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On integro-differential equations of parabolic and elliptic type

by H. Ugowski (Gdańsk)

Krzyżański ([5], [6]) considered linear integro-differential equations of parabolic type and showed that the weak maximum principle known for a linear parabolic partial differential equation can be extended to the integro-differential equation. As a consequence he obtained a uniqueness theorem for solutions of the first Fourier problem for the integro-differential equation in an unbounded region. Some more general theorems concerning a system of non-linear integro-differential inequalities of parabolic type were proved by Łojczyk-Królikiewicz and Szarski ([8], [9]). Similar results for parabolic differential inequalities containing functionals can be found in reference [10].

In this paper we are dealing with the existence of solutions of the first initial-boundary value problem for the following system of parabolic integro-differential equations:

$$(0.1)\qquad \sum_{i,j=1}^{n} a_{ij}^k(x,t)u_{x_ix_j}^k+\sum_{i=1}^{n} b_i^k(x,t)u_{x_i}^k+c^k(x,t)u^k-u_t^k$$

$$=f^k\Big(x,t,u^1,\ldots,u^N,u_{x_1}^1,\ldots,u_{x_n}^1,\ldots,u_{x_1}^N,\ldots,u_{x_n}^N\int_{G_t}u^1(y,t)\mu^1(x,t;dy),\ldots$$

$$\ldots,\int_{G_t}u^N(y,t)\mu^N(x,t;dy)\Big),\qquad k=1,\ldots,N.$$

We take advantage of well-known theorems on existence and Schauder estimates of solutions of a single linear parabolic equation.

At first we consider a system more general than (0.1) containing some operators $B^k u$ on the right-hand side. For bounded regions we formulate two existence theorems. The proof of Theorem 1 is based on Schauder's fixed point theorem and is patterned on the proof given by A. Friedman for a single semilinear parabolic partial differential equation ([4], p. 204). Theorem 2 is proved by means of the successive approximations method, making use of the Banach fixed point theorem. The latter method, ensuring uniqueness as well, is also employed to solve the Cauchy problem for the system in question. The theorems mentioned above involve as a particular case system (0.1).

Some of the results obtained for parabolic equations are carried over to elliptic equations in section 6.

I wish to express my best thanks to P. Besala for suggesting the problem and giving me valuable hints.

1. Preliminary lemmas for the parabolic case in bounded domains. Let G be a bounded open domain of the Euclidean space E_{n+1} of the variables $(x, t) = (x_1, \ldots, x_n, t)$, enclosed by two domains R_0 and R_T lying on the planes $t = 0$ and $t = T = \text{const} > 0$ respectively, and by a manifold S situated in the strip $0 < t \leqslant T$. The parabolic distance of points $P(x, t), P'(x', t') \in E_{n+1}$ is defined as

$$d(P, P') = (|x-x'|^2 + |t-t'|)^{1/2}, \quad \text{where } |x-x'| = \Big[\sum_{i=1}^{n} (x_i - x_i')^2\Big]^{1/2}.$$

We shall make use of the definition of Hölder continuity with exponent a, $0 < a < 1$, included in [4].

Let us introduce the following norms:

$$|u|_0^G = \sup_{P \in G} |u(P)|, \qquad |u|_a^G = |u|_0^G + \sup_{P, P' \in G} \frac{|u(P) - u(P')|}{[d(P, P')]^a},$$

$$|u|_{1+a}^G = |u|_a^G + \sum_{i=1}^{n} |u_{x_i}|_a^G,$$

$$|u|_{2+a}^G = |u|_a^G + \sum_{i=1}^{n} |u_{x_i}|_a^G + \sum_{i,j=1}^{n} |u_{x_i x_j}|_a^G + |u_t|_a^G \qquad (0 < a < 1).$$

Denote by $C_{k+a}(G)$ $(k = 0, 1, 2)$ the set of all functions u for which $|u|_{k+a}^G < \infty$. The following norms will also be needed:

$$|u|_{1-0}^G = |u|_0^G + \sup_{P, P' \in G} \frac{|u(P) - u(P')|}{|x-x'| + |t-t'|}, \qquad |u|_{2-0}^G = |u|_{1-0}^G + \sum_{i=1}^{n} |u_{x_i}|_{1-0}^G.$$

The set of all functions u for which $|u|_{k-0}^G < \infty$ $(k = 1, 2)$ will be denoted by $C_{k-0}(G)$.

Suppose that for every point P of the closure $\bar{S}$ there exists an $(n+1)$-dimensional neighbourhood V such that $V \cap \bar{S}$ can be represented, for some i $(1 \leqslant i \leqslant n)$, by an equation of the form

$$x_i = h(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, t).$$

If the functions h belong to C_ω ($\omega = a,\ 1+a,\ 2+a,\ 1-0,\ 2-0$), then we say that S is of class C_ω. If $S \in C_{2+a}$ and the derivatives $h_{x_i t}$ exist and are continuous, then we say that S is of class $\bar{C}_{2+a}$; if, moreover, h_{tt} exist and are continuous, then S is said to belong to class $\bar{\bar{C}}_{2+a}$. For the manifold S of class C_ω there exist a finite number of balls V^k covering $\bar{S}$ such that $S^k = \bar{S} \cap V^k$ can be represented by the equation

$$x_{i_k} = h^k(x_1, \ldots, x_{i_k-1}, x_{i_k+1}, \ldots, x_n, t)\,, \tag{1.1}$$

where $h^k \in C_\omega$.

Let $v(x, t)$ be a function defined on the manifold S of class C_ω. Using (1.1) we can write the function $v(x, t)$ on S^k as a function of variables $(x_1, \ldots, x_{i_k-1}, x_{i_k+1}, \ldots, x_n, t)$ in a certain region S_0^k. We then define

$$|v|_\omega^S = \max_k |v|_\omega^{S_0^k}$$

and we say that $v \in C_\omega(S)$ if $|v|_\omega^S < \infty$.

A function $\varphi(x, t)$ defined on the parabolic boundary $\Sigma = \bar{R}_0 \cup S$ is said to be of class $C_{2+a}(G)$ if there exists a function $\Phi \in C_{2+a}(G)$ such that $\Phi = \varphi$ on Σ. We then define $|\varphi|_{2+a}^G = \inf\limits_{\Phi} |\Phi|_{2+a}^G$. If $S \in C_{2+a}$, then for any extension Φ of φ, the derivatives $\Phi_{x_i}, \Phi_{x_i x_j}, \Phi_t$ are uniquely defined (by continuity) on the boundary ∂R_0 of domain R_0, and the definition is independent of Φ. We denote these derivatives (on ∂R_0) by $\varphi_{x_i}, \varphi_{x_i x_j}, \varphi_t$ [1].

Let us consider the first initial-boundary value problem for the linear parabolic equation:

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x, t) u_{x_i x_j} + \sum_{i=1}^{n} b_i(x, t) u_{x_i} + c(x, t) u - u_t = f(x, t)\,, \qquad (x, t) \in \bar{G} \setminus \Sigma\,, \tag{1.2}$$

$$u(x, t) = \varphi(x, t)\,, \qquad (x, t) \in \Sigma\,. \tag{1.3}$$

By a solution of equation (1.2) we shall always understand a regular solution, i.e. continuous in the domain $\bar{G}$ and possessing in $\bar{G} \setminus \Sigma$ continuous derivatives appearing in Lu.

The following assumptions will be needed:

(A) For any $(x, t) \in \bar{G}$ and $\xi \in E_n$ we have $a_{ij}(x, t) = a_{ji}(x, t)$,

$$\sum_{i,j=1}^{n} a_{ij}(x, t) \xi_i \xi_j \geqslant K_0 |\xi|^2 \qquad (K_0 > 0)\,.$$

(B) The coefficients of L and the function $f(x, t)$ are uniformly Hölder continuous (exponent a) in $\bar{G}$.

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(1) All the definitions stated above are taken from [4] (pp. 61-65, 190).

Then there exists a constant $K_1 > 0$ such that

$$(1.4)\qquad |a_{ij}|_a^G\,,\quad |b_i|_a^G\,,\quad |c|_a^G \leqslant K_1\,.$$

(C) The coefficients a_{ij} are uniformly Hölder continuous (exponent a) in $\bar{G}$ and belong to $C_{1-0}(S)$; b_i and c are continuous in $\bar{G}$.

Thus for some constants $K_2, K_3 \geqslant 0$

$$(1.5)\qquad \sum_{i,j=1}^{n} |a_{ij}|_a^G + \sum_{i=1}^{n} |b_i|_0^G + |c|_0^G \leqslant K_2\,,\qquad \sum_{i,j=1}^{n} |a_{ij}|_{1-0}^S \leqslant K_3\,.$$

Now we state two lemmas included in [4].

LEMMA 1 ([4], p. 65). *Let assumptions* (A) *and* (B) *hold true. If* $\varphi \in C_{2+a}$, $S \in \bar{\bar{C}}_{2+a}$ *and* $L\varphi = f$ *on* ∂R_0, *then there exists a unique solution* u *of problem* (1.2), (1.3), *and furthermore* $u \in C_{2+a}(G)$. *Moreover, there exists a constant* K *depending only on* K_0, K_1, a *and domain* G *such that*

$$|u|_{2+a}^G \leqslant K(|\varphi|_{2+a}^G + |f|_a^G)\,.$$

LEMMA 2 ([4], p. 191). *Assume that* $S \in C_{2+a} \cap C_{2-0}$ *and that* (A), (C) *hold true. Let* $f(x,t)$ *be a continuous function in* $\bar{G}$ *vanishing on* ∂R_0 *and let* $u(x,t)$ *be a solution of the problem*

$$Lu = f(x,t) \quad \text{in } \bar{G}\setminus\Sigma\,,\qquad u = 0 \quad \text{on } \Sigma\,.$$

Then for any β, $0 < \beta < 1$, *there exists a constant* K' *depending only on* β, K_0, K_2, K_3 *and* G *such that*

$$|u|_{1+\beta}^G \leqslant K'|f|_0^G\,.$$

Remark. From the proof of this lemma follows the existence of a constant $\bar{K}$ depending on the same parameters as K' and such that

$$|u|_{1+\beta}^{G^\tau} \leqslant \bar{K}\tau^{(1-\beta)/2}|f|_0^{G^\tau}\,,\qquad \text{where } G^\tau = G \cap \{(x,t)\colon\ t < \tau\}\,,\ 0 < \tau \leqslant T\,.$$

Now we shall prove some lemmas on functions defined by an integral. Put $G_t = \{x\colon (x,t) \in \bar{G}\setminus\bar{S}\}$ and denote by m the n-dimensional Lebesgue measure.

LEMMA 3. *If the manifold* S *is of class* C_{2+a}, *then there exists a constant* $\varkappa$ *depending only on* S *and such that*

$$m(G_t\setminus G_{t'}) \leqslant \varkappa|t-t'|\,.$$

Proof. Denote by W^k any $(n+1)$-dimensional cube

$$\{\varepsilon_k \leqslant x_i \leqslant \delta_k\ (i = 1,\ldots,n)\,,\ \bar{\varepsilon}_k \leqslant t \leqslant \bar{\delta}_k\}\qquad (\delta_k - \varepsilon_k = \bar{\delta}_k - \bar{\varepsilon}_k\,,\ \bar{\varepsilon}_k \geqslant 0)\,.$$

The assumption of the lemma implies the existence of a finite number of cubes W^k $(k = 1, \ldots, k_1)$ with centres on $\bar{S}$, covering $\bar{S}$ and such that every manifold $S^k = W^k \cap \bar{S}$ can be represented by (1.1), where $h^k \in C_{2+\alpha}$. Hence, writing

$$S^k_{\tau\tau'} = S^k \cap \{(x, t)\colon \tau \leqslant t \leqslant \tau'\} \qquad (\tau < \tau')$$

and using the formula for the surface area, we immediately obtain

$$(1.6) \qquad m(S^k_{\tau\tau'}) \leqslant (\tau' - \tau)(\delta_k - \varepsilon_k)^{n-1} \cdot \sup \Big[1 + \sum_{\substack{i=1 \\ i \neq i_k}}^{n} (h^k_{x_i})^2 + (h^k_t)^2\Big]^{1/2} = \varkappa_k(\tau' - \tau)\,.$$

It is easy to see that the set $(G_\tau \cup G_{\tau'}) \backslash (G_\tau \cap G_{\tau'})$ is contained in the projection of the manifold

$$S_{\tau\tau'} = S \cap \{(x, t)\colon \tau \leqslant t \leqslant \tau'\}$$

on the plane $t = 0$. Since the n-dimensional Lebesgue measure of any n-dimensional manifold is not less than that of the projection of the manifold on a plane of the same dimension, therefore by (1.6) we have

$$m([G_\tau \cup G_{\tau'}] \backslash [G_\tau \cap G_{\tau'}]) \leqslant m(S_{\tau\tau'}) \leqslant m\Big(\sum_{k=1}^{k_1} S^k_{\tau\tau'}\Big) \leqslant \sum_{k=1}^{k_1} m(S^k_{\tau\tau'})$$

$$\leqslant (\tau' - \tau) \sum_{k=1}^{k_1} \varkappa_k = \varkappa(\tau' - \tau)$$

and the lemma follows.

Denote by $\mathcal{M}$ the σ-field of all Lebesgue-measurable subsets of the domain

$$D_0 = \overline{\bigcup_{0 \leqslant t \leqslant T} G_t}\,.$$

We shall make use of a non-negative measure $\mu(x, t; D)$ (depending on $(x, t) \in \bar{G}$) defined on $\mathcal{M}$ which satisfies the following conditions:

(1) there is a constant $M_1 > 0$ such that for any $(x, t) \in \bar{G}$

$$\mu(x, t; D_0) \leqslant M_1;$$

(2) there exists a finite non-negative measure $\bar{\mu}$ defined on $\mathcal{M}$ such that for any $D \in \mathcal{M}$ and any points $P(x, t)$, $P'(x', t')$ of the domain $\bar{G}$ we have

$$|\mu(x, t; D) - \mu(x', t'; D)| \leqslant \bar{\mu}(D)[d(P, P')]^\gamma\,,$$

where $0 < \gamma < 1$ is a constant.

If G is not a cylindrical domain (i.e. a domain which cannot be represented as the topological product of a domain in E_n by an interval of variable t), we additionally assume that

(3) for any $D \in \mathcal{M}$ there is a positive constant M_2 such that

$$\mu(x, t; D) \leqslant M_2 m(D),$$

$m(D)$ being the Lebesgue measure of D.

As a simple example of a measure satisfying conditions (1) and (2) may serve a measure given by the formula

$$\mu(x, t; D) = \varrho(x, t)\bar{\mu}(D), \quad (x, t) \in \bar{G}, \ D \in \mathcal{M},$$

$\varrho(x, t)$ being a function non-negative and uniformly Hölder continuous in $\bar{G}$.

LEMMA 4. *We assume that the measure μ satisfies conditions* (1) *and* (2), *or, if G is not a cylindrical domain, we assume that μ satisfies conditions* (1), (2) *and* (3) *and that $S \in C_{2+a}$. Moreover, suppose that $u(x, t)$ is uniformly Hölder continuous with exponent a in $\bar{G}$.*

Then the function

$$v(x, t) = \int_{G_t} u(y, t)\mu(x, t; dy)$$

is uniformly Hölder continuous with exponent $\delta = \min(a, \gamma)$ *in* $\bar{G}$.

Proof. Since $u \in C_a(G)$, the integral

$$\int_{G_t} u(y, t)\mu(x, t; dy)$$

exists. Making use of the known properties of integrals, we obtain

$$v(x, t) - v(x', t') = I_1 + I_2 - I_3 + I_4 - I_5, \tag{1.7}$$

where

$$I_1 = \int_{G_t \cap G_{t'}} [u(y, t) - u(y, t')]\mu(x, t; dy), \quad I_2 = \int_{G_t \cap G_{t'}} u(y, t')\mu(x, t; dy),$$

$$I_3 = \int_{G_t \cap G_{t'}} u(y, t')\mu(x', t'; dy), \quad I_4 = \int_{G_t \setminus G_{t'}} u(y, t)\mu(x, t; dy),$$

$$I_5 = \int_{G_{t'} \setminus G_t} u(y, t')\mu(x', t'; dy) \text{ [2]}.$$

Further, we have

$$|I_1| \leqslant \int_{G_t \cap G_{t'}} |u(y, t) - u(y, t')|\mu(x, t; dy) \leqslant M_3|t - t'|^{a/2} \int_{G_t \cap G_{t'}} \mu(x, t; dy)$$
$$= M_3|t - t'|^{a/2}\mu(x, t; G_t \cap G_{t'}) \leqslant M_1 M_3 T^{(a-\delta)/2}[d(P, P')]^{\delta},$$

where $P = (x, t)$, $P' = (x', t')$, $M_3 = \sup_{P, P' \in G} \dfrac{|u(P) - u(P')|}{[d(P, P')]^a}$.

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[2] If G is a cylindrical domain, then for any $t, t' \in [0, T]$ we have $G_t = G_{t'} = D_0$ and consequently $I_4 = I_5 = 0$.

Direct application of the definition of integral and condition (2) yield the estimate

$$|I_2 - I_3| \leqslant \bar{\mu}(G_t \cap G_{t'}) \sup |u| [d(P, P')]^{\gamma} \leqslant \bar{\mu}(D_0) |u|_0^G [d(P, P')]^{\gamma} .$$

If $G_t \setminus G_{t'} \neq O$, then it follows from Lemma 3 and from condition (3) ([3]) that

$$|I_4| \leqslant M_2 |u|_0^G \varkappa |t - t'| \leqslant M_2 |u|_0^G \varkappa T^{1-\delta/2} [d(P, P')]^{\delta}$$

and the same inequality is true for I_5. The estimates obtained and (1.7) imply the inequality

$$|v(x, t) - v(x', t')| \\ \leqslant \big(M_1 M_3 T^{(\alpha-\delta)/2} + \bar{\mu}(D_0) |u|_0^G R^{\gamma-\delta} + 2M_2 |u|_0^G T^{1-\delta/2}\big) [d(P, P')]^{\delta} ,$$

R being the diamater in the parabolic distance of domain G. This completes the proof.

2. Existence theorems for parabolic equations in bounded domains. Set $u(x,t) = \big(u^1(x,t), \dots, u^N(x,t)\big)$, $\bar{u}(x,t) = \big(\overline{u^1}(x,t), \dots, \overline{u^N}(x,t)\big)$, where $u^k(x, t)$, $\overline{u^k}(x, t) \in C_{1+\varepsilon}(G)$. We define the following operations and norm:

$$u + \bar{u} = (u^1 + \overline{u^1}, \dots, u^N + \overline{u^N}) , \qquad \eta u = (\eta u^1, \dots, \eta u^N)$$

$$(\eta \text{ is a real number}) ,$$

$$|u|_{l+\varepsilon}^G = \sum_{k=1}^{N} |u^k|_{l+\varepsilon}^G .$$

Then the set $C_{l+\varepsilon}^N(G)$ of all vector-functions $u(x, t)$ for which $|u|_{l+\varepsilon}^G < \infty$ is a Banach space.

For every $0 < \tau \leqslant T$ let B^k $(k = 1, \dots, N)$ be an operator defined on the set of all vector-functions $u = \{u^k\}$ regular in $\overline{G^\tau}$ with values belonging to the set of all functions defined in $\overline{G^\tau} \setminus \Sigma^\tau$, where $G^\tau = G \cap \cap \{(x, t) \colon 0 < t < \tau\}$ and $\Sigma^\tau = \Sigma \cap \{(x, t) \colon 0 < t \leqslant \tau\}$.

We shall consider the first boundary problem

$$\text{(2.1)} \qquad L^k u^k \equiv \sum_{i,j=1}^{n} a_{ij}^k(x, t) u_{x_i x_j}^k + \sum_{i=1}^{n} b_i^k(x, t) u_{x_i}^k + c^k(x, t) u^k - u_t^k = B^k u ,$$

$$(x, t) \in \overline{G^\tau} \setminus \Sigma^\tau ,$$

$$\text{(2.2)} \qquad u^k(x, t) = \varphi^k(x, t) , \quad (x, t) \in \Sigma^\tau \quad (k = 1, \dots, N) .$$

([3]) If G is a cylindrical domain, then assumption $S \in C_{2+\alpha}$ and condition (3) are superfluous.

The following assumptions are introduced $(i, j = 1, \ldots, n;\ k = 1, \ldots, N;\ 0 < \tau \leqslant T)$:

I. For any $(x, t) \in \bar{G}$ and $\xi \in E_n$ we have $a_{ij}^k(x, t) = a_{ji}^k(x, t)$, $\sum_{i,j=1}^{n} a_{ij}^k(x, t)\xi_i\xi_j \geqslant K_0|\xi|^2$ (K_0 is a positive constant).

II. The coefficients of L^k satisfy the uniform Hölder condition with exponent a in $\bar{G}$ and, moreover, $a_{ij}^k \in C_{1-0}(S)$.

III. The manifold S belongs both to $\bar{\bar{C}}_{2+a}$ and to C_{2-0}.

IV. The functions φ^k are of class $C_{1+\beta} \cap C_{2+a}$ $(a < \beta < 1)$.

V. If $\Phi \in C_{1+\beta}^N(G^\tau) \cap C_{2+a}^N(G^\tau)$ and $\Phi = \varphi$ on Σ^τ, then $B^k\Phi = L^k\varphi^k$ on ∂R_0.

VI. Operators B^k map the space $C_{1+a}^N(G^\tau)$ into the set $\bigcup_{0<\varepsilon<1} C_\varepsilon(G^\tau)$ and there are constants $A_1, A_2, A_3 \geqslant 0$, $0 \leqslant \lambda < 1$ such that for any $u \in C_{1+a}^N(G^\tau)$ the following inequality holds:

$$|B^k u|_0^{G^\tau} \leqslant A_1 + A_2(|u|_1^{G^\tau})^\lambda + A_3|u|_1^{G^\tau}, \tag{2.3}$$

where

$$|u|_1^{G^\tau} = \sum_{k=1}^{N} |u^k|_0^{G^\tau} + \sum_{k=1}^{N}\sum_{i=1}^{n} |u_{x_i}^k|_0^{G^\tau}.$$

VII. Operators B^k are continuous in the space $C_{1+a}^N(G^\tau)$; more precisely, if $u, u_l \in C_{1+a}^N(G^\tau)$ and $\lim_{l\to\infty} |u_l - u|_{1+a}^{G^\tau} = 0$, then $\lim_{l\to\infty} |B^k u_l - B^k u|_0^{G^\tau} = 0$.

By assumption II inequalities (1.4) and (1.5) hold true for coefficients of L^k $(k = 1, \ldots, N)$.

Let $F^k(x, t)$ be continuous functions in G^τ and let $L^k\varphi^k = F^k$ $(k = 1, \ldots, N)$ on ∂R_0. Then it follows from assumptions I-IV and from the remark to Lemma 2 that there is a constant $\bar{K}(\theta)$ $(\theta = a, \beta)$ depending only on θ, K_0, K_2, K_3 and domain G such that for any solution $u^k(x, t)$ $(k = 1, \ldots, N)$ of the problem

$$L^k u^k = F^k \quad \text{in } \overline{G^\tau}\setminus\Sigma^\tau, \qquad u^k = \varphi^k \quad \text{on } \Sigma^\tau \tag{2.4}$$

we have

$$|u^k|_{1+\theta}^{G^\tau} \leqslant \bar{K}(\theta)\tau^{(1-\theta)/2}(|F^k|_0^{G^\tau} + |L^k\Phi^k|_0^{G^\tau}) + |\Phi^k|_{1+\theta}^{G^\tau}, \tag{2.5}$$

where $\Phi^k \in C_{1+\beta}(G^\tau) \cap C_{2+a}(G^\tau)$ is any extension of φ^k.

THEOREM 1. *If assumptions* I-VII *are satisfied and*

$$\bar{K}(a)NA_3\tau^{(1-a)/2} < 1, \tag{2.6}$$

then there exists a solution $u(x, t) = \{u^k(x, t)\}$ of problem (2.1), (2.2); *furthermore, $u \in C_{1+\beta}^N(G^\tau) \cap C_{2+\varepsilon}(G^\tau)$ for some ε, $0 < \varepsilon < 1$.*

Proof. Denote by C_M the set of all functions $u(x,t) \in C^N_{1+\alpha}(G^\tau)$ such that $|u|^{G^\tau}_{1+\alpha} \leqslant M$ and $u(x,t) = \varphi(x,t)$ on Σ^τ, where the constant $M > 0$ will be specified later. Now for $u \in C_M$ consider the problem

$$L^k v^k = B^k u \equiv F^k(x,t)\,, \quad (x,t) \in \overline{G^\tau}\setminus\Sigma^\tau\,, \tag{2.7}$$

$$v^k(x,t) = \varphi^k(x,t)\,, \quad (x,t) \in \Sigma^\tau \quad (k = 1, \dots, N)\,. \tag{2.8}$$

By virtue of assumptions I-VI and Lemma 1, problem (2.7), (2.8) possesses a unique solution $v(x,t)$ which belongs to $C^N_{2+\varepsilon}(G^\tau)$ for some $0 < \varepsilon < 1$. Moreover, by (2.5), $v(x,t)$ belongs to $C^N_{1+\beta}(G^\tau)$.

Now we define on C_M a transformation Z setting $v = Zu$. Using Schauder's theorem, we shall prove that Z has a fixed point. We first show that Z maps C_M into itself.

In view of (2.3) we have

$$|F^k|^{G^\tau}_0 \leqslant A_1 + A_2 M^\lambda + A_3 M \quad (k = 1, \dots, N)\,. \tag{2.9}$$

Hence, by (2.5), we immediately obtain

$$\begin{aligned} |v|^{G^\tau}_{1+\theta} \leqslant {} & \overline{K}(\theta) N A_2 M^\lambda \tau^{(1-\theta)/2} + \overline{K}(\theta) N A_3 M \tau^{(1-\theta)/2} + \\ & + [\overline{K}(\theta)\tau^{(1-\theta)/2}(A_1 + |L^k \Phi^k|^{G^\tau}_0) + |\Phi^k|^{G^\tau}_{1+\theta}] N\,. \end{aligned} \tag{2.10}$$

Using (2.6) we now select M so that for $\theta = \alpha$ both the first and the third terms on the right-hand side of (2.10) are equal to or less than the expression

$$\tfrac{1}{2}[1 - \overline{K}(\alpha) N A_3 \tau^{(1-\alpha)/2}] M\,.$$

Thus, by (2.10), we have $|v|^{G^\tau}_{1+\alpha} \leqslant M$, i.e. Z maps C_M into itself.

It also follows from (2.10) that the set $\{Zu\colon u \in C_M\}$ is bounded in the space $C^N_{1+\beta}(G^\tau)$, whence, by Theorem 1 of [4] (p. 188), this set is a precompact subset of $C^N_{1+\alpha}(G^\tau)$.

Note further that Z is continuous, i.e. $|u_l - u|^{G^\tau}_{1+\alpha} \xrightarrow[l\to\infty]{} 0$ implies $|Zu_l - Zu|^{G^\tau}_{1+\alpha} \xrightarrow[l\to\infty]{} \infty$. Indeed, by the definition of Z, we have $v = Zu$ and $v_l = Zu_l$, where

$$L^k v^k = F^k(x,t)\,, \quad L^k v^k_l = B^k u_l \equiv F^k_l(x,t)\,, \quad (x,t) \in \overline{G^\tau}\setminus\Sigma^\tau\,,$$

$$v^k = v^k_l = \varphi^k \text{ on } \Sigma^\tau \ (k = 1, \dots, N)\,.$$

Hence

$$L^k(v^k_l - v^k) = F^k_l(x,t) - F^k(x,t)\,, \quad (x,t) \in \overline{G^\tau}\setminus\Sigma^\tau\,, \tag{2.11}$$

$$v^k_l - v^k = 0 \quad \text{on } \Sigma^\tau \ (k = 1, \dots, N)\,. \tag{2.12}$$

Assumption VII and Lemma 2 applied to (2.11), (2.12) yield the relation

$$\lim_{l\to\infty} |v_l^k - v^k|_{1+a}^{G^\tau} = 0 , \quad \text{i.e.} \quad \lim_{l\to\infty} |Zu_l - Zu|_{1+a}^{G^\tau} = 0 .$$

Finally, since C_M is a closed convex set of $C_{1+a}^N(G^\tau)$, then by Schauder's theorem Z has a fixed point u. Observe that u satisfies (2.1), (2.2) and $u \in C_{1+\beta}^N(G^\tau) \cap C_{2+\varepsilon}^N(G^\tau)$ for some $0 < \varepsilon < 1$, which was to be proved.

Now we derive corollaries from Theorem 1 concerning some special cases of operators B^k. We introduce the following assumptions:

1. Let $\Psi^k(x, t; z(\cdot, t))$ ($(x, t) \in \overline{G^\tau}, 1 \leqslant k \leqslant N$) be a functional defined, for every $0 < \tau \leqslant T$, on a set of all functions $z(x, t)$ regular in $\overline{G^\tau}$ such that for any $z, \bar{z} \in C_{1+a}(G^\tau)$ we have the inequality

$$|\Psi^k(x, t; z(\cdot, t)) - \Psi^k(x, t; \bar{z}(\cdot, t))|_0^{G^\tau} \leqslant M_1|z-\bar{z}|_0^{G^\tau} ,$$

$M_1 > 0$ being a certain constant. Besides, for any $z \in C_{1+a}(G^\tau)$, functions $g^k(x, t) = \Psi^k(x, t; z(\cdot, t))$ satisfy a uniform Hölder condition (with exponent a' which may depend on z and k) in G^τ.

2. Let functions $f^k(x, t, p, q, r)$ $(k = 1, \dots, N)$, defined on $\bar{G} \times \times E_{N+nN+N}$, satisfy a uniform Hölder condition in every bounded set $\bar{G} \times H$ $(H \subset E_{N+nN+N})$ and

$$f^k\big(x, 0, \varphi, \varphi_x, \Psi(x, 0; \varphi(\cdot, 0))\big) = L^k\varphi^k \quad \text{on } \partial R_0 , \tag{2.13}$$

where

$$u_x = (\overset{1}{u}_{x_1}, \dots, \overset{1}{u}_{x_n}, \dots, \overset{N}{u}_{x_1}, \dots, \overset{N}{u}_{x_n}) ,$$

$$\Psi(x, t; u(\cdot, t)) = \big(\Psi^1(x, t; u^1(\cdot, t)), \dots, \Psi^N(x, t; u^N(\cdot, t))\big) .$$

Moreover, there exist constants $N_1, N_2, N_3 \geqslant 0$, $0 \leqslant \lambda < 1$, such that

$$|f^k(x, t, p, q, r)| \leqslant N_1 + N_2|(p, q, r)|^\lambda + N_3|(p, q, r)| ,$$

where

$$|(p, q, r)| = \sum_{i=1}^N |p^i| + \sum_{i=1}^N \sum_{j=1}^n |q^{ij}| + \sum_{i=1}^N |r^i| .$$

Corollary 1. *If assumptions* I-IV, 1, 2 *are fulfilled and*

$$\bar{K}(a) N N_3 (M_1+1) \tau^{(1-a)/2} < 1 , \tag{2.14}$$

then Theorem 1 *holds true in the case*

$$B^k u = f^k\big(x, t, u, u_x, \Psi(x, t; u(\cdot, t))\big) \text{ [4]} . \tag{2.15}$$

[4] Expressions of this form occur in differential inequalities of parabolic type treated by Szarski in [10].

In order to prove this corollary it suffices to observe that conditions (1), (2), (2.14) imply assumptions V-VII, (2.6) respectively, and to apply Theorem 1.

COROLLARY 1'. *Let assumptions* I-IV, 2 *and* (2.14) *be fulfilled and let the measures* $\mu^k(x, t; D)$ *satisfy all the conditions imposed in section* 1. *Then Theorem* 1 *is true in the case*

$$B^k u = f^k\Big(x, t, u, u_x, \int_{G_t} u(y, t)\mu(x, t; dy)\Big), \tag{2.16}$$

where

$$\int_{G_t} u(y, t)\mu(x, t; dy) = \Big(\int_{G_t} u^1(y, t)\mu^1(x, t; dy), \ldots, \int_{G_t} u^N(y, t)\mu^N(x, t; dy)\Big).$$

Proof of this corollary follows immediately by applying Lemma 4 and Corollary 1.

At present, under stronger assumptions than those of Theorem 1, we shall prove the existence and uniqueness of solutions of the problem (2.1), (2.2) and the convergence of successive approximations. We retain assumptions I-V whereas VI-VII are replaced by the following ones:

VI'. Operators B^k $(k = 1, \ldots, N)$ map the space $C^N_{1+\alpha}(G^\tau)$ into $C_\alpha(G^\tau)$.

VII'. There exists a constant $A'_1 \geqslant 0$ such that for any $u, \bar{u} \in C^N_{1+\alpha}(G^\tau)$ we have

$$|B^k u - B^k \bar{u}|_0^{G^\tau} \leqslant A'_1 |u - \bar{u}|_1^{G^\tau}.$$

THEOREM 2. *If assumptions* I-V [5], VI'-VII' *are satisfied and*

$$\bar{K}(\alpha) N A'_1 \tau^{(1-\alpha)/2} < 1, \tag{2.17}$$

then problem (2.1), (2.2) *has a unique solution* $u = \{u^k\}$ *in the space* $C^N_{1+\alpha}(G^\tau)$. *Moreover,* $u \in C^N_{1+\beta}(G^\tau) \cap C^N_{2+\alpha}(G^\tau)$.

Proof. Denote by Λ the set of all functions $u(x, t) \in C^N_{1+\alpha}(G^\tau)$ such that $u(x, t) = \varphi(x, t)$ on Σ^τ. In view of Lemma 1 problem (2.7), (2.8) has, for $u \in \Lambda$, a unique solution $v = \{v^k\}$ in the class $C^N_{2+\alpha}(G^\tau)$. Moreover, by (2.5), $v \in C^N_{1+\beta}(G^\tau)$.

Now we define on Λ a transformation Z setting $v = Zu$. We shall prove that Z is a contraction in $C^N_{1+\alpha}(G^\tau)$. Indeed, let $\bar{v} = Z\bar{u}$. Then $L^k(v^k - \bar{v}^k) = F^k(x, t) - \bar{F}^k(x, t)$ for $(x, t) \in \bar{G}^\tau \setminus \Sigma^\tau$, $v^k - \bar{v}^k = 0$ on Σ^τ. Setting $\Phi^k \equiv 0$ in (2.5) and making use of assumption VII' we obtain

$$|v^k - \bar{v}^k|_{1+\alpha}^{G^\tau} \leqslant \bar{K}(\alpha) A'_1 \tau^{(1-\alpha)/2} |u - \bar{u}|_{1+\alpha}^{G^\tau}.$$

---

[5] The condition $\alpha < \beta < 1$ may be replaced by $\alpha \leqslant \beta < 1$.

Thus

$$|Zu - Z\bar{u}|_{1+\alpha}^{G^\tau} \leqslant \bar{K}(\alpha) N A_1' \tau^{(1-\alpha)/2} |u - \bar{u}|_{1+\alpha}^{G^\tau},$$

which means, by (2.17), that Z is a contraction. Since Λ is the closed set of $C_{1+\alpha}^N(G^\tau)$ and Z maps Λ into itself, then by the Banach fixed point theorem Z has a unique fixed point u. At the same time we have proved that $u \in C_{1+\beta}^N(G^\tau) \cap C_{2+\alpha}^N(G^\tau)$. Obviously the function u is a (unique) solution of problem (2.1), (2.2), and the proof is completed.

The solution of the above problem can be obtained by the method of successive approximations. For this purpose we construct a sequence $\{u_l\} = \{u_l^1, \ldots, u_l^N\}$ putting

$$(2.18) \qquad u_0 = \Phi, \quad u_{l+1} = Zu_l \quad (l = 0, 1, 2, \ldots).$$

It follows from the proof of the Banach theorem (see e.g. [4], p. 59) that the sequence $\{u_l\}$ is convergent in $C_{1+\alpha}^N(G^\tau)$ to a unique fixed point u of the transformation Z. Thus we have relations

$$\lim_{l\to\infty} u_l^k = u^k \quad \text{and} \quad \lim_{l\to\infty} (u_l^k)_{x_i} = u_{x_i}^k \quad (k = 1, \ldots, N;\ i = 1, \ldots, n),$$

where the convergence is uniform. Suppose additionally that

VI″. Operators B^k map every bounded subset of $C_{1+\alpha}^N(G^\tau)$ $(0 < \tau \leqslant T)$ into a bounded subset of $C_\alpha(G^\tau)$.

Then, by Lemma 1, the sequence $\{|u_l^k|_{2+\alpha}^{G^\tau}\}$ is bounded as well. Hence, in view of Arzela's theorem, there is a subsequence $\{u_{l'}^k\}$ such that

$$(2.19) \qquad \lim_{l'\to\infty} (u_{l'}^k)_{x_i x_j} = u_{x_i x_j}^k \quad \text{and} \quad \lim_{l'\to\infty} (u_{l'}^k)_t = u_t^k,$$

where the convergence is uniform.

In order to derive corollaries from Theorem 2 for cases (2.15) and (2.16) we make the following assumptions, instead of assumptions 1, 2:

1′. Assumption 1 with $\alpha' = \alpha$.

2′. Functions $f^k(x, t, p, q, r)$ satisfy a uniform Hölder condition with exponent α in $(x, t) \in \bar{G}$, uniformly with respect to $(p, q, r) \in E_{N+nN+N}$ and condition (2.13) holds true. Furthermore, there is a constant $N_1' \geqslant 0$ such that

$$(2.20) \qquad |f^k(x, t, p, q, r) - f^k(x, t, \bar{p}, \bar{q}, \bar{r})| \leqslant N_1' |(p - \bar{p}, q - \bar{q}, r - \bar{r})|.$$

COROLLARY 2. *If assumptions* I-IV, 1′, 2′ *are satisfied and*

$$(2.21) \qquad \bar{K}(\alpha) N N_1' (M_1 + 1) \tau^{(1-\alpha)/2} < 1,$$

then Theorem 2 *remains true in case* (2.15).

COROLLARY 2'. *Let assumptions* I-IV, 2' *and* (2.21) *be fulfilled and let the measures* $\mu^k(x, t; D)$ *satisfy all the conditions imposed in section* 1 *with* $\gamma = a$. *Then Theorem* 2 *is true in case* (2.16).

Since assumption VI'' is fulfilled in cases (2.15) and (2.16), there exists a subsequence $\{u_{l'}\}$ of sequence $\{u_l\}$ defined by (2.18) such that (2.19) holds.

3. Lemmas for the parabolic case in an unbounded zone. Now let G be an unbounded zone $E_n \times (0, T)$. We preserve the definition of norms $|u|_{l+a}^{G}$ $(l = 0, 1, 2)$ introduced in section 1.

Consider the Cauchy problem

$$(3.1) \qquad Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x, t) u_{x_i x_j} + \sum_{i=1}^{n} b_i(x, t) u_{x_i} + c(x, t) u - u_t = f(x, t) , \quad (x, t) \in G' = E_n \times (0, T] ,$$

$$(3.2) \qquad u(x, 0) = \varphi(x) , \quad x \in E_n .$$

We make the following assumptions:

(A) The operator L is uniformly parabolic in G' (see [4], p. 3), whereas its coefficients are bounded and uniformly Hölder continuous, with exponent a, in G'.

(B) The function φ together with its first and second order derivatives are bounded in E_n. Moreover, φ and φ_{x_i} are uniformly Hölder continuous with exponent a in E_n, while derivatives $\varphi_{x_i x_j}$ are locally Hölder continuous with exponent a in E_n.

(C) The function $f(x, t)$ is bounded in G' and satisfies a uniform Hölder condition with exponent a in every bounded domain $H \times (0, T]$ $(H \subset E_n)$.

LEMMA 5. *If assumptions* (A), (B) *and* (C) *are satisfied, then problem* (3.1), (3.2) *has a unique regular solution* $u(x, t)$. *Moreover,* u *and all its derivatives appearing in* Lu *satisfy a uniform Hölder condition with exponent* a *in every bounded domain* $H \times [\sigma, T]$ $(H \subset E_n,\ 0 < \sigma < T)$.

Proof. The existence and uniqueness are immediate consequences of Theorem 12 of [4] (p. 25) and of Theorem 16 of [4] (p. 29) respectively, whereas the Hölder continuity of u and its derivatives follows from the proof of Theorem 10 of [4] (p. 72).

LEMMA 6. *If assumptions* (A), (B) *and* (C) *are satisfied, then any regular solution* $u(x, t)$ *of problem* (3.1), (3.2) *belongs to* $C_{1+a}(G)$. *Moreover, there exists a constant* K *depending only on* a, n, T *and* L *such that*

$$(3.3) \qquad |u|_{1+a}^{G^\tau} \leqslant K\tau^{(1-a)/2}(|f|_0^{G^\tau} + |L\varphi|_0^{G^\tau}) + |\varphi|_{1+a}^{E_n} ,$$

where $G^\tau = E_n \times (0, \tau)$, $0 < \tau \leqslant T$.

Proof. Proceeding as in the proof of Lemma 2 of [4] (p. 193) one can derive for a solution $v(x, t)$ of problem $Lv = f(x, t) - L\varphi(x)$ for $(x, t) \in G'$, $v(x, 0) = 0$ in E_n the following estimate

$$|v|_{1+\alpha}^{G^\tau} \leqslant K\tau^{(1-\alpha)/2}|f - L\varphi|_0^{G^\tau},$$

which easily implies (3.3).

4. Existence and uniqueness theorem for parabolic equations in an unbounded zone. In this section we prove the existence and uniqueness of solutions of the Cauchy problem

(4.1) $$L^k u^k = B^k u \ (^6), \quad (x, t) \in G'^\tau = E_n \times (0, \tau].$$

(4.2) $$u^k(x, 0) = \varphi^k(x), \quad x \in E_n \ (k = 1, \ldots, N).$$

We make the following assumptions:

I. Operators L^k and functions φ^k $(k = 1, \ldots, N)$ satisfy conditions (A) and (B) of the previous section.

II. For every $0 < \tau \leqslant T$ and for any $u \in C^N_{1+\alpha}(G^\tau)$ functions $B^k u$ are bounded in G'^τ and satisfy a uniform Hölder condition with exponent α in every bounded domain $H \times (0, \tau]$ $(H \subset E_n)$.

III. Assumption VII' of section 2.

Let functions $F^k(x, t)$ $(k = 1, \ldots, N)$ satisfy assumption (C) of section 3. Then it follows from assumption I and from Lemma 6 that there exists a constant K depending only on α, n, T and operators L^k (considered for $(x, t) \in G'$) such that for any regular solution $u^k(x, t)$ $(k = 1, \ldots, N)$ of the problem $L^k u^k = F^k$ in G'^τ, $u^k(x, 0) = \varphi^k(x)$ in E_n we have

(4.3) $$|u^k|_{1+\alpha}^{G^\tau} \leqslant K\tau^{(1-\alpha)/2}(|F^k|_0^{G^\tau} + |L^k\varphi^k|_0^{G^\tau}) + |\varphi^k|_{1+\alpha}^{E_n}.$$

THEOREM 3. *If assumptions* I-III *are fulfilled and*

(4.4) $$NKA_1'\tau^{(1-\alpha)/2} < 1,$$

then problem (4.1), (4.2) *has a unique regular solution* $u(x, t) = \{u^k(x, t)\}$ *in class* $C^N_{1+\alpha}(G^\tau)$. *Moreover, derivatives* $u^k_{x_i x_j}$ *and* u^k_t *are uniformly Hölder continuous of exponent* α *in every bounded domain* $H \times [\sigma, \tau]$ $(H \subset E_n$, $0 < \sigma < \tau)$.

Proof. The proof is similar to that of Theorem 2. Namely, let us denote by Λ the set of all functions $u(x, t) \in C^N_{1+\alpha}(G^\tau)$ such that $u(x, 1) = \varphi(x)$ in E_n and consider, for $u \in \Lambda$, the problem

$$\begin{aligned} L^k v^k &= B^k u, \quad (x, t) \in G'^\tau, \\ v^k(x, 0) &= \varphi^k(x), \quad x \in E_n, \end{aligned} \qquad (k = 1, \ldots, N).$$

---

(6) We retain the notation of section 2.

By Lemmas 5 and 6 the above problem has a unique regular solution $v^k(x,t)$ $(k = 1, \ldots, N)$, where $v^k \in C_{1+a}(G^\tau)$, whereas $v^k_{x_ix_j}$ and v^k_t are uniformly Hölder continuous of exponent a in every bounded domain $H\times[\sigma,\tau]$ $(0<\sigma<\tau)$. This enables us to define on the set Λ a transformation Z by formula $v = Zu$.

Let $\bar{v} = Z\bar{u}$. Then we have

$$\begin{aligned} &L^k(v^k-\bar{v}^k) = B^k u - B^k\bar{u}\,, && (x,t)\in G'^\tau\,, \\ &v^k(x,0)-\bar{v}^k(x,0) = 0\,, && x\in E_n\,, \end{aligned} \qquad (k = 1, \ldots, N)$$

and hence, by (4.3) and assumption III,

$$|Zu - Z\bar{u}|^{G^\tau}_{1+a} \leqslant NKA'_1\tau^{(1-a)/2}|u-\bar{u}|^{G^\tau}_{1+a}\,.$$

Thus, by (4.4), Z is a contraction. Since the remaining assumptions of the Banach theorem are also fulfilled, Z has a unique fixed point which completes the proof.

The solution of problem (4.1), (4.2) can be obtained as a limit of the sequence of successive approximations defined similarly as in section 2.

Now we formulate Theorem 3 for cases (2.15) and (2.16). For this purpose instead of assumption 2′ of section 2 we introduce the following one:

2″. Functions $f^k(x,t,p,q,r)$ $(k = 1, \ldots, N)$ are uniformly Hölder continuous with exponent a in (x,t) in every bounded domain $H\times(0,T]$ uniformly with respect to $(p,q,r)\in E_{N+nN+N}$. Moreover, functions $f^k(x,t,0,0,0)$ $(k = 1, \ldots, N)$ are bounded in G' and condition (2.20) is fulfilled.

COROLLARY 3. *If assumptions* I, 2″ *and* 1′ *of section* 2 *are satisfied and*

$$NKN'_1(M_1+1)\tau^{(1-a)/2} < 1\,, \tag{4.5}$$

then Theorem 3 *remains true for case* (2.15).

COROLLARY 3′. *Let assumptions* I, 2″, (4.5) *be satisfied and let the measures* $\mu^k(x,t;D)$ *fulfil conditions* (1) *and* (2) (*with* $\gamma = a$) *of section* 1. *Under these assumptions Theorem* 3 *is true for case* (2.16).

5. Lemmas for the elliptic case. The elliptic problem will only be treated in a bounded domain. So let G be an open bounded domain of the Euclidean space E_n of the variables $x = (x_1, \ldots, x_n)$. Following Friedman [4] we formulate several definitions concerning norms, sets of functions, properties of the boundary ∂G of domain G and functions defined on ∂G.

The following norms are introduced:

$$|u|_0^G = \sup_{x \in G} |u(x)| , \qquad |u|_a^G = |u|_0^G + \sup_{x,x' \in G} \frac{|u(x)-u(x')|}{|x-x'|^a} ,$$

$$|u|_{1+a}^G = |u|_0^G + \sum_{i=1}^n |u_{x_i}|_a^G , \qquad |u|_{2+a}^G = |u|_0^G + \sum_{i=1}^n |u_{x_i}|_0^G + \sum_{i,j=1}^n |u_{x_i x_j}|_a^G$$

$$(0 < a < 1) .$$

The set of all functions u with finite norm $|u|_{l+a}^G$ $(l = 0, 1, 2)$ will be denoted by $C_{l+a}(G)$.

For every point $x \in \partial G$ let there exist an n-dimensional neighbourhood V such that $V \cap \partial G$ can be represented, for some i $(1 \leqslant i \leqslant n)$, by an equation of the form

$$x_i = h(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) .$$

If the functions h belong to C_{2+a}, then we say that ∂G is of class C_{2+a}.

A function φ defined on ∂G is said to be of class C_{2+a} if it has an extension $\Phi \in C_{2+a}(G)$. Then we define

$$|\varphi|_{2+a}^G = \inf_{\Phi} |\Phi|_{2+a}^G .$$

Consider the Dirichlet problem:

$$Lu = \sum_{i,j=1}^n a_{ij}(x) u_{x_i x_j} + \sum_{i=1}^n b_i(x) u_{x_i} + c(x) u = f(x) , \qquad x \in G , \tag{5.1}$$

$$u(x) = \varphi(x) , \qquad x \in \partial G . \tag{5.2}$$

We shall need the following assumptions:

(A) For any $x \in G$ and $\xi \in E_n$ we have

$$a_{ij}(x) = a_{ji}(x) , \qquad c(x) \leqslant 0 , \qquad \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geqslant K_0 |\xi|^2 \qquad (K_0 = \text{const} > 0) .$$

(B) The coefficients of L and the function f are uniformly Hölder continuous with exponent a in G. Thus for some constant $K_1 \geqslant 0$

$$|a_{ij}|_a^G , \qquad |b_i|_a^G , \qquad |c|_a^G \leqslant K_1 .$$

From [4], p. 86, and [3], p. 808 (see also [1]) we obtain the following

LEMMA 7. *If assumptions* (A) *and* (B) *hold true, the boundary* ∂G *is of class* $C_{2+a'}$ $(a < a' < 1)$ *and* $\varphi \in C_{2+a}$, *then problem* (5.1), (5.2) *has a unique regular* (7) *solution* $u(x)$ *in the class* $C_{2+a}(G)$. *Moreover, there*

(7) I.e. continuous in $\bar{G}$ and possessing continuous first and second order derivatives in G.

exist constants K, β (depending only on L and G) and K' (depending only on K_0, K_1, a and G) such that

$$(5.3)\qquad |u|^G_{1+\beta} \leqslant K(|f|^G_0 + |L\Phi|^G_0) + |\Phi|^G_{1+\beta} \text{ (8)} .$$

and

$$|u|^G_{2+a} \leqslant K'(|\varphi|^G_{2+a} + |f|^G_a) ,$$

where $\Phi \in C_{2+a}(G)$ is any extension of φ.

Now denote by $\mathcal{M}$ the σ-field of all Lebesgue-measurable subsets of the closure $\bar{G}$. Let $\mu(x; D)$ be a non-negative measure defined on $\mathcal{M}$, depending on $x \in \bar{G}$, and satisfying the following conditions:

(1) there is a constant $M_1 > 0$ such that

$$\mu(x; \bar{G}) \leqslant M_1$$

for any $x \in \bar{G}$;

(2) there exists a finite non-negative measure $\bar{\mu}$ defined on $\mathcal{M}$ such that for any $D \in \mathcal{M}$ and $x, x' \in \bar{G}$

$$|\mu(x; D) - \mu(x'; D)| \leqslant \bar{\mu}(D)|x - x'|^\gamma ,$$

where $0 < \gamma < 1$ is a constant.

By an argument similar to that used in the proof of Lemma 4, one can prove the following

LEMMA 8. *If a function $u(x)$ is uniformly Hölder continuous with exponent a in G, then the function*

$$v(x) = \int_G u(y)\mu(x; dy)$$

is uniformly Hölder continuous with the exponent $\bar{a} = \min(a, \gamma)$ in the domain G.

6. Existence theorems for elliptic equations. In the present section we discuss the existence of solutions of the Dirichlet problem

$$(6.1)\qquad L^k u^k \equiv \sum_{i,j=1}^{n} a^k_{ij}(x) u^k_{x_i x_j} + \sum_{i=1}^{n} b^k_i(x) u^k_{x_i} + c^k(x) u^k = B^k u , \qquad x \in G ,$$

$$(6.2)\qquad u^k(x) = \varphi^k(x) , \qquad x \in \partial G \ (k = 1, \ldots, N) ,$$

where B^k is an operator defined on the set of all vector-functions $u = \{u^k\}$ regular in $\bar{G}$ with values belonging to the set of all functions defined in G.

For $i, j = 1, \ldots, n$; $k = 1, \ldots, N$ we need the following assumptions:

(8) It is easy to see that relation $\Phi \in C_{2+a}(G)$ implies $\Phi \in C_{1+\beta}(G)$ for any $0 < \beta < 1$.

I. For any $x \in G$ and $\xi \in E_n$ we have

$$a_{ij}^k(x) = a_{ji}^k(x)\,, \quad c^k(x) \leqslant 0\,, \quad \sum_{i,j=1}^{n} a_{ij}^k(x)\xi_i\xi_j \geqslant K_0|\xi|^2 \quad (K_0 = \text{const} > 0)\,.$$

II. The coefficients of L^k are uniformly Hölder continuous with exponent α in G. Hence, for some constant $K_1 > 0$, we have

$$|a_{ij}^k|_\alpha^G\,, \quad |b_i^k|_\alpha^G\,, \quad |c^k|_\alpha^G \leqslant K_1\,.$$

III. The boundary ∂G is of class $C_{2+\alpha'}$ $(\alpha < \alpha' < 1)$ and $\varphi^k \in C_{2+\alpha}$. In order to formulate further assumptions we introduce some symbols. Assume that functions $F^k(x)$ $(k = 1, \ldots, N)$ are uniformly Hölder continuous with exponent α in G. Then, by assumptions I-III and by Lemma 7, the problem

$$L^k v^k = F^k \quad \text{in } G,\ v^k = \varphi^k \text{ on } \partial G \ (k = 1, \ldots, N)$$

has a unique solution $v^k(x)$. Moreover, there exist constants K and β (depending only on L^k and domain G) such that

$$|v^k|_{1+\beta}^G \leqslant K(|F^k|_0^G + |L^k\Phi^k|_0^G) + |\Phi^k|_{1+\beta}^G\,, \tag{6.3}$$

where $\Phi^k \in C_{2+\alpha}(G)$ is any extension of φ^k. Now let A_1 and β' be such constants that

$$0 \leqslant A_1 < (KN)^{-1}\,, \tag{6.4}$$

$$R^{\beta-\beta'} \leqslant 1 + \Delta/2\,, \quad 0 < \beta' < \beta\,, \tag{6.5}$$

where R is the diameter of the domain G and $\Delta = 1 - KNA_1$.

IV. Operators B^k map the space $C_{1+\beta'}^N(G)$ (⁹) into the set $\bigcup_{0<\varepsilon<1} C_\varepsilon(G)$ and there exist constants $A_2, A_3 \geqslant 0$, $0 \leqslant \lambda < 1$ such that for any $u \in C_{1+\beta'}^N(G)$ we have the inequality

$$|B^k u|_0^G \leqslant A_1|u|_1^G + A_2(|u|_1^G)^\lambda + A_3\,. \tag{6.6}$$

V. Operators B^k are continuous in the space $C_{1+\beta'}^N(G)$, i.e. if $u, u_l \in C_{1+\beta'}^N(G)$ and $\lim_{l\to\infty} |u_l - u|_{1+\beta'}^G = 0$, then $\lim_{l\to\infty} |B^k u_l - B^k u|_0^G = 0$.

Theorem 4. *If assumptions* I-V *are satisfied, then problem* (6.1), (6.2) *has a solution* $u = \{u^k\}$, *which belongs to* $C_{2+\varepsilon}^N(G)$ *for some* $0 < \varepsilon < 1$.

Proof. Denote by C_M the set of all functions $u(x) \in C_{1+\beta'}^N(G)$ such that $|u|_{1+\beta'}^G \leqslant M$ and $u(x) = \varphi(x)$ on ∂G, where $M > 0$ is a constant to be conveniently chosen.

(⁹) We retain, with obvious modifications, the notation introduced in section 2.

Consider now, for $u \in C_M$, the problem

$$(6.7)\qquad L^k v^k = B^k u \equiv F^k(x)\,, \quad x \in G\,, \quad v^k(x) = \varphi^k(x)\,, \quad x \in \partial G \quad (k = 1, \dots, N)\,.$$

This problem, by Lemma 7, has a unique solution $v = \{v^k\}$ belonging to the class $C^N_{2+\varepsilon}(G)$ for some $0 < \varepsilon < 1$. Now we define a transformation Z setting $v = Zu$. Applying (6.3) we obtain

$$(6.8)\qquad |v^k|^G_{1+\beta'} \leqslant \max(1, R^{\beta-\beta'})[K(|F^k|^G_0 + |L^k\Phi^k|^G_0) + |\Phi^k|^G_{1+\beta}]\,.$$

Hence, by (6.6) and (6.5),

$$|v|^G_{1+\beta'} \leqslant (1+\Delta/2)N[K(A_3 + |L^k\Phi^k|^G_0) + |\Phi^k|^G_{1+\beta}] + \\ + (1+\Delta/2)KA_2NM^\lambda + (1+\Delta/2)KA_1NM\,.$$

Choosing the constant M so that the sum of the first and the second term on the rigth-hand side of the last inequality is less than or equal to $\dfrac{\Delta^2+\Delta}{2}M$, we get, by the definition of Δ, $|v|^G_{1+\beta'} \leqslant M$. Thus Z maps C_M into itself. The further argumentation is the same as in the proof of Theorem 1.

As in section 2, we shall consider now some special cases of operators B^k for which Theorem 4 holds true. We introduce the following assumptions:

1. Let $\Psi^k(x; z(\cdot))$ $(x \in G,\ k = 1, \dots, N)$ be a functional defined on the set of all functions $z(x)$ regular in $\bar{G}$ such that for any $z, \bar{z} \in C_{1+\beta'}(G)$ we have the inequality

$$|\Psi^k(x; z(\cdot)) - \Psi^k(x; \bar{z}(\cdot))|^G_0 \leqslant M_1 |z - \bar{z}|^G_0\,,$$

$M_1 > 0$ being a certain constant. Besides, for any $z \in C_{1+\beta'}(G)$ the function $g^k(x) = \Psi^k(x; z(\cdot))$ satisfies a uniform Hölder condition (with exponent $\bar{\beta}$ which may depend on z and k) in G.

2. Let functions $f^k(x, p, q, r)$ $(k = 1, \dots, N)$, defined on $G \times E_{N+nN+N}$, satisfy a uniform Hölder condition in every bounded set $G \times H$ $(H \subset E_{N+nN+N})$. Moreover, there are constants $N_1, N_2 \geqslant 0$ and $0 \leqslant \lambda < 1$ such that

$$|f^k(x, p, q, r)| \leqslant A_1(M_1+1)^{-1}|(p, q, r)| + N_1|(p, q, r)|^\lambda + N_2\,.$$

COROLLARY 4. *If assumptions* I-III *and* 1, 2 *are satisfied, then the assertion of Theorem* 4 *remains valid with*

$$(6.9)\qquad B^k u = f^k(x, u, u_x, \Psi(x; u(\cdot)))\,.$$

COROLLARY 4′. *If assumptions* I-III, 2 *and* 1, 2 *with* $(\mu = \mu^k)$ *of section* 5 *are satisfied, then Theorem* 4 *is true for operators* B^k *given by the formulas*

$$B^k u = f^k\Big(x, u, u_x, \int_G u(y)\mu(x; dy)\Big). \tag{6.10}$$

Now, under stronger assumptions than those of Theorem 4, we shall prove the existence and uniqueness of solutions of problem (6.1), (6.2) and the convergence of the successive approximations. Instead of assumptions IV, V we make the following ones (assumptions I-III being retained):

IV′. Operators B^k $(k = 1, \ldots, N)$ map the space $C^N_{1+\beta}(G)$ into $C_\delta(G)$, where $\delta = \min(a, \beta)$.

V′. There is a constant $A_1' \geqslant 0$ such that for any $u, \bar{u} \in C^N_{1+\beta}(G)$ we have

$$|B^k u - B^k \bar{u}|_0^G \leqslant A_1' |u - \bar{u}|_1^G.$$

THEOREM 5. *Let assumptions* I-III [10], IV′, V′ *be fulfilled and let*

$$A_1' < (KN)^{-1}. \tag{6.11}$$

Under these assumptions problem (6.1), (6.2) *has a unique solution* $u = \{u^k\}$ *of class* $C^N_{1+\beta}(G)$. *Moreover,* $u \in C^N_{2+\delta}(G)$.

Proof. The proof is similar to that of Theorem 2. Namely, let us denote by Λ the set of all functions $u(x) \in C^N_{1+\beta}(G)$, such that $u(x) = \varphi(x)$ on ∂G and consider problem (6.7) for $u \in \Lambda$. Next, using the theorems on the existence and uniqueness of solutions of this problem we define the transformation $v = Zu$. We show that Z maps Λ into $\Lambda \cap C^N_{2+\delta}(G)$ and that Z is a contraction in Λ. This enables us to apply the Banach fixed point theorem, from which the proof of the theorem follows.

As in the parabolic case the solution of problem (6.1), (6.2) can be obtained by the method of successive approximations.

To end this section, we formulate Theorem 5 for cases (6.9) and (6.10). For this purpose we make, instead of assumptions 1, 2, the following ones:

1′. Assumption 1 with $\beta' = \beta$ and $\bar{\beta} = \delta$.

2′. Functions $f^k(x, p, q, r)$ are uniformly Hölder continuous with exponent δ in $x \in G$, uniformly with respect to $(p, q, r) \in E_{N+nN+N}$. Moreover, there is such a constant $N_1' > 0$ that

$$|f^k(x, p, q, r) - f^k(x, \bar{p}, \bar{q}, \bar{r})| \leqslant N_1' |(p - \bar{p}, q - \bar{q}, r - \bar{r})|.$$

COROLLARY 5. *Let assumptions* I-III, 1′, 2′ *be satisfied and let*

$$N_1' < [KN(M_1 + 1)]^{-1}. \tag{6.12}$$

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[10] In III, instead of assumption $\varphi^k \in C_{2+a}$, we may only assume that $\varphi^k \in C_{2+\delta}$.

Then Theorem 5 *holds true in case* (6.9).

COROLLARY 5′. *If assumptions* I-III, 2′, 1, 2 (*with* $\mu = \mu^k$ *and* $\gamma = \delta$) *of section* 5 *and condition* (6.12) *are satisfied, then the conclusion of Theorem* 5 *holds true in the case of the operators given by* (6.10).

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