi-norms n_B, where B ranges over a fundamental system of equicontinuous subsets of E, n over a fundamental system of continuous; pseudosemi-norms on F, and

$$n_B(u) = \sup\{n(ux) \mid x \in B\}.$$

A continuous semi-norm ν on E, a pseudo-semi-norm n_a of exponent q_a on F, and a real number r being given, we must find M, n', and B in such a way that

$$\nu^r \otimes_r n^{r/q} \leqslant M n'_B.$$

The construction is classical. We let $n' = n$ and choose $\nu' \geqslant \nu$ in such a way that the identity induces a map $E_{\nu'} \to E_\nu$ which can be represented by an element of $(E_\nu)^* \hat{\otimes}_r E_{\nu'}$. A proof, which is classical when $r = 1$, i.e. in the locally convex case, and still valid now, shows that we have the required inequality if B is the polar of the set of $x \in E$, such that $\nu(x) \leqslant 1$.

The result is hardly new, but does not seem to have been stated yet. Since E has the approximation property, the elements of finite rank are dense in $\mathscr{L}_e(E^*, F)$.

COROLLARY 1. *Let F be complete. $\mathscr{L}_e(E^*, F)$ is then the completed, locally pseudo-convex, projective tensor product of E and F.*

COROLLARY 2. *The strongest admissible locally pseudo-convex topology on the tensor product of a nuclear and a locally p-convex space is locally p-convex.*

COROLLARY 2'. *The strongest admissible locally pseudo-convex topology on the tensor product of a nuclear and a locally convex space is locally convex.*

Corollary 2 is obvious, since $\mathscr{L}_e(E^*, F)$ is then locally p-convex. Corollary 2' is the special case $p = 1$ of corollary 2.

4. It is worthwhile noting that the hypotheses have been used. Nuclearity of E, or some similar property is used in the proof of corollary 2. J. P. Kahane [a] has studied the spectral properties of rings $C(X_1) \hat{\otimes}_p \dots \hat{\otimes}_p C(X_r)$, with $rp > 1$. It follows easily from his results that $l_1 \hat{\otimes}_p l_1$ is not normable, and that $l_p \hat{\otimes}_{p'} l_p$ is not p-normable when $p' < p$ (observation by P. Turpin). The problem of whether the strongest admissible locally pseudoconvex topology on $l_1 \otimes l_1$ is locally p-convex for some p is still open, but it seems likely that the answer is no.

It is possible to find a nuclear space, a locally convex space, say E and F, and a non-locally convex admissible topology on $E \otimes F$. This topology cannot be locally pseudo-convex. The construction uses some unpublished results of the author, and will not be given here.

This is a good place to remember that the problem of whether the projective tensor product of two p-normed spaces is p-normed seems to be open. And in a similar spirit, the autor does not know whether an admissible Hausdorff topology can always be found on the tensor product of two Hausdorff topological vector spaces.

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Colloquium on
Nuclear Spaces and Ideals in Operator Algebras

Injektive Operatorenideale über der Gesamtheit aller Banachräume und ihre topologische Erzeugung

von

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1. Einführung in den Begriff des Operatorenideals. Für je zwei Banachräume E und F sei $\boldsymbol{L}(E,F)$ der lineare Raum aller linearen und stetigen Abbildungen von E in F, und mit $\boldsymbol{L}$ werde die Gesamtheit der linearen stetigen Abbildungen zwischen Banachräumen schlechthin bezeichnet. Im folgenden geht es um die Aussonderung spezieller Abbildungstypen aus der Gesamtheit $\boldsymbol{L}$. Diesem Zweck dient der *Begriff des Operatorenideals* (vgl. [5]), durch den in der Gesamtheit $\boldsymbol{L}$ eine Teilgesamtheit $\boldsymbol{A}$ axiomatisch festgelegt wird, und zwar in folgender Weise:

(A_1) Für je zwei Banachräume E und F ist $\boldsymbol{A}(E,F)$ ein linearer Teilraum von $\boldsymbol{L}(E,F)$. Dabei gibt es mindestens ein Paar $\tilde{E},\tilde{F}$, so daß $\boldsymbol{A}(\tilde{E},\tilde{F})$ eine Transformation $\tilde{T}_0$ mit $\tilde{T}_0 \neq 0$ enthält.

(A_2) (a) Aus $T \in \boldsymbol{A}(E,F)$ und $R \in \boldsymbol{L}(F,G)$ folgt $RT \in \boldsymbol{A}(E,G)$.
(b) Aus $T \in \boldsymbol{A}(F,G)$ und $R \in \boldsymbol{L}(E,F)$ folgt $TR \in \boldsymbol{A}(E,G)$.

Einen Operator T, der für ein Paar von Banachräumen E,F zur Klasse $\boldsymbol{A}(E,F)$ gehört, werden wir gelegentlich auch eine *Abbildung vom Typ* $\boldsymbol{A}$ nennen.

Es zeigt sich, daß für ein beliebiges Operatorenideal $\boldsymbol{A}$ und je zwei Banachräume E,F die Gesamtheit $\boldsymbol{L}_0(E,F)$ der ausgearteten Abbildungen von E in F in $\boldsymbol{A}(E,F)$ enthalten ist. Umgekehrt erfüllt das System $\boldsymbol{L}_0$ der ausgearteten Abbildungen bereits die Bedingungen (A_1) und (A_2), d.h. $\boldsymbol{L}_0$ kann als das kleinste Operatorenideal angesprochen werden.

Tritt zu den Forderungen (A_1) und (A_2) noch die Forderung nach einer Norm α auf jedem der linearen Räume $\boldsymbol{A}(E,F) \subset \boldsymbol{L}(E,F)$ hinzu und wird (A_2) durch die Aussagen

(N) (a) $\alpha(RT) \leqslant \|R\|\,\alpha(T)$ für $T \in \boldsymbol{A}(E,F)$ und $R \in \boldsymbol{L}(F,G)$,
(b) $\alpha(TR) \leqslant \|R\|\,\alpha(T)$ für $T \in \boldsymbol{A}(F,G)$ und $R \in \boldsymbol{L}(E,F)$

ergänzt, so soll das vorliegende Operatorenideal ein *normiertes Operatorenideal* oder kurz *Normideal* genannt werden. Die auf den einzelnen linearen