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On a generalization of the convolution

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Let $\mathcal{C}(a, \infty)$ be the set of all continuous real functions f of a real variable $x \geqslant a > -\infty$. Let $K(x, t) \geqslant a$ be a real-valued continuous function in the domain D: $a \leqslant t \leqslant x < \infty$, and let $\varphi(x)$ be a real function of bounded variation in every finite interval $a \leqslant x \leqslant X$. For $f, g \in \mathcal{C}(a, \infty)$ let $f \ast g$ be the function whose value at x is

$$f \ast g(x) = \int_a^x f[K(x, t)]g(t)\,d\varphi(t)\,. \tag{1}$$

If $K(x, t) = x-t$, $\varphi(t) = t$ and $a = 0$, then (1) reduces to the convolution

$$\int_0^x f(x-t)g(t)\,dt\,. \tag{1.1}$$

Therefore we will use for $f \ast g$ the term "generalized convolution".

It is well known that the set $\mathcal{C}(0, \infty)$ forms a commutative ring with respect to addition and multiplication in the sense of (1.1). It follows from Titchmarsh's theorem that this ring has no zero divisor [1].

We raise the following question: For which functions K and φ does the set $\mathcal{C}(a, \infty)$ form a commutative ring without divisor of zero, where the generalized convolution (1) is taken as the ring multiplication?

In connection with this problem we prove here the following

THEOREM. *Let $K(x, t)$ be strictly monotonic in the variable t for every fixed x and continuous in D: $a \leqslant t \leqslant x < \infty$, and let $\varphi(x)$ be a normalized* [1] *monotonic function in $a \leqslant x < \infty$.*

If $\mathcal{C}(a, \infty)$ has no zero divisor with respect to the generalized convolution (1) *and if*

$$1 \ast f = f \ast 1 \in \mathcal{C}(a, \infty) \tag{2}$$

for every $f \in \mathcal{C}(a, \infty)$, then

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[1] $\varphi(x)$ is said to be *normalized* in $[a, \infty)$ if $\varphi(a) = 0$ and

$$\varphi(x) = \frac{\varphi(x+)+\varphi(x-)}{2} \quad (a \leqslant x < \infty)\,.$$

(a) *$\varphi(x)$ is continuous and strictly monotonic in $a \leqslant x < \infty$.*

(b) *The function $K(x, t)$ can be expressed by the form*

$$K(x, t) = \varphi^{-1}[\varphi(x) - \varphi(t)], \tag{3}$$

where $\varphi^{-1}(x)$ is the inverse function of $\varphi(x)$.

(c) *$\mathcal{C}(a, \infty)$ is a commutative ring with respect to the generalized convolution as ring multiplication.*

We give the proof in several steps.

§ 1. *$\varphi(x)$ is continuous in $a \leqslant x < \infty$.*

Proof. Since $f(t) \equiv 1 \in \mathcal{C}(a, \infty)$, we get by (2) ([2])

$$\varphi(x) = \int_a^x d\varphi(t) = 1 * 1 \in \mathcal{C}(a, \infty).$$

§ 2. *$\varphi(x)$ is strictly monotonic in $a \leqslant x < \infty$.*

Proof. If $\varphi(x)$ were constant in an interval $a \leqslant \alpha \leqslant x \leqslant \beta < \infty$, we should have

$$1 * f = \int_a^x f(t)\,d\varphi(t) \equiv 0 \qquad (a \leqslant x < \infty)$$

for the function

$$f(x) = \begin{cases} 0 & \text{if } x \notin [\alpha, \beta], \\ (x-\alpha)(x-\beta) & \text{if } x \in [\alpha, \beta]. \end{cases}$$

This is a contradiction since, by hypothesis, $\mathcal{C}(a, \infty)$ has no zero divisor with respect to operation (1).

§ 3. *If $a \leqslant t \leqslant x < \infty$, then*

$$a \leqslant K(x, t) \leqslant x. \tag{4}$$

Proof. We make use of hypothesis (2). This may be written in detail by (1)

$$\int_a^x f[K(x, t)]\,d\varphi(t) = \int_a^x f(t)\,d\varphi(t). \tag{5}$$

Proceeding to the proof of (4), suppose that it is not true. Then there is an x_1 and $t_1 < x_1$ such that

$$K(x_1, t_1) > x_1. \tag{6}$$

([2]) It is enough to assume only $1 * 1 \in \mathcal{C}(a, \infty)$ and the supposition $f * 1 \in \mathcal{C}(a, \infty)$ for all $f \in \mathcal{C}(a, \infty)$ can be left aside.

Then, by the continuity and the strict monotonity of the function $K(t) = K(x_1, t)$, there exists a largest interval $[a, \beta] \subseteq [a, x_1]$ containing t_1, so that

$$K(t) = K(x_1, t) > x_1 \tag{7}$$

for all $a < t < \beta$, and

$$K(t) < x_1$$

if $t \notin [a, \beta]$. Let $A = \min[K(a), K(\beta)]$ and $B = \max[K(a), K(\beta)]$. We have, by strict monotonity of $K(t)$, $x_1 = A \leqslant K(t) \leqslant B$, $A < B$ $(t \in [a, \beta])$.

Let $\mathcal{C}(A, B)$ be the space of the functions g continuous in $[A, B]$. Clearly,

$$Lg = \int_a^\beta g[K(t)]\,d\varphi(t) \tag{8}$$

is a linear bounded functional on $\mathcal{C}[A, B]$. We write (8) by the formula for the change of the variable of a Stieltjes integral

$$Lg = \pm\int_A^B g(\lambda)\,d\varphi[K^{-1}(\lambda)], \tag{9}$$

where $K^{-1}(\lambda)$ is the inverse function of $K(t)$.

The set of all elements $g_0 \in \mathcal{C}(A, B)$ for which $g_0(A) = g_0(B) = 0$ will be denoted by $\mathcal{C}_0(A, B)$. $\mathcal{C}_0(A, B)$ is naturally a subspace of $\mathcal{C}(A, B)$. Let g_0 be an arbitrary fixed function of $\mathcal{C}_0(A, B)$. Define

$$f_0(t) = \begin{cases} g_0(t) & \text{if } t \in [A, B], \\ 0 & \text{otherwise}. \end{cases}$$

Then $f_0 \in \mathcal{C}(a, \infty)$ and we have by (5) for $x = x_1 = A$

$$\int_a^\beta g_0[K(t)]\,d\varphi(t) = \int_a^{x_1} f_0(t)\,d\varphi(t) = 0.$$

Hence

$$Lg = \pm\int_A^B g(\lambda)\,d\varphi[K^{-1}(\lambda)] = 0 \tag{10}$$

for any $g \in \mathcal{C}_0(A, B)$.

It follows from the continuity of the function $a(\lambda) = \varphi[K^{-1}(\lambda)]$ that (10) holds for any $g \in \mathcal{C}(A, B)$. To prove this define the functions $g_{0,n}(\lambda)$ for enough large n as follows:

$$g_{0,n}(\lambda) = \begin{cases} ng\left(A+\frac{1}{n}\right)\cdot(\lambda - A) & \text{if } \lambda \in \left[A, A+\frac{1}{n}\right], \\ g(\lambda) & \text{if } \lambda \in \left[A+\frac{1}{n}, B-\frac{1}{n}\right], \\ ng\left(B-\frac{1}{n}\right)\cdot(B-\lambda) & \text{if } \lambda \in \left[B-\frac{1}{n}, B\right]. \end{cases}$$

Since $g_{0,n} \in \mathcal{C}_0(A, B)$, we get by (10)

$$0 = \int_A^B g_{0,n}(\lambda)\,da(\lambda) \tag{11}$$

$$= \int_A^{A+\frac{1}{n}} ng\left(A+\frac{1}{n}\right)(\lambda - A)\,da(\lambda) + \int_{A+\frac{1}{n}}^{B-\frac{1}{n}} g(\lambda)\,da(\lambda) +$$

$$+ \int_{B-\frac{1}{n}}^{B} ng\left(B-\frac{1}{n}\right)(B-\lambda)\,da(\lambda)\,.$$

It is easy to see by the continuity of g and a that

$$\int_A^{A+\frac{1}{n}} ng\left(A+\frac{1}{n}\right)(\lambda - A)\,da(\lambda) \to 0 \quad \text{if } n \to \infty\,,$$

$$\int_{B-\frac{1}{n}}^{B} ng\left(B-\frac{1}{n}\right)(B-\lambda)\,da(\lambda) \to 0 \quad \text{if } n \to \infty\,,$$

$$\int_{A+\frac{1}{n}}^{B-\frac{1}{n}} g(\lambda)\,da(\lambda) \to \int_A^B g(\lambda)\,da(\lambda) \quad \text{if } n \to \infty\,.$$

In virtue of (11) we obtain

$$\int_A^B g(\lambda)\,da(\lambda) = 0 \tag{12}$$

for any $g \in \mathcal{C}(A, B)$.

Thus, in virtue of the theorem of F. Riesz concerning linear functionals (3), (12) implies that $a(\lambda) = \varphi[K^{-1}(\lambda)] = c =$ constant in $[A, B]$, i.e. $\varphi(t) = c$ in $[\alpha, \beta]$, which leads to contradiction, since by § 2. $\varphi(x)$ has no points of invariability.

§ 4. *The equation*

$$K(x, t) = \lambda \tag{13}$$

has a solution in $a \leqslant t \leqslant x$ for each fixed x and $a \leqslant \lambda \leqslant x$.

(3) See, for example, [2], p. 102.

Proof. Let us suppose on the contrary that there exist an interval $[a, x_1]$ and a number $\lambda_1 \in [a, x_1]$ such that

$$K(x_1, t) \neq \lambda_1$$

if $t \in [a, x_1]$. Since the function $K(x_1, t)$ is continuous, we may assume $a < \lambda_1 < x_1$.

By the continuity of $K(x_1, t)$ we have only the following two possibilities:

(i) $K(x_1, t) > \lambda_1$ for all $t \in [a, x_1]$,

(ii) $K(x_1, t) < \lambda_1$ for all $t \in [a, x_1]$.

Let

$$f_\lambda(x) = \begin{cases} \lambda - x & \text{if } a \leqslant x \leqslant \lambda, \\ 0 & \text{if } \lambda < x. \end{cases}$$

If $\lambda \leqslant \lambda_1 < K(x_1, t)$, then $f_\lambda[K(x_1, t)] = 0$ and thus by (5)

$$0 = \int_a^{x_1} f_\lambda[K(x_1, t)]d\varphi(t) = \int_a^{x_1} f_\lambda(t)d\varphi(t) = \int_a^{\lambda} (\lambda - t)d\varphi(t) = \int_a^{\lambda} \varphi(t)dt$$

for each $\lambda \in [a, \lambda_1]$. By the continuity of $\varphi(x)$ this implies that $\varphi(\lambda) = 0$ in $[a, \lambda_1]$. Since $\varphi(x)$ has no points of invariability, case (i) is not possible.

If $K(x_1, t) < \lambda_1 \leqslant \lambda \leqslant x_1$, then $f_\lambda[K(x_1, t)] = \lambda - K(x_1, t)$. Consequently according to (5)

$$\begin{aligned} \int_a^{\lambda} \varphi(t)dt &= \int_a^{\lambda} (\lambda - t)d\varphi(t) = \int_a^{x_1} f_\lambda(t)d\varphi(t) = \int_a^{x_1} f_\lambda[K(x_1, t)]d\varphi(t) \\ &= \int_a^{x_1} [\lambda - K(x_1, t)]d\varphi(t) = \lambda\varphi(x_1) - \int_a^{x_1} K(x_1, t)d\varphi(t), \end{aligned}$$

i.e.

$$\int_a^{\lambda} \varphi(t)dt = \lambda\varphi(x_1) - \int_a^{x_1} K(x_1, t)d\varphi(t).$$

Thus we have after derivation $\varphi(\lambda) = \varphi(x_1)$ for all $\lambda \in [\lambda_1, x_1]$, but this is impossible by § 2.

This contradiction proves the statement.

§ 5. It follows from our supposition that for $K(x, t)$ only the following two cases are possible:

(I) $K(x, t)$ is, for each fixed x, increasing in t,

(II) $K(x, t)$ is, for each fixed x, decreasing in t.

The following statements are obvious according to § 3, and § 4:

(14) $$K(x, a) = a \quad \textit{and} \quad K(x, x) = x \quad \textit{for all } x \geqslant a$$

in case (I),

(15) $\qquad K(x, a) = x \quad$ *and* $\quad K(x, x) = a \quad$ *for all* $x \geqslant a$

in case (II).

§ 6. *$K(x, t)$ is (for each fixed x) decreasing in t.*

Proof. Suppose that $K(x, t)$ is increasing in t. The solution of equation (13) will be denoted by $t = K^{-1}(x, \lambda)$.

It follows from (5) by (14) that

$$\int_a^x f(t)\,d\varphi(t) = \int_a^x f[K(x, t)]\,d\varphi(t) = \int_a^x f(\lambda)\,d\varphi[K^{-1}(x, \lambda)]\,.$$

Since the function $\varphi[K^{-1}(x, \lambda)]$ is continuous and $\varphi[K^{-1}(x, a)] = \varphi(a) = 0$, both $\varphi(\lambda)$ and $\varphi[K^{-1}(x, \lambda)]$ are normalized in $[a, x]$. Thus by the theorem of F. Riesz we obtain $\varphi[K^{-1}(x, \lambda)] = \varphi(\lambda)$, i.e.

$$\varphi(t) = \varphi[K(x, t)] \qquad (t \in [a, x])\,. \tag{16}$$

Since the function $\varphi(t)$ is strictly monotonic, it follows that $K(x, t) \equiv t$. The generalized convolution (1) has in this case the form

$$f * g = \int_a^x f(t)g(t)\,d\varphi(t)\,.$$

It is easy to see that $C(a, \infty)$ has zero divisors with respect to this multiplication.

Thus $K(x, t)$ may be only decreasing in t.

§ 7. *We have*

$$K(x, t) = \varphi^{-1}[\varphi(x) - \varphi(t)]\,. \tag{3}$$

Proof. Fix x. Since $K(x, t)$ is decreasing, it follows from (15) that $K^{-1}(x, x) = a$ and $K^{-1}(x, a) = x$. Thus using (5) we obtain

$$\begin{aligned}\int_a^x f(\lambda)\,d\varphi(\lambda) &= \int_a^x f(t)\,d\varphi(t) = \int_a^x f[K(x, t)]\,d\varphi(t) = \int_x^a f(\lambda)\,d\varphi[K^{-1}(x, \lambda)] \\ &= -\int_a^x f(\lambda)\,d\varphi[K^{-1}(x, \lambda)] = \int_a^x f(\lambda)\,d\{-\varphi[K^{-1}(x, \lambda)]\} \\ &= \int_a^x f(\lambda)\,d\{\varphi(x) - \varphi[K^{-1}(x, \lambda)]\}\,.\end{aligned}$$

Since

$$\varphi(x)-\varphi[K^{-1}(x, a)] = \varphi(x)-\varphi(x) = 0 ,$$

the continuous function $a(\lambda) = \varphi(x)-\varphi[K^{-1}(x, \lambda)]$ is a normalized function of bounded variation. Hence by F. Riesz's theorem

$$\varphi(\lambda) = \varphi(x)-\varphi[K^{-1}(x, \lambda)] .$$

Substituting $t = K^{-1}(x, \lambda)$, i.e. $\lambda = K(x, t)$, in this equation, we get

$$\varphi[K(x, t)] = \varphi(x)-\varphi(t) .$$

Denote by $\varphi^{-1}(x)$ the inverse function of $\varphi(x)$; then we obtain (3).

§ 8. $\mathcal{C}(a, \infty)$ *is a commutative ring with respect to the generalized convolution*

$$f \ast g = \int_a^x f\{\varphi^{-1}[\varphi(x)-\varphi(t)]\} g(t)\, d\varphi(t) \tag{17}$$

as ring multiplication.

Proof. It is obvious, by the continuity of φ, that $f \ast g \in \mathcal{C}(a, \infty)$ for all $f, g \in \mathcal{C}(a, \infty)$. Similarly the proof of the distributivity of this multiplication is omitted. Thus it remains to prove only the commutativity and the associativity of the product (17).

Let $f(x)$ be an arbitrary element of $\mathcal{C}(a, \infty)$. Then, on the supposition that $\varphi(x)$ is increasing ([4]),

$$F(y) = f[\varphi^{-1}(y)] \in \mathcal{C}(0, \infty) \qquad (0 \leqslant y < \infty) \tag{18}$$

and (18) establishes a one-to-one correspondence between the sets $\mathcal{C}(a, \infty)$ and $\mathcal{C}(0, \infty)$, in which the product (17) corresponds to the ordinary convolution product (1.1). Indeed, by substituting $t = \varphi^{-1}(\tau)$, $\varphi(x) = y$ in (17) we get

$$f \ast g[\varphi^{-1}(y)] = \int_a^y F(y-\tau) G(\tau)\, d\tau ,$$

where $F(y)$ is defined by (18) and similarly $G(y) = g[\varphi^{-1}(y)]$. Thus the product (17) is commutative and associative, since the ordinary con-

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([4]) If $\varphi(x)$ is decreasing, then $\psi(x) = -\varphi(x)$ is increasing. Therefore by substituting $\varphi(t) = -\psi(t)$ in (17) we get $-(f \ast g)$ instead of $f \ast g$.

volution is commutative and associative. This asserts the isomorphism of rings $\mathcal{C}(a, \infty)$ and $\mathcal{C}(0, \infty)$.

The proof of our theorem is thus complete.

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References

[1] J. Mikusiński, *Operational calculus*, 1959.
[2] F. Riesz und B. Sz.-Nagy, *Vorlesungen über Funktionalanalisis*, Berlin 1956.

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