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On meromorphic solutions of a functional equation

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We consider the problem of the existence and uniqueness of meromorphic solutions in a domain U for the equation

$$\varphi[f(z)]-g(z)\varphi(z) = h(z)\,, \tag{1}$$

where $\varphi(z)$ is the unknown function and $f(z)$, $g(z)$ and $h(z)$ are known functions of one complex variable z.

We assume that:

(I) *The function $f(z)$ is analytic in a domain U, $f(U)\subset U$ and the boundary of the domain U contains at least two finite points.*

(II) *$f(z) = z$ for $z \in U$ if and only if $z = a$, $a \in U$, and*

$$0 < |c| < 1\,, \qquad \textit{where } c = f'(a);$$

(III) *$g(z)$ and $h(z)$ are meromorphic functions in the domain U.*

This problem was investigated by Raclis [3] in the case where $g(z) \equiv -1$ and by Pranger [2] in the case where $g(z) = \text{const}$ under a little more restrictive assumptions concerning the function $f(z)$.

Our considerations will be based on the following theorem, which has been proved by Smajdor [4]:

THEOREM I. *Let us suppose that the function $f(z)$ is analytic at the point $z = a$*

$$f(z) = a+c(z-a)+\sum_{k=2}^{\infty} b_k(z-a)^k \qquad \textit{for } |z-a| < r\,,$$

where $|c| < 1$, and $h(z, w)$ is an analytic function of two complex variables z and w at the point (a, b) with the expansion

$$h(z, w) = \sum_{n,m=0}^{\infty} a_{nm}(z-a)^n(w-b)^m\,, \qquad a_{00} = b$$

valid for $|z-a| \leqslant r_0$, $|w-b| \leqslant R_0$. Moreover, let $1-c^n a_{01} \neq 0$ for every n $(n = 1, 2, \ldots)$.

Then there exists a unique solution φ of the equation

$$\varphi(z) = h\big(z, \varphi[f(z)]\big)$$

analytic at the point $z = a$ and such that $\varphi(a) = b$.

Let us denote by $f^n(z)$ the n-th iteration of the function $f(z)$:

$$f^0(z) = z\,, \quad f^{n+1}(z) = f[f^n(z)]\,.$$

Now we prove the following

LEMMA 1. *Let hypotheses* (I), (II) *be fulfilled. Then the sequence $\{f^n(z)\}$ of the iterates of the function $f(z)$ tends to a uniformly on every compact $K \subset U$.*

Proof. Since $f^n(U) \subset U$ for every positive integer n, the sequence $f^n(z)$ in U omits at least two values, namely those lying on the boundary of U. On account of Montel's theorem [1] the sequence $\{f^n(z)\}$ is a normal family in U. We shall show that the sequence $\{f^n(z)\}$ tends to a constant function equal to a in a neighbourhood of the point $z = a$. There exists a number $\vartheta < 1$ such that $|f'(a)| < \vartheta$ and there exists a number $r > 0$ such that the inequality

$$|f(z) - a| = |f(z) - f(a)| < \vartheta\,|z-a|$$

holds for $z \in U$ and $|z-a| < r$. By induction we obtain the inequality

$$|f^n(z) - a| < \vartheta^n |z-a| \tag{2}$$

for $z \in U$ and $|z-a| < r$. From (2) it follows that $\lim\limits_{n\to\infty} f^n(z) = a$ for $z \in U$ and $|z-a| < r$. Since $f^n(z)$ is a normal family, this completes the proof of Lemma 1.

LEMMA 2. *Let the function $f(z)$ fulfil hypotheses* (I), (II) *and let the functions $g_1(z)$ and $h_1(z)$ be a meromorphic in the domain U. Moreover, let us suppose that the functions $g_1(z)$ and $h_1(z)$ are analytic at the point $z = a$ and $g_1(a) \neq 0$.*

If there exists an analytic function $\varphi(z)$ in the neighbourhood U_a of the point $z = a$ such that

$$(z-a)^r \varphi[f(z)] - g_1(z)\varphi(z) = h_1(z) \quad \text{for } z \in U_a \ (r \in N)\,, \tag{3}$$

then there exists a unique meromorphic function $\psi(z)$ such that

$$(z-a)^r \psi[f(z)] - g_1(z)\psi(z) = h_1(z) \quad \text{for } z \in U\,, \tag{4}$$

$$\psi(z) = \varphi(z) \quad \text{for } z \in U_a\,. \tag{5}$$

Proof. First we suppose that $g_1(z)$ and $h_1(z)$ are analytic functions in the domain U and $g(z) \neq 0$ for $z \in U$. Let $K \subset U$ be any compact. Let us put $K_0 = K$, $K_1 = f(K)$, $K_n = f(K_{n-1}) = f^n(K)$. On account of Lemma 1 it follows that there exists a positive integer n_0 such that $f^n(K) \subset U_a$ for $n \geqslant n_0$. Since $f(z) \in K_n \subset U_a$ for $z \in K_{n-1}$, we can define the function $\varphi_1(z)$ in the following manner:

$$\varphi_1(z) = \frac{(z-a)^r \varphi[f(z)] - h_1(z)}{g_1(z)} \quad \text{for } z \in K_{n-1}.$$

Similary we define the function $\varphi_2(z)$:

$$\varphi_2(z) = \frac{(z-a)^r \varphi_1[f(z)] - h_1(z)}{g_1(z)} \quad \text{for } z \in K_{n-2},$$

$$\vdots$$

$$\varphi_n(z) = \frac{(z-a)^r \varphi_{n-1}[f(z)] - h_1(z)}{g_1(z)} \quad \text{for } z \in K_0 = K.$$

Now we put

$$\psi_K(z) = \varphi_n(z) \quad \text{for } z \in K.$$

We shall prove that $\psi_K(z)$ is independent of the choice of n. Let m be a positive integer, $m \geqslant n_0$ and let $\psi_1(z), \ldots, \psi_m(z)$ be functions defined on K in the same manner as $\varphi_1(z), \ldots, \varphi_n(z)$. We can suppose that $m < n$. Hence and from (3) we obtain

$$\varphi_{n-m}(z) = \varphi(z) \quad \text{for } z \in K_m = f^m(K)$$

or

$$\varphi_{n-m}[f^m(z)] = \varphi[f^m(z)] \quad \text{for } z \in K.$$

It follows that

$$\frac{(z-a)^r \varphi_{n-m}[f^m(z)] - h_1(z)}{g_1(z)} = \frac{(z-a)^r \varphi[f^m(z)] - h_1(z)}{g_1(z)};$$

thus

$$\varphi_{n-m+1}[f^{m-1}(z)] = \psi_1[f^{m-1}(z)].$$

Hence in the same manner we obtain

$$\varphi_{n-m+2}[f^{m-2}(z)] = \psi_2[f^{m-2}(z)].$$

Finally $\varphi_n(z) = \psi_m(z)$ for $z \in K$.

Now we take a compact set $K^* \subset U$ and the function $\psi_{K^*}(z)$. We must prove that $\psi_K(z) = \psi_{K^*}(z)$ for $z \in K \cap K^*$. Let p be a positive integer such that $f^p(K \cup K^*) = f^p(K) \cup f^p(K^*) \subset U_a$ and take the function $\psi_{K \cup K^*}(z)$. Since we may take $n = p$, it follows that $\psi_K(z) = \psi_{K \cup K^*}(z)$ for $z \in K$. Similary we get $\psi_{K^*}(z) = \psi_{K \cup K^*}(z)$ for $z \in K^*$, so $\psi_K(z) = \psi_{K^*}(z)$ for $z \in K \cap K^*$.

Let $\{K_n\}$ be a sequence of compact sets such that $K_n \subset K_{n+1}$ for every n and $\bigcup_{n=1}^{\infty} K_n = U$. We put

$$\psi(z) = \psi_{K_n}(z) \quad \text{for } z \in K_n.$$

Evidently, $\psi(z)$ is an analytic function. It remains to prove that (4) and (5) hold. If $z \in U$, then there exist positive integers p and q such that $z \in K \overset{\mathrm{df}}{=} K_p$ and $f(K) \subset K_q$ and we obtain

$$\begin{aligned}\psi(z) = \psi_K(z) &= \frac{(z-a)^r \varphi_{n-1}[f(z)] - h_1(z)}{g_1(z)} \\ &= \frac{(z-a)^r \psi_{f(K)}[f(z)] - h_1(z)}{g_1(z)} \\ &= \frac{(z-a)^r \psi_{K_q}[f(z)] - h_1(z)}{g_1(z)} \\ &= \frac{(z-a)^r \psi[f(z)] - h_1(z)}{g_1(z)}\,.\end{aligned}$$

Taking $K \subset U^a$ it is easy to verify that (5) holds.

Now we assume that $g_1(z)$ and $h_1(z)$ are meromorphic functions and their poles are different from a. Then, as in the preceding case, we shall obtain meromorphic function $\psi(z)$ in the domain U. In any compact $K \subset U$ the function $\psi(z)$ has a finite number of poles as the quotient of meromorphic functions. This completes the proof of Lemma 2.

By N we denote the set of positive integers. For a function $a(z)$ we denote by $Z(a)$ the order of the zero of $a(z)$ at the point $z = a$ and by $P(a)$ the order of the pole of $a(z)$ at the point $z = a$.

Now we shall prove the following

THEOREM 1. *Let hypotheses* (I)-(III) *be fulfilled. Let us suppose that the functions* $g(z)$ *and* $h(z)$ *are analytic at the point* $z = a$ *and, moreover, that*

$$g(a) \neq 0, 1, c^k \quad \textit{for } k = \pm 1, \pm 2, \ldots$$

Then there exists exactly one meromorphic solution of equation (1) *in the domain* U. *This solution is analytic at the point* $z = a$ *and* $\varphi(a) = \dfrac{h(a)}{1-g(a)}$.

Proof. It is easy to verify that the function

$$h(z, w) = \frac{w - h(z)}{g(z)}$$

is analytic at the point (a, b), where $b = \dfrac{h(a)}{1-g(a)}$ and $h(a, b) = b$. Since $g(a) \neq c^n$, we have

$$1 - a_{01} c^n = 1 - \frac{c^n}{g(a)} \neq 0 \quad \text{for } n \in N\,.$$

According to Theorem 1 there exists exactly one analytic solution of equation (1) in a neighbourhood of the point $z = a$. By Lemma 2 we may extend this solution onto the whole domain U. This is a meromorphic solution of equation (1) in the domain U.

Now suppose that there exists another solution. It must have the form

$$\varphi(z) = \frac{\varphi_1(z)}{(z-a)^r}, \tag{6}$$

where $\varphi_1(z)$ is analytic at the point $z = a$, $\varphi_1(a) \neq 0$ and $r \in N$. We have also

$$f(z) = a + (z-a)f_1(z), \tag{7}$$

where $f_1(z)$ is an analytic function, and $f_1(a) = c$. Putting (6) and (7) in equation (1) we obtain

$$\frac{\varphi_1[f(z)]}{(z-a)^r[f_1(z)]^r} - g(z)\frac{\varphi_1(z)}{(z-a)^r} = h(z). \tag{8}$$

Since $g(a) \neq c^{-r}$, the left-hand side of (8) has at $z = a$ a pole of order r, whereas the right-hand side is analytic at the point $z = a$, which is impossible. This completes the proof.

THEOREM 1'. *Let hypotheses* (I)-(III) *be fulfilled. Let us suppose that* $g(z)$ *and* $h(z)$ *are analytic at* $z = a$ *and,*

$$g(a) = c^{-p}, \quad \textit{where } p \in N.$$

Then there exists in the domain U:

(a) *exactly one meromorphic solution of equation* (1) *that is analytic at* $z = a$;

(b) *a one-parameter family* F *of meromorphic solutions of equation* (1) *such that for every* $\varphi \in F$ *we have* $P(\varphi) = p$.

There are no other meromorphic solutions of equation (1) *in* U.

Proof. (a) The proof is analogous to that of Theorem 1.

(b) Suppose that there exists another meromorphic solution of equation (1). It follows from (a) that it must have form (6), $\varphi_1(a) \neq 0$. Putting (6) in equation (1) we obtain equation (8).

Hence we conclude that necessarily $r = p$. Putting $r = p$ in (8), we get the equation

$$\varphi_1(z) = \frac{\varphi_1[f(z)] - (z-a)^p[f_1(z)]^p h(z)}{g(z)[f_1(z)]^p} \tag{9}$$

$(g(a)[f_1(a)]^p = 1$, and $1 - c^n a_{01} = 1 - c^n \neq 0$ for $n \in N)$.

Theorem I says that there exists a one-parameter family of local analytic solutions of equation (9) (here b may be arbitrary). We can extend these solutions onto the whole domain U. If $\varphi_1(z)$ is a meromorphic solution of equation (9), then $\varphi(z) = \varphi_1(z)/(z-a)^p$ is a meromorphic solution of equation (1). Evidently these are all the meromorphic solutions of equation (1).

THEOREM 2. *Let hypotheses* (I)-(III) *be fulfilled. Let us suppose that* $g(z)$ *is analytic at the point* $z = a$, *and* $h(z) = h_1(z)/(z-a)^q$, *where* $h_1(z)$ *is analytic at the point* $z = a$, $h_1(a) \neq 0$ *and* $q \in N$. *Moreover, let*

$$g(a) \neq 0\,,\ c^k \quad \text{for } k = 0, \pm 1, \pm 2, \ldots$$

Then there exists exactly one meromorphic solution φ *of equation* (1) *in* U, *and* $P(\varphi) = q$.

Proof. If a function $\varphi(z)$ is a meromorphic solution of equation (1), then it must have form (6), where $\varphi_1(a) \neq 0$ and $r \in N$ [1].

Putting (6) and (7) into equation (1), we obtain

$$\frac{\varphi_1[f(z)]}{(z-a)^r[f_1(z)]^r} - g(z)\frac{\varphi_1(z)}{(z-a)^r} = \frac{h_1(z)}{(z-a)^q}\,. \tag{10}$$

Since $g(a) \neq [f_1(a)]^{-r} = c^{-r}$, we conclude that $r = q$. Putting $r = q$ in (10), we obtain for $\varphi_1(z)$ the equation

$$\varphi_1(z) = \frac{\varphi_1[f(z)] - [f_1(z)]^q h_1(z)}{g(z)[f_1(z)]^q}\,. \tag{11}$$

Here $1 - c^n a_{01} = 1 - \dfrac{c^{n-q}}{g(a)} \neq 0$. By Theorem I we get the existence of exactly one solution φ_1 of equation (11) that is analytic at the point $z = a$. By Lemma 2 we can extend this solution onto the whole domain U. The function $\varphi(z) = \varphi_1(z)/(z-a)^q$ is the only meromorphic solution of equation (1).

Remark. The example of the equation

$$\varphi(z^2) - \varphi(z) = 1/z$$

shows that in the case where $c = 0$ there may be no meromorphic solution of equation (1). Indeed, if $\varphi(z)$ is a meromorphic solution of this equation, then $\varphi(z)$ has a form $\varphi(z) = \varphi_1(z)/z^r$, where $\varphi_1(0) \neq 0$ and $r \in N$. But then

[1] Every meromorphic solution of equation (1) must have a pole at $z = a$, for otherwise the left-hand side of (1) would be analytic at $z = a$, whereas the right-hand side has a pole at $z = a$.

the left-hand side has a pole of order $2r$ and the right-hand side has a pole of order one.

THEOREM 2'. *Let hypotheses* (I)-(III) *be fulfilled. Let us suppose that* $g(z)$ *is analytic at* $z = a$, *and* $h(z) = h_1(z)/(z-a)^q$, *where* $h_1(z)$ *is analytic at* $z = a$, $h_1(a) \neq 0$ *and* $q \in N$.

Then (a) *in the case where*

$$g(a) = c^{-q}$$

there exists no meromorphic solution of equation (1).

(b) *in the case where*

$$g(a) = c^{-p}, \quad \text{where } p > q,$$

there exists exactly one meromorphic solution of equation (1) *with* $P(\varphi) = q$ *and there exists a one-parameter family* F *of meromorphic solutions such that we have* $P(\varphi) = p$ *for* $\varphi \in F$.

Proof. The same argument as in the proof of Theorem 2 shows that if $\varphi(z)$ is a solution of equation (1), then $\varphi(z)$ must have form (6), and for $\varphi_1(z)$ we obtain the following equation:

$$\varphi_1(z) = \frac{\varphi_1[f(z)] - h_1(z)[f_1(z)]^q}{g(z)[f_1(z)]^q}. \tag{12}$$

Putting $z = a$ we obtain $\varphi_1(a) = \varphi_1(a) - c^{-q}h_1(a)$, which is impossible. This completes the proof of case (a).

(b) Evidently, every meromorphic solution of equation (1) has form (6). For $\varphi_1(z)$ we obtain equation (8). It is easy to see that a necessary condition of the existence of an analytic solution $\varphi_1(z)$ of equation (8) is $r = q$ or $r = p$.

Let $r = q$; putting $h(z) = h_1(z)/(z-a)^q$ and $r = q$ in (8) we get for $\varphi_1(z)$ equation (12).

Here $1 - c^n a_{01} = 1 - c^n c^{p-q} \neq 0$ for $n = 1, 2, 3, \ldots$ By Theorem I we get exactly one analytic solution of equation (1). By Lemma 2 we may extend this solution onto the whole domain U. Let $\varphi_1(z)$ be a meromorphic solution of equation (12). Then $\varphi(z) = \varphi_1(z)/(z-a)^q$ is a meromorphic solution of equation (1) and, evidently, $P(\varphi) = q$. Putting $h(z) = h_1(z)/(z-a)^q$ and $r = p$ in (8) we get the equation

$$\varphi_1(z) = \frac{\varphi_1[f(z)] - h_1(z)[f_1(z)]^p (z-a)^{p-q}}{g(z)[f_1(z)]^p}. \tag{13}$$

As in case (b) of Theorem 1, we get a one-parameter family of solutions of equation (13). By Lemma 2 we may extend these solutions onto the whole domain U. Let $\varphi_1(z)$ be a meromorphic solution of equation (13). Then $\varphi(z) = \varphi_1(z)/(z-a)^p$ is a meromorphic solution of equation (1). This completes the proof.

THEOREM 3. *Let hypotheses* (I)-(III) *be fulfilled. Suppose that* $g(z)$ *has a pole and* $h(z)$ *is analytic at* $z = a$. *Then there exists exactly one meromorphic solution* φ *of equation* (1). *This solution is analytic at* $z = a$ *and fulfils the condition* $Z(\varphi) = P(g)+Z(h)$.

Proof. Suppose that $\varphi(z)$ is a meromorphic solution of equation (1) that is analytic at $z = a$. Putting in (1) $g(z) = g_1(z)/(z-a)^p$ and $h(z) = (z-a)^q h_1(z)$, where $g_1(z)$ and $h_1(z)$ are analytic at $z = a$, $g_1(a) \neq 0$, $h_1(a) \neq 0$, we see that $Z(\varphi) = p+q$. Hence $\varphi(z)$ must have the form $\varphi(z) = (z-a)^{p+q}\varphi_1(z)$, $\varphi_1(a) \neq 0$. For $\varphi_1(z)$ we get the equation

$$\varphi_1(z) = \frac{(z-a)^p[f_1(z)]^{p+q}\varphi_1[f(z)]-h_1(z)}{g_1(z)} . \tag{14}$$

From Theorem I and Lemma 2 it follows that there exists exactly one meromorphic solution in U of equation (14). Let $\varphi_1(z)$ be this solution. Then $\varphi(z) = (z-a)^{p+q}\varphi_1(z)$ is a meromorphic solution of equation (1). Suppose that there exists another meromorphic solution of equation (1). Since we have already found all solutions that are analytic at $z = a$, it must have form (6), where $\varphi_1(a) \neq 0$ and $r \in N$. Thus we get for $\varphi_1(z)$ the equation

$$\frac{\varphi_1[f(z)]}{(z-a)^r[f_1(z)]^r} - \frac{g_1(z)}{(z-a)^p} \cdot \frac{\varphi_1(z)}{(z-a)^r} = h(z) .$$

On the left-hand side there is a pole of order $p+q$ and on the right-hand side there is an analytic function at $z = a$. This is impossible, which shows that there are no other meromorphic solutions and completes the proof of the theorem.

Similary we can prove the following

THEOREM 4. *Let* (I)-(III) *be fulfilled. Moreover, suppose that* $g(z) = g_1(z)/(z-a)^p$, $h(z) = h_1(z)/(z-a)^q$, *where* $g_1(a) \neq 0$, $h_1(a) \neq 0$, *and* $p, q \in N$.

Then there exists exactly one meromorphic solution $\varphi(z)$ *of equation* (1), *and*

(a) *in the case of* $p = q$ *it is analytic at the point* $z = a$ *and* $\varphi(a) = -h_1(a)/g_1(a)$;

(b) *in the case of* $p < q$ *it has a pole at* $z = a$ *and* $P(\varphi) = q-p$;

(c) *in the case of* $p > q$ *it is analytic at the point* $z = a$ *and* $Z(\varphi) = p-q$.

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