ANNALES
POLONICI MATHEMATICI
XXIV (1970)

The boundary value problems for ordinary non-linear differential and difference equations of the fourth order

by Zdzisław Denkowski (Kraków)

1. Introduction. In the present note we deal with two boundary value problems formulated, respectively, for differential and difference equations of certain type. The first of them consists in the search for a solution of the differential equation

$$y^{(IV)} = g(t, y, y', y''), \tag{1.1}$$

satisfying the boundary condition of the form

$$y(0) = r_{00}, \quad y(h) = r_{0h}, \quad y'(0) = r_{10}, \quad y'(h) = r_{1h}, \tag{1.2}$$

where $r_{00}, r_{0h}, r_{10}, r_{1h}$ are arbitrary fixed real numbers, and the second — in search for a solution of the difference equation

$$\nabla\Delta\nabla\Delta v_i = \tilde{g}(i, v_i, \Delta v_i, \nabla\Delta v_i) \quad (i = 2, \ldots, n-2), \tag{1.3}$$

satisfying the boundary condition of the form

$$v_0 = \tilde{r}_{00}, \quad v_n = \tilde{r}_{0n}, \quad \Delta v_0 = \tilde{r}_{10}, \quad \nabla v_n = \tilde{r}_{1n}, \tag{1.4}$$

where Δv_i, ∇v_i, $\nabla\Delta v_i$, $\nabla\Delta\nabla\Delta v_i$ denote the difference operators and $\tilde{r}_{00}, \tilde{r}_{0n}, \tilde{r}_{10}, \tilde{r}_{1n}$ — arbitrary fixed real numbers.

In the sequel we shall assume that the function $g\colon [0, h]\times R^3 \to R$ satisfies the Carathéodory condition (i.e. 1° for every fixed $z\in R^3$ it is measurable on $[0, h]$, 2° for every fixed $t\in[0, h]$ it is continuous with respect to z) and the following inequality

$$|g(t, z_1, z_2, z_3)| \leqslant p_0(t)+\sum_{j=1}^{3} P_j|z_j|, \tag{1.5}$$

where $p_0\colon [0, h] \to R$ is a function sommable on $[0, h]$ and P_j $(j = 1, 2, 3)$ are fixed real numbers.

It is easy to see that condition (1.5) is fulfilled if the function g satisfies the Lipschitz condition

$$|g(t, z_1, z_2, z_3)-g(t, \tilde{z}_1, \tilde{z}_2, \tilde{z}_3)| \leqslant \sum_{j=1}^{3} P_j|z_j-\tilde{z}_j| \tag{1.6}$$

and the function $[0, h] \ni t \to g(t, 0, 0, 0) \in R$ is sommable on $[0, h]$. It is to notice that the general linear equation of the fourth order

$$y^{(IV)} + p_4(t)y''' + p_3(t)y'' + p_2(t)y' + p_1(t)y + p_0(t) = 0$$

with the coefficients of class C^3 can be transformed to an equation of form (1.1) with Lipschitzien right-hand side by a linear transformation $y(t) = z(t)\exp\left(-\frac{1}{4}\int p_4(t)\,dt\right)$.

Similarly, we shall assume that the function $\tilde{g}\colon \{2, \ldots, n-2\} \times \times R^3 \to R$ is continuous and satisfies the inequality

$$|\tilde{g}(i, w_1, w_2, w_3)| \leqslant P_0 + \sum_{j=1}^{3} \tilde{P}_j |w_j| \qquad (i = 2, \ldots, n-2), \tag{1.7}$$

or the more restrictive Lipschitz condition

$$|\tilde{g}(i, w_1, w_2, w_3) - \tilde{g}(i, \tilde{w}_1, \tilde{w}_2, \tilde{w}_3)| \leqslant \sum_{j=1}^{3} \tilde{P}_j |w_j - \tilde{w}_j| \tag{1.8}$$
$$(i = 2, \ldots, n-2, (w_1, w_2, w_3), (\tilde{w}_1, \tilde{w}_2, \tilde{w}_3) \in R^3),$$

where $\tilde{P}_j$ $(j = 0, \ldots, 3)$ are fixed real numbers.

The aim of this paper is to present some theorems concerning the existence and uniqueness of solutions of problem (1.1) and (1.2), to give analogous theorems for discrete problem (1.3) and (1.4), and to prove a theorem concerning the convergence of solutions of appropriately defined discrete problems of form (1.3) and (1.4) to the solution of problem (1.1) and (1.2).

The proofs of the existence and uniqueness theorems for problem (1.1) and (1.2) will be based on theorems of Lasota [3] (Theorem 2.1 and Corollary 2.1) for the general linear problems in the theory of differential equations. Similarly, the proof of the existence and uniqueness theorems for problem (1.3) and (1.4) will be based on discrete analogues of the theorems of Lasota given in [2] (Theorem 4.1 and Theorem 4.2).

All these theorems, as well for continuous case as for discrete one, allow us, roughly speaking, to deduce the existence or uniqueness of solutions of a linear problem for vectorial non-linear differential or difference equations with single-valued right-hand sides if a homogeneous problem for appropriately associated differential or difference equations with multi-valued right-hand sides (the contigent equations) has only the trivial solution. Thus, in order to apply these theorems, it is necessary to fix in advance conditions which assure the required uniqueness of solutions of mentioned problems for suitable contingent equations. In case of problems (1.1) and (1.2), and (1.3) and (1.4) such conditions may be given owing to à priori estimates for solutions of some differential and difference inequalities. For to obtain these estimates we make use of inequalities of Wirtinger type which, both continuous and discrete cases, are given in [1].

Finally, let us notice that in recent years an intensive development of boundary problems for difference equations may be observed, generally as numerical aids to problems for differential equations (see for instance [4] and [6]). In this situation it seems useful to give a theorem concerning the possibility of approximating solutions of boundary value problems for differential equations by solutions of boundary value problems for suitably chosen difference equations.

In Section 2 we give à priori estimates for solutions of some differential and difference inequalities. Section 3 contains the existence and uniqueness theorems and Section 4 deal with the above mentioned approximation problem. Finally, some final remarks are given in Section 5.

Throughout the paper we shall make use of generally accepted notations and notions which are explained in detail in [1], [2] and [3].

2. A priori estimates. Let $z_1 = \frac{3}{2}\pi + \varepsilon$ $(\varepsilon > 0)$ be the smallest positive root of the equation $chz = \sec z$ (see also Section 2 of [1]).

THEOREM 2.1. *If $y\colon [0, h] \to R$ is a solution of the differential inequality*

$$y^{(IV)} \leqslant P_3|y''| + P_2|y'| + P_2|y| + P_0, \tag{2.1}$$

satisfying the boundary condition

$$y(0) = y'(0) = y'(h) = y(h) = 0, \tag{2.2}$$

and if the non-negative constants P_j $(j = 1, 2, 3)$ fulfil the inequality

$$\varrho = P_3\left(\frac{2h}{3\pi+2\varepsilon}\right)^2 + P_2\frac{h}{\pi}\left(\frac{2h}{3\pi+2\varepsilon}\right)^2 + P_1\left(\frac{2h}{3\pi+2\varepsilon}\right)^4 < 1, \tag{2.3}$$

then

$$\begin{aligned} \|y''\| &\leqslant \left(\frac{2h}{3\pi+2\varepsilon}\right)^2 \frac{P_0\sqrt{h}}{1-\varrho}, \\ \|y'\| &\leqslant \frac{h}{\pi}\left(\frac{2h}{3\pi+2\varepsilon}\right)^2 \frac{P_0\sqrt{h}}{1-\varrho}, \\ \|y\| &\leqslant \left(\frac{2h}{3\pi+2\varepsilon}\right)^4 \frac{P_0\sqrt{h}}{1-\varrho}. \end{aligned} \tag{2.4}$$

($\| \ \|$ — denotes the usual norm in the space $L^2([0, h])$ of all square sommable functions defined on $[0, h]$).

Proof. Multiplying by $|y|$ and then integrating on interval $[0, h]$ both sides of inequality (2.1), we get

$$\int_0^h |y^{(IV)}\cdot y|\,dt \leqslant P_3\int_0^h |y''\cdot y|\,dt + P_2\int_0^h |y'\cdot y|\,dt + P_1\int_0^h y^2\,dt + P_0\int_0^h |y|\,dt. \tag{2.5}$$

Integrating by parts and applying inequalities (2.6), (2.8) from [1] and the Schwarz inequality, we obtain

$$\int_0^h |y^{(\mathrm{IV})}y|\,dt \geqslant \Big|\int_0^h y^{(\mathrm{IV})}y\Big|\,dt = \int_0^h (y'')^2 dt,$$

$$\int_0^h |y''|\,|y|\,dt \leqslant \Big(\int_0^h (y'')^2 dt\Big)^{1/2}\Big(\int_0^h y^2 dt\Big)^{1/2} \leqslant \Big(\frac{2h}{3\pi+2\varepsilon}\Big)^2 \int_0^h (y'')^2 dt,$$

$$\int_0^h |y'|\,|y|\,dt \leqslant \Big(\int_0^h (y')^2 dt\Big)^{1/2}\Big(\int_0^h y^2 dt\Big)^{1/2} \leqslant \frac{h}{\pi}\Big(\frac{2h}{3\pi+2\varepsilon}\Big)^2 \int_0^h (y'')^2 dt,$$

$$\int_0^h |y|\,dt \leqslant \sqrt{h}\,\Big(\int_0^h y^2 dt\Big)^{1/2} \leqslant \sqrt{h}\,\Big(\frac{2h}{3\pi+2\varepsilon}\Big)^2 \int_0^h (y'')^2 dt.$$

Now, applying the above estimates and the mentioned inequality (2.8) from [1] to inequality (2.5), and then dividing it by $\big(\int_0^h (y'')^2 dt\big)^{1/2}$ (the case when $\|y''\| = 0$ is trivial since in this case condition (2.2) implies $\|y\| = \|y'\| = 0$) we get

$$(1-\varrho)\Big(\int_0^h (y'')^2 dt\Big)^{1/2} \leqslant P_0\sqrt{h}\,\Big(\frac{2h}{3\pi+2\varepsilon}\Big)^2 .$$

Hence by (2.3) we obtain immediately the first of estimates (2.4). We obtain the second from the first and from the also mentioned Wirtinger inequality (2.6) (from [1]) applied to y'. Similarly, we obtain the third of inequalities (2.4) from the first and from inequality (2.8) in [1]. Thus the proof of the theorem is completed.

Let the operator $T_n\colon K_4^n \to R^{n+1}$, where

$$K_4^n = \{v \in R^{n+1}\colon\ v_0 = v_1 = v_{n-1} = v_n = 0\} \quad (n \geqslant 4),$$

be defined by the formula:

$$T_n v = \nabla^{(2)} \Delta^{(2)} v, \tag{2.6}$$

and let λ_1^n denote its smallest positive eigenvalue (see also Section 3 in [1]).

In the sequel the set $\{0, \ldots, n\}$ will be denoted by N.

The theorem we state below is an exact discrete analogue of Theorem 2.1.

THEOREM 2.2. *If the vector $v \in R^{n+1}$ is a solution of the difference inequality*

$$|\nabla\Delta\nabla\Delta v_i| \leqslant \tilde{P}_3|\nabla\Delta v_i| + \tilde{P}_2|\Delta v_i| + \tilde{P}_1|v_i| + \tilde{P}_0 \quad (i \in N), \tag{2.7}$$

satisfying the boundary condition

$$v_0 = v_1 = v_{n-1} = v_n = 0, \tag{2.8}$$

and if non-negative constants $\tilde{P}_j$ $(j = 1, 2, 3)$ *fulfil the inequality*

$$\varrho_n = \tilde{P}_3 \frac{1}{\sqrt{\lambda_1^n}} + \tilde{P}_2 \frac{1}{2\sqrt{\lambda_1^n}\sin\frac{\pi}{2n}} + \tilde{P}_1 \frac{1}{\lambda_1^n} < 1, \tag{2.9}$$

then

$$\begin{gathered} |\nabla\Delta v| \leqslant \frac{1}{\sqrt{\lambda_1^n}} \frac{\tilde{P}_0\sqrt{n-3}}{1-\varrho_n}, \\ |\Delta v| \leqslant \frac{1}{2\sqrt{\lambda_1^n}\sin\frac{\pi}{2n}} \frac{\tilde{P}_0\sqrt{n-3}}{1-\varrho_n}, \\ |v| \leqslant \frac{1}{\lambda_1^n} \frac{\tilde{P}_0\sqrt{n-3}}{1-\varrho_n} \end{gathered} \tag{2.10}$$

($|\ |$ denotes the Euclidean norm in R^{n+1}).

Proof. The idea of getting suitable estimates in discrete case is the same as in continuous one. As there are, however, some subtle differences in calculations, we repeat the whole reasoning. Multiplying inequality (2.7) by $|v_i|$ $(i \in N)$ and summing with respect to i, we get

$$\begin{aligned} &\Big|\sum_{i=0}^{n} \nabla\Delta\nabla\Delta v_i \cdot v_i\Big| \\ &\qquad \leqslant \tilde{P}_3 \sum_{i=0}^{n} |\nabla\Delta v_i \cdot v_i| + \tilde{P}_2 \sum_{i=0}^{n} |\Delta v_i \cdot v_i| + \tilde{P}_1 \sum_{i=0}^{n} v_i^2 + \tilde{P}_0 \sum_{i=0}^{n} |v_i|. \end{aligned} \tag{2.11}$$

Applying the summation by parts formula — formula (1.3) in [1] (the possibility of applying step by step formula (1.3) follows from analogous reasons as the possibility of summation by parts of scalar product $\langle T_n v_1, v_2\rangle$ in the proof of Theorem 2 of [1]), Schwarz inequality for the vectors of R^{n+1} and inequalities (3.6), (4.2) given in [1] we easily obtain

$$\sum_{i=0}^{n} |\nabla\Delta\nabla\Delta v_i \cdot v_i| \geqslant \Big| - \sum_{i=0}^{n} \nabla\Delta\nabla v_i \cdot v_i\Big| = \sum_{i=0}^{n} (\nabla\Delta v_i)^2,$$

$$\sum_{i=0}^{n} |\nabla\Delta v_i \cdot v_i| \leqslant \Big(\sum_{i=0}^{n} (\nabla\Delta v_i)^2\Big)^{1/2} \Big(\sum_{i=0}^{n} v_i^2\Big)^{1/2} \leqslant \frac{1}{\sqrt{\lambda_1^n}} \sum_{i=0}^{n} (\nabla\Delta v_i)^2,$$

$$\sum_{i=0}^{n} |\Delta v_i \cdot v_i| \leqslant \Big(\sum_{i=0}^{n} (\Delta v_i)^2\Big)^{1/2} \Big(\sum_{i=0}^{n} v_i^2\Big)^{1/2} \leqslant \frac{1}{2\sin\frac{\pi}{2n}} \frac{1}{\sqrt{\lambda_1^n}} \sum_{i=0}^{n} (\nabla \Delta v_i)^2,$$

$$\sum_{i=0}^{n} |v_i| = \sum_{i=2}^{n-2} |v_i| \leqslant \sqrt{n-3}\,\Big(\sum_{i=0}^{n} v_i^2\Big)^{1/2} \leqslant \sqrt{n-3}\,\frac{1}{\sqrt{\lambda_1^n}} \sum_{i=0}^{n} (\nabla \Delta v_i)^2.$$

Now, applying these estimates and inequality (4.2) from [1] to (2.11), and then dividing it by $\big(\sum_{i=0}^{n} (\nabla\Delta v_i)^2\big)$ (the case if $|\nabla\Delta v| = 0$ is trivial since in this case condition (2.8) implies $|v| = |\Delta v| = 0$), we get

$$(1-\varrho_n)\Big(\sum_{i=0}^{n} (\nabla\Delta v_i)^2\Big)^{1/2} \leqslant \frac{1}{\sqrt{\lambda_1^n}} \tilde{P}_0\sqrt{n-3}\,.$$

Hence by (2.9) we obtain immediately the first of inequalities (2.10). The second may be obtained from the first applying to Δv inequality (3.6) from [1], what really can be done since condition (2.8) implies $|\Delta^{(2)} v| = |\nabla\Delta v|$. At last, the third of estimates (2.10) may be obtained from the first by a simple application of inequality (4.2) from [1]. This completes the proof.

3. Existence and uniqueness theorems. In this section we state four theorems from which the first two are related to problem (1.1) and (1.2), and remaining two to problem (1.3) and (1.4).

THEOREM 3.1. *If a real function g, the right-hand side of equation* (1.1), *satisfies in the set $[0,h]\times R^3$ the Carathéodory condition, condition* (1.5) *with a sommable function p_0: $[0,h]\to R$ and if the non-negative constants P_j $(j=1,2,3)$ in* (1.5) *fulfil inequality* (2.3), *then the boundary value problem* (1.1) *and* (1.2) *has at least one solution.*

Proof. Setting $x_1 = y, x_2 = x_1', x_3 = x_2', x_4 = x_3'$ we can write problem (1.1) and (1.2) in vector notation

$$x' = f(t, x), \tag{3.1}$$

$$Lx = r, \tag{3.2}$$

where $x = (x_1, x_2, x_3, x_4)\in C_4([0,h])$ — space of all continuous function x: $[0,h]\to R^4$ with norm $\|x\| = \max\{|x(t)|: t\in[0,h]\}$, $r = (r_{00}, r_{0h}, r_{10}, r_{1h}) \in R^4$ and the mappings f: $[0,h]\times R^4 \to R^4$, L: $C_4([0,h])\to R^4$ are defined by the formulae:

$$f(t, z_1, z_2, z_3, z_4) = \big(z_2, z_3, z_4, g(t, z_1, z_2, z_3)\big),$$

$$Lx = \big(x_1(0), x_1(h), x_2(0), x_2(h)\big).$$

Let F: $[0, h]\times R^4 \to cf(R^4)$ be the map defined by the formula

$$F(t, z_1, z_2, z_3, z_4) = \Big\{q\in R^4\colon\ q_1 = z_2, q_2 = z_3, q_3 = z_4, |q_4| \leqslant \sum_{j=1}^{3} P_j|z_j|\Big\}.$$

From this formula it follows immediately that $F(t, z_1, z_2, z_3, z_4)$ is a closed and convex subset of R^4.

Notice that problem (3.1) and (3.2) is a particular case of general problem (2.2) and (2.3) considered in [3]. Thus, owing to Theorem 2.1 of Lasota [3] mentioned in Introduction, it is sufficient for the proof to verify that

1° F satisfies the Carathéodory condition (in the sense given in [3]), for every fixed $t\in[0, h]$ it is homogeneous with respect to $z = (z_1, z_2, z_3, z_4)$ (i.e. $F(t, \lambda z) = \lambda F(t, z)$ for $\lambda\in R$), and the function φ: $[0, h] \to R$ defined by the formula $\varphi(t) = \sup\{|F(t, z)|\colon |z| = 1\}$ is sommable on $[0, h]$,

2° f satisfies the Carathéodory condition and

$$\lim_{k\to\infty}\frac{1}{k}\int_0^h \sup_{|z|\leqslant k}\delta\big(f(t, z),\ F(t, z)\big)\,dt = 0 \tag{3.3}$$

($\delta\big(f(t, z), F(t, z)\big)$ denotes the distance of vector $f(t, z)$ to set $F(t, z)$),

3° L is continuous and homogeneous (i.e. $L\lambda x = \lambda Lx$ for $\lambda\in R$),

4° the homogeneous problem

$$x'\in F(t, x), \tag{3.4}$$

$$Lx = 0 \tag{3.5}$$

has only trivial solution.

For to prove 1°, notice that for every fixed $t\in[0, h]$ the conditions of homogeneouity and continuity with respect to z are simple consequences of the definition of F.

Next, notice that for any closed fixed set $A \subset R^4$ and arbitrary fixed $z\in R^4$ the set $\{t\in[0, h]\colon F(t, z) \cap A \neq \emptyset\}$ is either empty or the whole interval $[0, h]$, so that it is measurable (in Lebesgue sense), what means that for any fixed $z\in R^4$, F is measurable with respect to t.

Finally, the sommability of φ on interval $[0, h]$ follows directly from its form

$$\varphi(t) = \sup\left\{\sqrt{z_2^2+z_3^2+z_4^2+\sum_{j=1}^{3}P_j|z_j|}\colon\ |z| = 1\right\} \qquad (t\in[0, h]).$$

The required in 2° Carathéodory condition for the map f follows easily from our assumptions about the function g.

Condition (3.3) may be verify by a strightforward calculation. Indeed, by the assumption of the sommability of p_0, we have

$$\lim_{k\to\infty}\frac{1}{k}\int_0^h \sup_{|z|\leqslant k}\delta\big(f(t,z),F(t,z)\big)dt=\lim_{k\to\infty}\frac{1}{k}\int_0^h p_0(t)dt=0.$$

Condition 3° is a consequence of definition L.

For to prove 4°, suppose that the vector $x=(x_1,x_2,x_3,x_4)\in C_4([0,h])$ is a solution of problem (3.4) and (3.5). Setting $y=x_1$, we notice that the function y is a solution of the differential inequality (2.1) satisfying condition (2.2).

Hence, in view of Theorem 2.1, we obtain from the third of estimates (2.4) with $P_0=0$ that $\|y\|=0$ what, by the absolute continuity of y, implies that $y(t)=0$ for $t\in[0,h]$.

Thus, we have $x=0$ and this completes the proof.

On an analogous way we obtain

THEOREM 3.2. *If a real function g, the right-hand side of equation* (1.1), *is for any fixed $(z_1,z_2,z_3)\in R^3$ measurable with respect to t, and satisfies in the set $[0,h]\times R^3$ the Lipschitz condition* (1.6), *where non-negative constants P_j $(j=1,2,3)$ fulfil inequality* (2.3), *and if the function $[0,h)\ni t\to g(t,0,0,0)\in R$ is sommable on interval $[0,h]$, then the boundary value problem* (1.1) *and* (1.2) *has exactly one solution.*

Proof. As in the proof of the preceding theorem we replace problem (1.1) and (1.2) by the problem of the form (3.1) and (3.2), and we define the mappings f, F, L. On a similar way one can verify that f, F, L satisfy conditions 1° and 4° defined in the proof of Theorem 3.1, and the two following conditions which replace conditions 2° and 3°.

5° f is measurable with respect to t for any fixed $z\in R^4$ and satisfies the condition

$$f(t,z)-f(t,w)\in F(t,z-w),\quad \int_0^h |f(t,0)|\,dt<+\infty, \tag{3.6}$$

6° L is a linear (additive and homogeneous) and continuous map.

Now, the assertion of our theorem is an immediate conclusion from Corollary 2.1 in [3].

Two following theorems are exact discrete analogues of Theorem 3.1 and 3.2.

THEOREM 3.3. *If a real function $\tilde{g}$, the right-hand side of equation* (1.3), *is continuous in the set $\{2,\ldots,n-2\}\times R^3$, satisfies condition* (1.7), *and if the non-negative constants $\tilde{P}_j$ $(j=1,2,3)$ in* (1.7) *fulfil inequality* (2.9), *then the boundary value problem* (1.3) *and* (1.4) *has at least one solution.*

Proof. Setting $u_i^1 = v_i$, $u_i^2 = \Delta u_i^1$, $u_1^3 = \nabla u_i^2$, $u_i^4 = \Delta u_i^3$ $(i \in N)$, we write equation (1.3) as system

$$\Delta u_i^1 = u_i^2, \quad \nabla u_i^2 = u_i^3, \quad \Delta u_i^3 = u_i^4, \quad \nabla u_i^4 = \tilde{g}(i, u_i^1, u_i^2, u_i^3)$$
$$(i = 2, \ldots, n-2),$$

which with the boundary condition (1.4) has in vector notation the following form

$$\Delta_s u_i = \tilde{f}(i, u_i) \quad (i \in N, s = (1, -1, 1, -1)), \tag{3.7}$$

$$\tilde{L}u = \tilde{r}, \tag{3.8}$$

where

$$\tilde{r} = (\tilde{r}_{00}, \tilde{r}_{0n}, \tilde{r}_{10}, \tilde{r}_{1n}) \in R^4, \quad \Delta_s u_i = (\Delta u_i^1, \nabla u_i^2, \Delta u_i^3, \nabla u_i^4) \quad (i \in N),$$

and the mappings $\tilde{f}$: $N \times R^4 \to R^4$, L: $(R^4)^{n+1} \to R^4$ are defined by the formulae

$$\tilde{f}(i, w^1, w^2, w^3, w^4) = \begin{cases} (w^2, 0, 0, 0), & i = 0, \\ (w^2, w^3, w^4, 0), & i = 1, \\ \big(w^2, w^3, w^4, \tilde{g}(i, w^1, w^2, w^3)\big), & i = 2, \ldots, n-2, \\ (w^2, w^3, 0, 0), & i = n-1, \\ (0, 0, 0, 0), & i = n, \end{cases}$$

$$\tilde{L}(u_0, \ldots, u_n) = (u_0^1, u_n^1, u_0^2, u_{n-1}^2).$$

Define mapping $\tilde{F}$: $N \times R^4 \to cf(R^4)$ putting

$$\tilde{F}(i, w^1, w^2, w^3, w^4) = \begin{cases} \{(w^2, 0, 0, 0)\}, & i = 0, \\ \{(w^2, w^3, w^4, 0)\}, & i = 1, \\ \{q \in R^4\colon q^1 = w^2, q^2 = w^3, q^3 = w^4, & \\ \qquad |q^4| \leqslant \sum_{j=1}^{3} \tilde{P}_j |w^j|\}, & i = 2, \ldots, n-2, \\ \{(w^2, w^3, 0, 0)\}, & i = n-1, \\ \{(0, 0, 0, 0)\}, & i = n \end{cases}$$

(the convexity and closedness of $\tilde{F}(i, w^1, w^2, w^3, w^4)$ for $i \in N$, $w = (w^1, w^2, w^3, w^4) \in R^4$, is evident).

Let us notice that problem (3.1) and (3.2) is a particular case of general problem (4.3) and (4.4) considered in [2] (namely, we have $m = 4$ and the components of multiindex s, which appear in definition of Δ_s are fixed as $s_1 = s_3 = 1$, $s_2 = s_4 = -1$).

Thus, in view of Theorem 4.1 in [2], it is sufficient for the proof of our theorem to verify that:

1° The components of $\tilde{F}$ satisfy the condition $\tilde{F}^k(0, w) = \{0\}$ if $s_k = -1$ and $\tilde{F}^k(n, w) = \{0\}$ if $s_k = 1$ $(w \in R^4, k = 1, \ldots, 4)$, $\tilde{F}$ is upper semi-continuous and homogeneous with respect to w $(\tilde{F}(i, \lambda w) = \lambda F(i, w)$ for $\lambda \in R)$.

2° The components of $\tilde{f}$ satisfy the condition $\tilde{f}^k(0, w) = 0$ if $s_k = -1$ and $\tilde{f}^k(n, w) = 0$ if $s_k = 1$ $(w \in R^4, k = 1, \ldots, 4)$, $\tilde{f}$ is continuous and satisfies the condition

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=0}^{n} \sup_{|w|\leqslant k} \delta\big(\tilde{f}(i, w), \tilde{F}(i, w)\big) = 0. \tag{3.9}$$

3° $\tilde{L}$ is continuous and homogeneous, i.e. $\tilde{L}\lambda u = \lambda \tilde{L} u$ $\big(\lambda \in R,\ u \in (R^4)\big)^{n+1}$.

4° The homogeneous problem

$$\Delta_s u_i \in \tilde{F}(i, u_i) \quad (i \in N), \tag{3.10}$$

$$\tilde{L}u = \tilde{r} \tag{3.11}$$

has only the trivial solution.

Conditions 1°, 2° and 3° follow simply from the definitions of mappings $\tilde{f}, \tilde{F}, \tilde{L}$ and from the assumptions of our theorem. Only for condition (3.9) we give a strightforward calculation

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=0}^{n} \sup_{|w|\leqslant k} \delta\big(\tilde{f}(i, w), \tilde{F}(i, w)\big)$$

$$= \lim_{k\to\infty} \frac{1}{k} \sum_{i=0}^{h} \sup_{|w|\leqslant k} \Big(\inf\Big\{|\tilde{g}(i, w^1, w^2, w^3) - q^4| : \ |q^4| \leqslant \sum_{j=1}^{3} \tilde{P}_j |w^j|\Big\}\Big)$$

$$= \lim_{k\to\infty} \frac{1}{k} (n+1) \tilde{P}_0 = 0.$$

For to prove 4°, suppose that $u \in (R^4)^{n+1}$ is a solution of problem (3.10) and (3.11). Then the components of vector u satisfy the conditions

$$\Delta u_i^1 = u_i^2, \quad \nabla u_i^2 = u_i^3, \quad \Delta u_i^3 = u_i^4, \quad |\nabla u_i^4| \leqslant \sum_{j=1}^{3} \tilde{P}_j |u_i^j| \quad (i \in N).$$

Hence, setting $u_i^1 = v_i$ $(i \in N)$, we conclude by the homogeneouity condition (3.11) that vector $v = (v_0, \ldots, v_n) \in R^{n+1}$ is a solution of difference inequality (2.7) satisfying condition (2.8).

Therefore, by Theorem 2.2, we obtain from the third of estimates (2.10) with $\tilde{P}_0 = 0$ that $|v| = 0$. Thus all components of v are equal to zero and in consequence $u = 0$.

This completes the proof.

In a similar way we obtain

THEOREM 3.4 *If a real function $\tilde{g}$, the right-hand side of equation* (1.3), *satisfies in set* $\{2, \ldots, n-2\} \times R^3$ *the Lipschitz condition* (1.8), *and if the non-negative constants* $\tilde{P}_j$ $(j = 1, 2, 3)$ *in* (1.8) *fulfil inequality* (2.9), *then the boundary value problem* (1.3) *and* (1.4) *has exactly one solution.*

Proof. We reconsider problem (1.3) and (1.4) in form (3.7) and (3.8), and we define the mappings $\tilde{f}$, $\tilde{F}$, $\tilde{L}$ as in the proof of the preceding theorem. Similarly as there one can verify that $\tilde{f}$, $\tilde{F}$, $\tilde{L}$ satisfies conditions 1° and 4° defined in the proof of Theorem 3.3 and the two following conditions which replace conditions 2° and 3°.

5° $\tilde{f}$ satisfies condition 2° defined in the proof of Theorem 3.3 with (3.9) replaced by

$$\tilde{f}(i, z) - \tilde{f}(i, w) \in \tilde{F}(i, z-w) \quad (i \in N,\ w, z \in R^4). \tag{3.12}$$

6° $\tilde{L}$ is linear (additive and homogeneous) and continuous.

Now, a strightforward application of Theorem 4.2 in [2] completes the proof.

4. Approximation theorem. Together with problem (1.1) and (1.2) we consider now a sequence (for $n = 1, 2, \ldots$) of discrete boundary value problems

$$\nabla\Delta\nabla\Delta v_i^n = h_n^4 g\left(t_i^n, v_i^n, \frac{\Delta v_i^n}{h_n}, \frac{\nabla\Delta v_i^n}{h_n^2}\right) \quad (i = 2, \ldots, n-2), \tag{4.1}$$

$$v_0^n = r_{00}, \quad v_n^n = r_{oh}, \quad \Delta v_0^n = h_n r_{10}, \quad \nabla v_n^n = h_n r_{1h}, \tag{4.2}$$

where the function g: $[0, h] \times R^3 \to R$ is the right-hand side of equation (1.1) and h_n, t_i^n are given by the formulae

$$h_n = \frac{h}{n}, \quad t_i^n = i \cdot h_n \quad (i \in N). \tag{4.3}$$

Let function $\tilde{g}_n$: $\{2, \ldots, n-2\} \times R^3 \to R$ be defined by the formula

$$\tilde{g}_n(i, w_1, w_2, w_3) = h_n^4 g\left(t_i^n, w_1, \frac{w_2}{h_n}, \frac{w_3}{h_n^2}\right) \quad (i = 2, \ldots, n-2). \tag{4.4}$$

The theorem we state below states the possibility of approximating (as $n \to \infty$) the solution of problem (1.1) and (1.2), if it is unique, by the solutions of problems (4.1) and (4.2).

THEOREM 4.1 *If the real function g, the right-hand side of equation* (1.1), *is continuous in* $[0, h) \times R^3$, *satisfies the Lipschitz condition* (1.6) *and if the non-negative constants* P_j $(j = 1, 2, 3)$ *in* (1.6) *fulfil inequality* (2.3), *then*

1° *for n sufficiently large there exists exactly one solution $v^n = (v_0^n, \ldots, v_n^n)$ of problem* (4.1) *and* (4.2),

2° $\lim_{n\to\infty} |v_i^n - y(t_i^n)| = 0$ $(i \in N)$, *where y denotes the solution of problem* (1.1) *and* (1.2), *and this convergence is uniform with respect to i.*

Proof. First of all, notice that the assumptions of our theorem imply the assumptions of Theorem 3.2, so problem (1.1) and (1.2) has exactly one solution. It is easy to see that the functions $\tilde{g}_n$ defined by (4.4) satisfy in set $\{2, \ldots, n-2\} \times R^3$ the Lipschitz condition

$$(4.5) \qquad |\tilde{g}_n(i, w_1, w_2, w_3) - \tilde{g}_n(i, \tilde{w}_1, \tilde{w}_2, \tilde{w}_3)| \leqslant P_1 h_n^4 |w_1 - \tilde{w}_1| + P_2 h_n^3 |w_2 - \tilde{w}_2| + P_3 h_n^2 |w_3 - \tilde{w}_3|.$$

We set

$$\varrho_n = P_3 \left(\frac{h}{n}\right)^2 \frac{1}{\sqrt{\lambda_1^n}} + P_2 \left(\frac{h}{n}\right)^3 \frac{1}{2\sqrt{\lambda_1^n}\, \sin\dfrac{\pi}{2n}} + P_1 \left(\frac{h}{n}\right)^4 \frac{1}{\lambda_1^n},$$

where λ_1^n is the smallest positive eigenvalue of the operator T_n defined by formula (2.6).

It is to notice that, by Theorem 4 in [1], we have

$$\lim_{n\to\infty} n^4 \lambda_1^n = h^4 \lambda_1,$$

where $\lambda_1 = (3\pi + 2\varepsilon/2h)^4$ (see also Section 2.2 of [1]).

Therefore, by the definition of ϱ_n, we obtain

$$(4.6) \qquad \lim_{n\to\infty} \varrho_n = P_3 \left(\frac{2h}{3\pi + 2\varepsilon}\right)^2 + P_2 \frac{h}{\pi} \left(\frac{2h}{3\pi + 2\varepsilon}\right)^2 + P_1 \left(\frac{2h}{3\pi + 2\varepsilon}\right)^4$$

what, by inequality (2.3), implies that $\varrho_n < 1$ for sufficiently large n.

Since problem (4.1) and (4.2) is a particular case of problem (1.3) and (1.4), and the assumptions of Theorem 3.4 are fulfilled for sufficiently large n, so for such n problem (4.1) and (4.2) has exactly one solution, what proves 1° of the assertion of our theorem.

For to prove 2°, we set $y_i^n = y(t_i^n)$ $(i \in N)$, where $y\colon [0, h] \to R$ is the (unique) solution of problem (1.1) and (1.2). By the Taylor formula, applied to the function y in a neighbourhood of t_i^n, we get

$$(4.7) \qquad \frac{1}{h_n} \Delta y_i^n = y'(\xi_i^n), \quad \frac{1}{h_n^2} \nabla\Delta y_i^n = y''(\eta_i^n), \quad \frac{1}{h_n^3} \Delta\nabla\Delta y_i^n = y'''(\sigma_i^n),$$

$$\frac{1}{h_n^4} \nabla\Delta\nabla\Delta y_i^n = y^{(\mathrm{IV})}(\tau_i^n) \qquad (i = 2, \ldots, n-2),$$

where $\xi_i^n, \eta_i^n, \sigma_i^n, \tau_i^n \in (t_{i-2}^n, t_{i+2}^n)$.

Hence, by the assumption that y is a solution of equation (1.1), we get

$$\nabla\Delta\nabla\Delta y_i^n = h_n^4 y^{(IV)}(\tau_i^n) = h_n^4 g\big(\tau_i^n, y(\tau_i^n), y'(\tau_i^n), y''(\tau_i^n)\big) \qquad (i = 2, \ldots, n-2),$$

which can be written in another form

$$\text{(4.8)} \quad \nabla\Delta\nabla\Delta y_i^n = h_n^4 g\left(t_i^n, y_i^n, \frac{\Delta y_i^n}{h_n}, \frac{\nabla\Delta y_i^n}{h_n^2}\right) + h_n^4\delta(t_i^n) \quad (i = 2, \ldots, n-2),$$

where

$$\delta(t_i^n) = g\big(\tau_i^n, y(\tau_i^n), y'(\tau_i^n), y''(\tau_i^n)\big) - g\left(t_i^n, y_i^n, \frac{\Delta y_i^n}{h_n}, \frac{\nabla\Delta y_i^n}{h_n^2}\right).$$

From (4.7) and from the definition of h_n we obtain

$$\lim_{n\to\infty}(\xi_i^n - t_i^n) = \lim_{n\to\infty}(\eta_i^n - t_i^n) = \lim_{n\to\infty}(\sigma_i^n - t_i^n) = \lim_{n\to\infty}(\tau_i^n - t_i^n) = 0$$

what, in view of the continuity of g and y, y', y'', implies

$$\text{(4.9)} \qquad \lim_{n\to\infty}\delta_n = 0,$$

where $\delta_n = \max_{i\in N}|\delta(t_i^n)|$.

Set

$$u_i^n = v_i^n - y_i^n \quad (i \in N).$$

Subtracting side by side equation (4.8) from equation (4.1) we obtain, by the Lipschitz condition and the definitions of δ_n, and difference operators, the following inequality

$$\text{(4.10)} \quad |\nabla\Delta\nabla\Delta u_i^n| \leqslant h_n^4 P_1|u_i^n| + h_n^3 P_2|\Delta u_i^n| + h_n^2 P_3|\nabla\Delta u_i^n| + h_n^4\delta_n \quad (i \in N),$$

and we observe that the boundary value condition

$$\text{(4.11)} \quad u_0^n = 0, \quad u_n^n = 0, \quad \Delta u_0^n = \alpha_n\cdot h_n, \quad \nabla u_n^n = \beta_n\cdot h_n$$

is fulfilled, where constants α_n, β_n are defined by the formulae

$$\alpha_n = r_{10} - y'(\theta_n\cdot h_n), \quad \beta_n = y'(h - \theta_n' h_n) - r_{1h} \quad (\theta_n, \theta_n' \in (0,1)).$$

From the above definition of α_n, β_n we obtain by (4.2)

$$\text{(4.12)} \qquad \lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\beta_n = 0.$$

Let p be a polynomial of the real variable t the third degree at most satisfying the condition

$$\text{(4.13)} \quad p(0) = 0, \quad p(h) = 0, \quad p(h_n) = \alpha_n h_n, \quad p(h - h_n) = \beta_n h_n.$$

It is easily seen that such polynomial p has to be of the form

$$p(t) = a_n t^3 + b_n t^2 + c_n t, \tag{4.14}$$

where

$$a_n = \frac{n^2(\alpha_n - \beta_n)}{(n-1)(n-2)h^2}, \quad b_n = \frac{n\big((n-1)\beta_n - (2n-1)\alpha_n\big)}{(n-1)(n-2)},$$

$$c_n = \frac{n\big((n-1)\alpha_n - \beta_n\big)}{(n-1)(n-2)},$$

and by (4.12) we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = 0. \tag{4.15}$$

Setting

$$w_i^n = u_i^n - p_i^n \quad (i \in N),$$

where $p_i^n = p(t_i^n)$, we obtain easily from (4.10)

$$|\nabla\Delta\nabla\Delta w_i^n| \leqslant h_n^4 P_1 |w_i^n + p_i^n| + h_n^3 P_2 |\Delta w_i^n + \Delta p_i^n| +$$
$$+ h_n^2 P_3 |\nabla\Delta w_i^n + \nabla\Delta p_i^n| + h_n^4 \delta_n \quad (i \in N),$$

what can be written also in the form

$$|\nabla\Delta\nabla\Delta w_i^n| \leqslant h_n^4 P_1 |w_i^n| + h_n^3 P_2 |\Delta w_i^n| + h_n^2 P_3 |\nabla\Delta w_i^n| + h_n^4 P_0^n \quad (i \in N),$$

where

$$P_0^n = \max_{i\in N}\left(\delta_n + P_1|p_i^n| + P_2\frac{|\Delta p_i^n|}{h_n} + P_3\frac{|\nabla\Delta p_i^n|}{h_n^2}\right).$$

Obviously, by (4.9), (4.14) and (4.15), we have

$$\lim_{n\to\infty} P_0^n = 0, \tag{4.16}$$

and from conditions (4.11) and (4.13) it follows that

$$w_0 = w_1 = w_{n-1} = w_n = 0.$$

Applying to vector $w = (w_0, \ldots, w_n) \in R^{n+1}$ Theorem 2.2 we get (by the second of estimates (2.10))

$$|\Delta w^n| \leqslant \frac{h_n^4}{2\sqrt{\lambda_1^n}\sin\dfrac{\pi}{2n}} \cdot \frac{P_0^n\sqrt{n-3}}{1-\varrho_n}. \tag{4.17}$$

Hence, using the evident identities

$$w_i^n = \sum_{k=0}^{i-1} \Delta w_k^n = \sum_{k=1}^{i-1} \Delta w_k^n, \quad w_i^n = -\sum_{k=i}^{n} \Delta w_k^n = -\sum_{k=i}^{n-2} \Delta w_k^n,$$

we obtain

$$(4.18)\qquad |w_i^n| \leqslant \tfrac{1}{2}\sum_{k=1}^{n-2}|\Delta w_k^n| \leqslant \tfrac{1}{2}\sqrt{n-2}\,|\Delta w^n| \leqslant Q_n\frac{P_0^n}{1-\varrho_n},$$

where

$$Q_n = \tfrac{1}{2}\sqrt{\frac{(n-2)(n-3)}{n^2}}\;\frac{1}{2n\sin\dfrac{\pi}{2n}}\;\frac{h^4}{\sqrt{n^4\lambda_1^n}}.$$

By Theorem 4 of [1] it is easy to see that

$$\lim_{n\to\infty} Q_n = \frac{1}{2\pi}\frac{h^2}{\sqrt{\lambda_1}},$$

what owing to (4.6) and the convergence of P_0^n to zero implies the uniform with respect to i convergence of w_i^n to zero, as $n\to\infty$.

Hence, by (4.14) and (4.15), part 2° of the assertion easily follows what completes the proof.

5. Final remarks. Some further questions are related to the ones considered in this paper. For instance, one can to ask if the methods used in this paper allow to obtain some similar results

1° for differential and difference equations of general form (i.e. if the functions g or $\tilde{g}$ depend on y''' and $\Delta\nabla\Delta v_i$, respectively),

2° for differential and difference equations of higher orders.

The difficulty in a giving a positive answer to this question lies in the impossibility of estimating from above the integral $\int_0^h |y'''y|\,dt$ by $\int_0^h (y'')^2dt$ (and in analogous difficulty for suitable sums in discrete case) in case 1° (for non-linear equations since, for linear this difficulty may be omitted — see Introduction), and in too much complicated computations leading to evaluations of λ_1 or λ_1^n in general inequalities (2.2) and (3.5) (for $l>2$) from [1] in case 2°.

Further question which concerns the problem of an optimal evaluation of the length of interval $[0,h]$ on which the a priori estimates of solutions of differential and difference inequality hold true is far much complicated. Another way leading to an estimation for $\int_0^h (y'')^2dt$ (or $|\nabla\Delta v|$) which consists in using first the Opial inequality — see [7] (or its discrete analogue — [4]) and next the Wirtinger inequality (or its discrete analogue — see inequalities (2.6) and (3.6) in [1]) gives less exact result. It seems, however, that some progress here may be obtained by applying the known principle of maximum of Pontriagin (see [5]).

References

[1] Z. Denkowski, *The inequalities of the Wirtinger's type and their discrete analogues*, Zeszyty Naukowe U. J., Prace Mat. (to appear).

[2] — *Linear problems for systems of difference equations*, this fasc., p. 77-86.

[3] A. Lasota, *Une généralisation du premier théorème de Fredholm et ses applications à la théorie des équations différentielles ordinaires*, Ann. Polon. Math. 18 (1966), p. 65-77.

[4] — *A discrete boundary value problem*, ibidem 20 (1968), p. 183-190.

[5] — et Z. Opial, *L'application du principe de Pontriagin à l'evaluation de l'intervalle d'existence et d'unicité des solutions d'un problème aux limites*, Bull. Acad. Polon. Sci., Sér. Sci. Math., Astr., Phys. 11 (1963), p. 41-46.

[6] M. Lees, *A boundary value problem for non-linear ordinary differential equations*, J. Math. Mech. 10 (1961), p. 423-430.

[7] Z. Opial, *Sur une inégalité*, Ann. Polon. Math. 8 (1960), p. 29-32.

Reçu par la Rédaction le 6. 9. 1969