Now the set of linear primes P of I with

$$\left(\frac{I(\zeta)/I}{P}\right) = \sigma_a$$

has positive density. But

$$p = \|P\|_I \equiv a \bmod m$$

and by Corollary II p splits principally in k.

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Bounds for solutions of diagonal equations

by

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1. Introduction. In the first part of this paper (§ 2 to § 7) I shall prove the following theorem for the case $k \geqslant 4$.

THEOREM 1. *Let k be an integer, $k \geqslant 2$, and let n be the integer defined by*

$$\begin{cases} n = 2^k+1 & \text{if} \quad 2 \leqslant k \leqslant 11, \\ n \geqslant 2k^2(2\log k + \log\log k + 3) + 1 > n-1 & \text{if} \quad k \geqslant 12. \end{cases} \tag{1}$$

Then for any $\theta > 0$ there exists a constant c_θ, depending only on θ and k, with the following property. If $\lambda_1, \ldots, \lambda_n$ are non-zero integers which are not all of the same sign if k is even, then the Diophantine equation

$$\lambda_1 x_1^k + \lambda_2 x_2^k + \ldots + \lambda_n x_n^k = 0 \tag{2}$$

has a solution in non-zero integers such that

$$|\lambda_1 x_1^k| + \ldots + |\lambda_n x_n^k| < c_\theta |\lambda_1 \lambda_2 \ldots \lambda_n|^{k\psi+\theta}, \tag{3}$$

where

$$\psi = \begin{cases} \tfrac{1}{2} & \text{if} \quad 2 \leqslant k \leqslant 11, \\ 1 & \text{if} \quad k \geqslant 12. \end{cases} \tag{4}$$

The case $k = 2$ of the theorem (which is a modified form of a theorem of Cassels [3]) was proved by Birch and Davenport [2] and was used in their proof of a corresponding result on diagonal quadratic inequalities [1]. The case $k = 3$ was proved by Pitman and Ridout [11] and used similarly in the proof of a corresponding result on cubic inequalities. The proof for the case $k \geqslant 4$ is a straightforward generalization of the proof of Theorem 1 of [11]. The theorem for $k \geqslant 4$ is an essential preliminary to my proof [10] of a theorem (Theorem A in § 9 below) which gives a bound for the least non-trivial solution of the Diophantine inequality

$$|\lambda_1 x_1^k + \ldots + \lambda_n x_n^k| < 1, \tag{5}$$

where n is defined by (1), and $\lambda_1, \ldots, \lambda_n$ are real numbers which satisfy $|\lambda_i| \geqslant 1$ for all i and which are not all of the same sign if k is even. From now on, by a *solution* of a Diophantine equation or inequality, I shall always mean a solution in integers $x_1, \ldots, x_n$, not all zero.

We note that for (2) to have a solution the following *congruence condition* must be satisfied: for any prime power p^s, the congruence

$$\lambda_1 x_1^k + \ldots + \lambda_n x_n^k \equiv 0 \pmod{p^s} \tag{6}$$

has a solution with $x_1, \ldots, x_n$ not all divisible by p. Following Davenport and Lewis [7], we define $G^*(k)$ to be the least number such that if $n \geqslant G^*(k)$ and the coefficients of (2) satisfy the congruence condition, then (2) has infinitely many solutions; and we define $\Gamma^*(k)$ to be the least number such that if $n \geqslant \Gamma^*(k)$ then every equation (2) satisfies the congruence condition.

Davenport and Lewis showed that $\Gamma^*(k) \leqslant k^2+1$, with equality whenever $k+1$ is a prime; Chowla and Shimura [4] showed that $\Gamma^*(k) = O(k\log k)$ if k is odd; and Dodson [8] has obtained further information about $\Gamma^*(k)$ which depends on the nature of k. Davenport and Lewis also showed that $G^*(k) = O(k\log k)$, and, in particular, that $G^*(k) \leqslant k^2+1$ for $k \geqslant 18$. It follows that for $k \geqslant 18$, (2) has infinitely many solutions if $n \geqslant k^2+1$, and, further, that for sufficiently large odd k, (2) has infinitely many solutions if $n > ck\log k$.

For $4 \leqslant k \leqslant 17$, the situation is less clear, though (2) certainly has infinitely many solutions if $n \geqslant 2^k+1$. Davenport and Lewis [7] indicated that the smallest value of n for which (2) always has a solution is exactly 17 for $k = 4$, at most 23 for $k = 5$, and exactly 37 for $k = 6$. They gave an upper bound for $G^*(k)$ for $12 \leqslant k \leqslant 17$, but expressed doubt as to whether existing methods would yield a proof that $G^*(k) \leqslant k^2+1$ for the remaining values of k, namely $7 \leqslant k \leqslant 17$.

We now see that the value of n in (1) is best possible for $k = 4$, but is far from best possible for large values of k. In § 8 I shall indicate how the methods of Davenport and Lewis [7] can be used to obtain an analogous theorem (Theorem 2) for $k \geqslant 12$ in which the size of n is more satisfactory (1). Unfortunately, as I shall explain in § 9, these methods do not apply to the inequality (5); in § 9 I shall also discuss the extent to which other related methods can be used to deal with (5) for smaller values of n.

This paper owes much to the helpful comments of the late Professor Davenport on an earlier version.

2. Notation and preliminary lemmas. Throughout this paper, k is a fixed integer, $k \geqslant 4$, and we write

$$\nu = 1/k.$$

The letters a, a_i always denote integers, the letters q, q_i always denote positive integers, and the symbol $\sum_i$ indicates summation from $i = 1$ to $i = n$. We use the permanent notation

$$e(y) = e^{2\pi i y}.$$

In this section we suppose that $\lambda_1, \ldots, \lambda_n$ are given real numbers which satisfy

$$|\lambda_i| \geqslant 1 \qquad (i = 1, \ldots, n)$$

and which are not all of the same sign if k is even. (In our applications in this paper, $\lambda_1, \ldots, \lambda_n$ will be integers, but it is convenient to give our preliminary results in a form which can be used in [10].) We write

$$\Pi = \prod_{i=1}^{n} |\lambda_i|, \qquad \Lambda = \max_i |\lambda_i|. \tag{7}$$

Let P be a large positive integer such that

$$P \geqslant |\lambda_i|^{\nu} \qquad (i = 1, \ldots, n). \tag{8}$$

We define

$$S_i(a) = \sum_{x_i} e(\lambda_i a x_i^k), \tag{9}$$

where x_i runs through all integral values in the range

$$P \leqslant |\lambda_i|^{\nu} x_i \leqslant 3P, \tag{10}$$

and we write

$$v(a) = \prod_{i=1}^{n} S_i(a). \tag{11}$$

We also define

$$S(a, q) = \sum_{x=1}^{q} e(ax^k/q), \tag{12}$$

$$I(\beta) = \sum_{m} \nu m^{-1+\nu} e(\beta m), \tag{13}$$

where m runs through all integral values in the interval

$$[P^k, (3P)^k]. \tag{14}$$

For convenience I collect together here some general lemmas which are needed for the proof of Theorem 1. In these lemmas, δ denotes a fixed small positive number, and ε denotes an arbitrarily small positive number which is not necessarily the same throughout. The constants implied by the notations $O, \ll, \gg$ are always independent of P and of the λ_i; in this section

(1) Added in proof. B. T. Birch has recently shown that for odd k the bound in (3) can be much improved by taking n very large. See Proc. London Math. Soc. (3) 21 (1970), pp. 12-18.

they depend only on δ and ε; in later sections they will depend on θ, δ and ε, and so ultimately on θ, since δ and ε will be determined by θ. (The notation $A \ll B$ means $A = O(B)$, and $A \gg B$ means $A > cB$ for some positive constant c.)

LEMMA 1. (i) *If p is prime and $(a,p)=1$, then*

$$|S(a,p)| \leqslant (k-1)p^{1/2}.$$

(ii) *If $(a,q)=1$, then*

$$S(a,q) \ll q^{1-\nu}.$$

Proof. See, for example, Lemmas 12 and 15 of Davenport [6].

LEMMA 2. *If $|\beta| \leqslant \frac{1}{2}$, then*

$$I(\beta) \ll P\min(1, P^{-k}|\beta|^{-1}).$$

Proof. See, for example, Lemma 4 of Davenport [5].

LEMMA 3. *Suppose that*

$$\lambda_i\alpha = (a_i/q_i)+\beta_i,$$

where

(15) $\quad (a_i,q_i)=1, \quad q_i \leqslant (|\lambda_i|^{-\nu}P)^{1-\delta}, \quad |\beta_i| \leqslant q_i^{-1}(|\lambda_i|^{-\nu}P)^{1-k-\delta}.$

Then

(16) $\quad S_i(\alpha) = |\lambda_i|^{-\nu} q_i^{-1} S(a_i,q_i) I(\pm\beta_i/\lambda_i) + O(q_i^{(3/4)+\varepsilon}),$

where $\pm$ is the sign of λ_i; and each of the two terms on the right-hand side of (16) *is*

$$\ll |\lambda_i|^{-\nu} q_i^{-\nu} P \min(1, P^{-k}|\lambda_i/\beta_i|).$$

Proof. By Lemma 8 of Davenport [5], with $|\lambda_i|^{-\nu}P$ in place of P, we have

$$S_i(\alpha) = q_i^{-1}S(a_i,q_i)J + O(q_i^{(3/4)+\varepsilon}),$$

where

$$J = \sum_m \nu m^{-1+\nu} e(\beta_i m)$$

and m runs through the integers of the range

$$|\lambda_i|^{-1}P^k \leqslant m \leqslant |\lambda_i|^{-1}(3P)^k.$$

The relation (16) follows from this, since it is easily deduced from (13) and the condition on β_i in (15) that

$$I(\pm\beta_i/\lambda_i) = J + O(1).$$

The last part of the lemma follows from Lemmas 1 and 2 and the fact that if (15) holds then

$$q_i^{(3/4)+\varepsilon} \ll |\lambda_i|^{-\nu} q_i^{-\nu} P \min(1, P^{-k}|\lambda_i/\beta_i|)$$

for all sufficiently small positive ε.

LEMMA 4. *Suppose that $\lambda_i\alpha = (a_i/q_i)+\beta_i$, where*

$$(a_i,q_i)=1, \quad (|\lambda_i|^{-\nu}P)^{1-\delta} \leqslant q_i \leqslant (|\lambda_i|^{-\nu}P)^{k-1+\delta}, \quad |\beta_i| \leqslant q_i^{-1}(|\lambda_i|^{-\nu}P)^{1-k-\delta}.$$

Then

$$S_i(\alpha) \ll (|\lambda_i|^{-\nu}P)^{1-\sigma+\delta},$$

where

(17) $$\sigma = \begin{cases} \dfrac{1}{2^{k-1}} & \text{if} \quad 4 \leqslant k \leqslant 11, \\ \dfrac{1}{2k^2(2\log k + \log\log k + 3)} & \text{if} \quad k \geqslant 12. \end{cases}$$

Proof. For $4 \leqslant k \leqslant 11$, the result follows from Weyl's inequality; see, for example, Lemma 11 of Davenport [5]. For $k \geqslant 12$, the result (actually with δ replaced by a smaller exponent) is a slight modification of Vinogradov's inequality as stated in Theorem 9 of Hua [9]; it is obtained by using Hua's Lemma 5.10 when q_i lies outside the range from $|\lambda_i|^{-\nu}P$ to $(|\lambda_i|^{-\nu}P)^{k-1}$.

LEMMA 5. *For any $X \geqslant 1$, we have*

(18) $$\int_0^X |S_i(\alpha)|^{n-1} d\alpha \ll X(|\lambda_i|^{-\nu}P)^{n-1-k+\varepsilon},$$

where n is defined by (1).

Proof. Let

$$S(\beta) = \sum_{P \leqslant x \leqslant 3P} e(\beta x^k).$$

Then for $4 \leqslant k \leqslant 11$, since $n = 2^k+1$, we have

(19) $$\int_0^1 |S(\beta)|^{n-1} d\beta \ll P^{n-1-k+\varepsilon}$$

by Hua's inequality (see, for example Davenport [6], Lemma 2). And for $k \geqslant 12$ (19) also holds, since in this case

$$n-1 \geqslant 2k^2(2\log k + \log\log k + 2.5)$$

and therefore (19) follows from Vinogradov's mean-value theorem as stated in Lemma 7.13 of Hua [9].

Inequality (18) follows from (19) by replacing P by $|\lambda_i|^{-\nu}P$, substituting $\beta = \lambda_i\alpha$, and using the fact that $S_i(\alpha)$ is periodic of period $|\lambda_i|^{-1}$. (See the proof of Lemma 5 of Pitman and Ridout [11].)

LEMMA 6. *Let*

$$Z = \sum_{m_1,\ldots,m_n} \nu^n (m_1 \ldots m_n)^{-1+\nu},$$

where the summation is over all integral n-tuples $(m_1, \ldots, m_n)$ such that each m_i is in the interval (14) *and*

$$m_1 - m_2 \pm m_3 \pm \ldots \pm m_n = 0,$$

where $\pm$ stands for exactly one of the signs $+$, $-$, but not necessarily the same one in each case. Then

$$Z \geqslant P^{n-k}.$$

Proof. The result follows from the fact that each integral $(n-1)$-tuple $m_2, \ldots, m_n$ such that

$$(1+2^k)P^k \leqslant m_2 \leqslant (3^k - 2^k)P^k,$$
$$P^k \leqslant m_i \leqslant (1+2^{-k})P^k \quad (i = 3, \ldots, n)$$

determines an integer

$$m_1 = m_2 \mp m_3 \mp \ldots \mp m_n$$

such that m_1 is in the interval (14).

3. Normalization of a diagonal equation. In order to deal with the singular series in the proof of Theorem 1, we need to make some additional assumptions about the coefficients λ_i. The following lemma shows that these assumptions involve no loss of generality.

LEMMA 7. *Let C, B be given positive numbers such that $B \geqslant 1$, and let n be an integer such that $n \geqslant 2k+1$. Suppose that* (2) *has a solution in non-zero integers such that*

$$\sum_i |\lambda_i x_i^k| < C\Pi^B, \tag{20}$$

whenever $\lambda_1, \ldots, \lambda_n$ satisfy the following conditions:

(21) *$\lambda_1, \ldots, \lambda_n$ are non-zero integers which are not all of the same sign if k is even;*

(22) *λ_i is k-th power free for all i;*

(23) *no prime divides more than $n-3$ of the λ_i.*

Then (2) *has a solution in non-zero integers such that* (20) *holds, whenever $\lambda_1, \ldots, \lambda_n$ are non-zero integers which satisfy the condition* (21).

Proof. The proof is similar to that of Lemma 6 of [11]. We introduce the following notation. For any diagonal form

$$F = \mu_1 x_1^k + \ldots + \mu_m x_m^k,$$

we write

$$\Pi_F = |\mu_1 \mu_2 \ldots \mu_m|,$$
$$\|F\| = |\mu_1 x_1^k| + \ldots + |\mu_m x_m^k|.$$

First we show by induction that the condition (23) can be removed. For $r = 0, 1, 2, \ldots$, we consider the condition

$(23)_r$ *at most r distinct primes divide more than $n-3$ of the λ_i.*

Suppose that (2) has a solution in non-zero integers such that (20) holds, whenever $\lambda_1, \ldots, \lambda_n$ satisfy (21), (22), and $(23)_r$; and consider a form F whose coefficients $\lambda_1, \ldots, \lambda_n$ satisfy (21), (22), and $(23)_{r+1}$, but not $(23)_r$. For some prime p, F can be written in the shape

$$F = F_0 + pF_1 + p^2F_2 + \ldots + p^{k-1}F_{k-1},$$

where the following conditions hold: each F_j is a diagonal form in n_j variables ($n_j \geqslant 0$ and the variables in distinct F_j are distinct);

$$n = n_0 + n_1 + \ldots + n_{k-1}, \quad n_0 \leqslant 2;$$

p does not divide any of the coefficients of the F_j; and at most r other primes divide more than $n-3$ of the coefficients of the F_j. Since $n \geqslant 2k+1$, at least one n_j is greater than 2; say $n_t \geqslant 3$ and $n_0, n_1, \ldots, n_{t-1} \leqslant 2$, $t \geqslant 1$. We consider the form

$$G = p^{k-t}F_0 + p^{k-t+1}F_1 + \ldots + p^{k-1}F_{t-1} + F_t + pF_{t+1} + \ldots + p^{k-t-1}F_{k-1},$$

and note that

$$p^tG = p^kF_0 + p^{k+1}F_1 + \ldots + p^{k+t-1}F_{t-1} + p^tF_t + \ldots + p^{k-1}F_{k-1}, \tag{24}$$

$$\|p^tG\| = p^t\|G\|, \tag{25}$$

$$\Pi_G p^{nt} = \Pi_F p^{k(n_0+n_1+\ldots+n_{t-1})} \leqslant \Pi_F p^{2kt}. \tag{26}$$

Since $n_t \geqslant 3$, the form G satisfies $(23)_r$, and so by our hypothesis there is a solution of $G = 0$ such that

$$\|G\| < C\Pi_G^B.$$

By using (24) and (25) and absorbing kth powers of p into the variables we obtain a solution of $F = 0$ such that

$$\|F\| < Cp^t\Pi_G^B.$$

Since $n \geqslant 2k+1$ and $t \geqslant 1$, it follows from (26) that

$$\Pi_G \leqslant p^{-t}\Pi_F.$$

Since $B \geqslant 1$, this implies that our solution of $F = 0$ satisfies

$$\|F\| < C\Pi_F^B.$$

Thus we have shown that under our hypothesis equation (2) has a solution in non-zero integers such that (20) holds, provided that $\lambda_1, \ldots, \lambda_n$ satisfy (21), (22), and $(23)_{r+1}$. Since we are given that (2) has a solution in non-zero integers such that (20) holds, provided that $\lambda_1, \ldots, \lambda_n$ satisfy (21), (22), and $(23)_0$, it now follows by induction that the same result holds provided that $\lambda_1, \ldots, \lambda_n$ satisfy (21), (22), and $(23)_r$ for some r; and this is equivalent to the removal of the restriction (23).

We now remove the restriction (22). We consider a form

$$F = \lambda_1 x_1^k + \ldots + \lambda_n x_n^k,$$

where (21) holds and for each i

$$\lambda_i = \mu_i^k \nu_i,$$

the ν_i being kth power free. The form

$$G = \nu_1 x_1^k + \ldots + \nu_n x_n^k$$

has non-zero integral coefficients which satisfy (21) and (22). Therefore, by the result of the last paragraph, there is a solution of $G = 0$ such that

$$\|G\| < C\Pi_G^B.$$

Let

$$\mu = \mu_1 \ldots \mu_n.$$

By considering the equation $\mu^k G = 0$ and absorbing the kth power of μ/μ_i into the ith variable, we obtain a solution of $F = 0$ such that

$$\|F\| = \mu^k \|G\| < C\mu^k \Pi_G^B \leqslant C(\mu^k \Pi_G)^B = C\Pi_F^B.$$

Thus (2) has a solution which satisfies (20). Since we have only assumed that (21) holds, this completes the proof of the lemma.

4. Minor arcs. We now assume that n is defined by (1) and that $\lambda_1, \ldots, \lambda_n$ are non-zero integers which satisfy conditions (21), (22), and (23). For any positive integer P such that (8) holds, we let $\mathcal{N}(P)$ denote the number of solutions of (2) such that (10) holds for all i. We suppose that θ is a given positive number and, further, that $\theta < 1$, which clearly involves no loss of generality. We shall show that there is a constant $D_\theta \geqslant 1$, independent of the λ_i, such that if $P^{1-\theta} > D_\theta \Pi^\psi$, then $\mathcal{N}(P) > 0$. Since we may take P so that

$$D_\theta \Pi^\psi < P^{1-\theta} < 2D_\theta \Pi^\psi,$$

this will imply the existence of a solution of (2) in non-zero integers such that

$$\sum_i |\lambda_i x_i^k| < n(2D_\theta \Pi^\psi)^{k/(1-\theta)}.$$

Since $1/(1-\theta) \downarrow 1$ as $\theta \downarrow 0$, this will prove Theorem 1 for the case where $\lambda_1, \ldots, \lambda_n$ satisfy the additional conditions (22) and (23); and the theorem for the general case without these restrictions will then follow from Lemma 7.

We have

$$\mathcal{N}(P) = \int_J v(a)\,da,$$

where J is the closed interval $[0, 1]$ and $v(a)$ is defined by (11). The main term in our estimate of $\mathcal{N}(P)$ will be of the form $c\Pi^{-\nu-\varepsilon}P^{n-k}$, where $c > 0$; so we must ensure, roughly speaking, that our error terms are substantially smaller than $\Pi^{-\nu}P^{n-k}$ whenever P is somewhat larger than Π^ψ. In order to estimate $v(a)$, and hence $\mathcal{N}(P)$, we must consider rational approximations to the $\lambda_i a$.

Let δ be a small positive number such that $0 < \delta < 1$; this number will be fixed throughout the argument and will eventually be chosen in terms of θ. For each $a \in J$, we consider rational approximations a_i/q_i to the $\lambda_i a$ such that

$$(27) \qquad \begin{cases} (a_i, q_i) = 1, \ \lambda_i a = (a_i/q_i) + \beta_i, \\ q_i \leqslant (|\lambda_i|^{-\nu}P)^{k-1+\delta}, \ |\beta_i| \leqslant q_i^{-1}(|\lambda_i|^{-\nu}P)^{-k+1-\delta}. \end{cases}$$

(The existence of such approximations follows from Dirichlet's theorem on Diophantine approximations and (8).) If for some i we have

$$(28) \qquad q_i \geqslant (|\lambda_i|^{-\nu}P)^{1-\delta}$$

for all such approximations a_i/q_i, so that $\lambda_i a$ belongs to a "minor arc" in the standard sense, then our only estimate of $S_i(a)$ is the upper bound given by Lemma 4. Therefore we must show that the contribution to $\mathcal{N}(P)$ from all a with this property is small enough to be permissible. Since this requirement determines the size of n and ψ in our theorem, we deal with it first, in the following lemma.

LEMMA 8. *Let K be the set of all a in J such that for some i inequality (28) holds for every approximation a_i/q_i which satisfies (27). Then*

$$(29) \qquad \int_K |v(a)|\,da \ll \Pi^{-\nu}P^{n-k} \cdot \Pi^{1/(n-1)}P^{-\sigma+2\delta}.$$

Proof. Let K_1 be the set of all a in J such that inequality (28) with $i = 1$ holds for every approximation a_1/q_1 which satisfies (27) with $i = 1$. Then, using the upper bound for $|S_1(a)|$ given by Lemma 4, we have

$$\int_{K_1} |v(a)|\,da \ll (|\lambda_1|^{-\nu}P)^{1-\sigma+\delta} \int_0^1 |S_2(a) \ldots S_n(a)|\,da,$$

where σ is defined by (17). By Hölder's inequality and Lemma 5 with $X = 1$, it follows that

$$\int_{K_1} |v(a)|\,da \ll \Pi^{-\nu} P^{n-k} \cdot |\lambda_1|^{\nu\sigma} |\lambda_2 \ldots \lambda_n|^{1/(n-1)} P^{-\sigma+2\delta}$$
$$\ll \Pi^{-\nu} P^{n-k} \cdot \Pi^{1/(n-1)} P^{-\sigma+2\delta},$$

since for all k by (1) and (17)

(30) $$\nu\sigma < 1/(n-1).$$

Corresponding results hold for the contributions from the sets $K_2, \ldots, K_n$ defined in the obvious way, and the lemma then follows from the fact that K is the union of the K_i.

The size of n in the theorem is determined by the need to use Lemma 5 (Hua's inequality or Vinogradov's mean value theorem), in the above argument. If we take n as small as possible and want our estimates to be in terms of Π, then clearly we cannot essentially improve the bound in (29) by the present methods. Roughly speaking, the condition that the error given by (29) is small enough to be permissible is

$$P \gg \Pi^{1/\{\sigma(n-1)\}}$$

which, by (1) and (17), is essentially $P \gg \Pi^{\nu}$. Thus the bound in the theorem cannot be decreased while we have an error term as large as the bound in (29).

It is clear from the proof of Lemma 8 that the same conclusion will follows if K is replaced by any subset of J in which the inequality

$$S_i(a) \ll |\lambda_i|^{-\nu+1/(n-1)} P^{1-\sigma+\delta}$$

always holds for at least one i. By Lemmas 3 and 4 and (30), this inequality holds if $\lambda_i a$ has an approximation a_i/q_i satisfying (27) such that at least one of the inequalities

$$q_i^{\nu} > |\lambda_i|^{-1/(n-1)} P^{\sigma}, \qquad |\beta_i/\lambda_i| > P^{\sigma-k-\delta}$$

holds.

Hence we dissect J into two subsets, M (the "major arcs") and $J-M$, where M is defined as the set of all a in J for which each $\lambda_i a$ has an approximation a_i/q_i such that (27) holds and

(31) $$q_i^{\nu} \leqslant |\lambda_i|^{-1/(n-1)} P^{\sigma}, \qquad |\beta_i/\lambda_i| \leqslant P^{\sigma-k-\delta}.$$

The main term in our estimate of $\mathcal{N}(P)$ will be the contribution from M, and it is clear from the preceding discussion that the contribution from $J-M$ has the same upper bound as the contribution from K:

COROLLARY TO LEMMA 8. *We have*

$$\int_{J-M} |v(a)|\,da \ll \Pi^{-\nu} P^{n-k} \cdot \Pi^{1/(n-1)} P^{-\sigma+2\delta}.$$

5. Major arcs: the main term. First we show that M can be regarded as a union of disjoint intervals.

LEMMA 9. *Suppose that $P > 2|\lambda_i\lambda_j|^{1/2}$ for $i, j = 1, \ldots, n$, $i \neq j$. For any positive integer q, define*

(32) $$\delta_i = \delta_i(q) = (\lambda_i, q) \qquad (i = 1, \ldots, n).$$

For each pair of integers a, q such that $(a, q) = 1$ and

(33) $$0 < q \leqslant \min_i \{\delta_i |\lambda_i|^{-k/(n-1)}\} P^{\sigma k},$$

let $J_{a,q}$ be the set of all a such that

(34) $$\left| a - \frac{a}{q} \right| \leqslant P^{\sigma-k-\delta}.$$

(i) *If $a \in J_{a,q}$, then there is exactly one pair a_i, q_i such that* (27) *and* (31) *hold, namely*

(35) $$a_i = \lambda_i a/\delta_i, \qquad q_i = q/\delta_i.$$

(ii) *We have*

$$\int_M v(a)\,da = \sum_{a,q} \int_{J_{a,q}} v(a)\,da,$$

where the summation is over all pairs a, q such that (33) *holds and*

(36) $$(a, q) = 1, \qquad 0 \leqslant a \leqslant q-1.$$

Proof. First we remark that for any a there is at most one pair a_i, q_i satisfying (27) and (31). For, if

$$\lambda_i a = \frac{a_i}{q_i} + \beta_i = \frac{a_i'}{q_i'} + \beta_i', \qquad \frac{a_i}{q_i} \neq \frac{a_i'}{q_i'},$$

where both approximations satisfy (27) and (31), then, exactly as in [11], Lemma 8 (i), we can deduce that

$$2|\lambda_i|^{1-2k/(n-1)} \geqslant P^{k-1-2k\sigma},$$

which is impossible under our assumptions. Part (i) of the lemma is easily deduced from (33) and (34) together with the above remark. It follows that $J_{0,1} \cap J$ and $J_{1,1} \cap J$ and the $J_{a,q}$ such that (36) holds and $q > 1$ are all subsets of M.

Suppose that $\alpha \in M$, and that, for each i, a_i/q_i satisfies (27) and (31). If for some $i \neq j$ we have $a_i/\lambda_i q_i \neq a_j/\lambda_j q_j$, then

$$\left|\frac{\beta_i}{\lambda_i} - \frac{\beta_j}{\lambda_j}\right| = \left|\frac{a_i}{\lambda_i q_i} - \frac{a_j}{\lambda_j q_j}\right| \geqslant \frac{1}{|\lambda_i \lambda_j| q_i q_j},$$

from which we deduce (by the same ideas as in [11], Lemma 8 (ii)) that

$$2|\lambda_i \lambda_j|^{1-k/(n-1)} \geqslant P^{k-1-2k\sigma},$$

which is impossible under our assumptions. Hence the $a_i/\lambda_i q_i$ are all equal, and so there exist integers a, q such that $(a, q) = 1$, $q > 0$, and

$$\frac{a_i}{\lambda_i q_i} = \frac{a}{q} \quad (i = 1, \ldots, n).$$

It follows that (35), (33), and (34) holds and hence that either $\alpha \in J_{1,1}$ or $\alpha \in J_{a,q}$ for some a, q such that (36) hold. Also the uniqueness of the a_i/q_i implies that the $J_{a,q}$ are disjoint. Since $J \cap (J_{0,1} \cup J_{1,1})$ is congruent to $J_{0,1}(\mathrm{mod}\, 1)$ and $v(\alpha+1) = v(\alpha)$ for all α, part (ii) now follows.

In order to be able to use Lemma 9, we shall assume from now on that

$$P > 2\Pi^{1/2}. \tag{37}$$

(There is no loss of generality here, since we shall eventually choose $P \gg \Pi^{\psi/(1-\theta)}$). We now estimate the contribution from M to $\mathcal{N}(P)$.

LEMMA 10. *Assuming that* (37) *holds, we have*

$$\int_M v(\alpha)\, d\alpha = \Pi^{-\nu}\mathfrak{S} R(P) + O(P^{n-k-1+(n-1)\delta}) + \\ + O\{\Pi^{-\nu} P^{n-k} \cdot (\Pi^{1/(n-1)} P^{-\sigma+\varepsilon})^{n-2k}\}, \tag{38}$$

where

$$\mathfrak{S} = \sum_{q=1}^{\infty} \sum_{\substack{a=0 \\ (a,q)=1}}^{q-1} (q_1 \ldots q_n)^{-1} S(a_1, q_1) \ldots S(a_n, q_n), \tag{39}$$

the a_i, q_i *are defined by* (35) *and* (32), *and*

$$R(P) \gg P^{n-k}.$$

Proof. Suppose that $\alpha \in J_{a,q}$, where a, q satisfy (33) and (36), and that a_i, q_i are defined by (35) and (32). Then it follows from Lemma 9 (i) and Lemma 3 that

$$v(\alpha) = \Pi^{-\nu}(q_1 \ldots q_n)^{-1} S(a_1, q_1) \ldots S(a_n, q_n) \prod_{i=1}^{n} \{I(\pm\beta)\} + E, \tag{40}$$

where, for each i, $\pm$ is the sign of λ_i, $\beta = \alpha - (a/q)$, and

$$E \ll \sum_{i=1}^{n} q_i^{(3/4)+\varepsilon} \prod_{j \neq i} \{|\lambda_j|^{-\nu} q_j^{-\nu} P \min(1, P^{-k}|\lambda_j/\beta_j|)\}$$
$$\ll q^{(3/4)+\varepsilon-(n-1)\nu} P^{n-1} \min(1, P^{-2k}|\beta|^{-2}),$$

since $|\lambda_i|^{-1} q \leqslant q_i \leqslant q$ by (35). Now

$$\int_{-1/2}^{1/2} \min(1, P^{-2k}|\beta|^{-2})\, d\beta \ll P^{-k}, \tag{41}$$

and

$$\sum_{q=1}^{\infty} \sum_{\substack{a=0 \\ (a,q)=1}}^{\infty} q^{(3/4)+\varepsilon-(n-1)\nu} \ll 1,$$

since ε may be taken arbitrarily small and $(n-1)\nu \geqslant 3$ for all $k \geqslant 4$. Hence, integrating (40) over $J_{a,q}$ and then summing over the pairs a, q we have, by Lemma 9 (ii),

$$\int_M v(\alpha)\, d\alpha = \sum_{a,q} \Pi^{-\nu}(q_1 \ldots q_n)^{-1} S(a_1, q_1) \ldots S(a_n, q_n) \int_{J_{a,q}} \prod_{i=1}^{n} I(\pm\beta)\, d\alpha + \\ + O(P^{n-1-k}), \tag{42}$$

where the ranges for a, q are given by (33) and (36).

The error caused by replacing $J_{a,q}$ by $[(a/q)-\frac{1}{2}, (a/q)+\frac{1}{2}]$ in (42) is

$$\ll \Pi^{-\nu} \sum_{a,q} (q_1 \ldots q_n)^{-\nu} \int_{J^*_{a,q}} P^n \min(1, P^{-k}|\beta|^{-1})^n\, d\alpha,$$

where the ranges for a, q are as before and $J^*_{a,q}$ is the set of all $\alpha = (a/q) + \beta$ such that

$$\tfrac{1}{2} \geqslant |\beta| \geqslant P^{\sigma-k-\delta}.$$

Now for any pair a, q,

$$\Pi^{-\nu}(q_1 \ldots q_n)^{-\nu} P^{n-kn} \int_{J^*_{a,q}} |\beta|^{-n}\, d\alpha \ll q^{-n\nu} P^{n-k-(n-1)(\sigma-\delta)};$$

and also

$$\sum_{a,q} q^{-n\nu} \ll \sum_{q=1}^{\infty} q^{-n\nu+1} \ll 1,$$

since $n\nu > 2$ for all k. As $(n-1)\sigma \geqslant 1$ by (1) and (17), we now obtain

$$\int_M v(\alpha)\, d\alpha = \Pi^{-\nu} \sum_{a,q} (q_1 \ldots q_n)^{-1} S(a_1, q_1) \ldots S(a_n, q_n) R(P) + \\ + O(P^{n-k-1+(n-1)\delta}), \tag{43}$$

where

$$R(P) = \int_{-1/2}^{1/2} \Big\{\prod_{i=1}^{n} I(\pm\beta)\Big\}\, d\beta, \tag{44}$$

and the ranges for a, q are given by (33) and (36).

By using Lemmas 1 and 2 and (41), it is easily shown that the error caused in (43) by extending the range for q to infinity is

$$\ll \Pi^{-\nu} P^{n-k} \sum_{q} \sum_{a} (q_1 \ldots q_n)^{-\nu},$$

where a runs over the range (36), and q over the range

$$q > \min_i \{\delta_i |\lambda_i|^{-k/(n-1)}\} P^{\sigma k}.$$

We consider a particular set of divisors $\delta_1, \ldots, \delta_n$ of $\lambda_1, \ldots, \lambda_n$, respectively, and suppose that

$$\min_i \{\delta_i |\lambda_i|^{-k/(n-1)}\} = \delta_j |\lambda_j|^{-k/(n-1)}.$$

The contribution to the above sum from the pairs a, q corresponding to $\delta_1, \ldots, \delta_n$ is

$$\ll \Pi^{-\nu} (\delta_1 \ldots \delta_n)^{\nu} P^{n-k} \sum_{q} q^{-(n\nu-1)},$$

where the range for q is

$$q > \delta_j |\lambda_j|^{-k/(n-1)} P^{\sigma k}.$$

Thus the contribution from these a, q is

$$\ll \Pi^{-\nu} P^{n-k} (\delta_1 \ldots \delta_n)^{\nu} \delta_j^{-(n\nu-2)} |\lambda_j|^{(n-2k)/(n-1)} P^{-(n-2k)\sigma}$$

$$\ll \Pi^{-\nu} P^{n-k} \Pi^{(n-2k)/(n-1)} P^{-(n-2k)\sigma}.$$

Since $\Pi \ll P^2$, the number of different possibilities for $\delta_1, \ldots, \delta_n$ is $O(P^{\varepsilon})$ and hence the total error satisfies the condition on the second error term in (38). Hence it follows from (43) that (38) holds with $R(P)$ given by (44).

By the definition (13) of $I(\beta)$, it follows that $R(P)$ is precisely a sum of the type Z considered in Lemma 6, since we may assume without loss of generality that $\lambda_1 > 0$, $\lambda_2 < 0$. Hence $R(P) \gg P^{n-k}$, and this completes the proof of the lemma.

6. The singular series. We now obtain a lower bound for $\mathfrak{S}$, the sum of the singular series defined by (39); the series is certainly absolutely convergent because, by Lemma 1 (ii),

$$(q_1 \ldots q_n)^{-1} S(a_1, q_1) \ldots S(a_n, q_n) \ll (q_1 \ldots q_n)^{-\nu} \ll \Pi^{\nu} q^{-n\nu}.$$

We may re-write the series as

$$\mathfrak{S} = \sum_{q=1}^{\infty} \sum_{\substack{a=1\\(a,q)=1}}^{q} q^{-n} S(\lambda_1 a, q) \ldots S(\lambda_n a, q).$$

As, for example, in Chapter 5 of Davenport [6], it can be shown that

$$\mathfrak{S} = \prod_{p} \chi(p), \tag{45}$$

where the product is over all primes p, and

$$\chi(p) = 1 + \sum_{s=1}^{\infty} \sum_{\substack{a=1\\p\nmid a}}^{p^s} p^{-ns} \prod_{i=1}^{n} S(\lambda_i a, p^s); \tag{46}$$

and that, furthermore,

$$\chi(p) = \lim_{s\to\infty} p^{-(n-1)s} M(p^s), \tag{47}$$

where $M(p^s)$ denotes the number of distinct solutions $(\bmod\, p^s)$ of the congruence (6). We shall need the lower bounds for $M(p^s)$ given by the following

LEMMA 11. (i) *Let p be a prime such that $p \nmid k$, and suppose that $\lambda_1, \ldots, \lambda_t$ are not divisible by p, while the remaining λ_i are all divisible by p. Then for all $s \geqslant 1$*

$$M(p^s) \geqslant p^{(n-1)(s-1)+(n-t)} N(p),$$

where $N(p)$ denotes the number of non-trivial solutions $(\bmod\, p)$ *of the congruence*

$$\lambda_1 x_1^k + \ldots + \lambda_t x_t^k \equiv 0 \pmod{p}.$$

(ii) *For each prime p there is a number $\gamma < 4k$ such that for all $s \geqslant \gamma$*

$$M(p^s) \geqslant p^{(n-1)(s-\gamma)} N(p^{\gamma}),$$

where $N(p^{\gamma})$ is the number of primitive solutions $(\bmod\, p^{\gamma})$ *of the congruence*

$$\lambda_1 x_1^k + \ldots + \lambda_n x_n^k \equiv 0 \pmod{p^{\gamma}}.$$

(*A primitive solution is one such that not all x_i are divisible by p.*)

Proof. The proof, which depends on Hensel's lemma, corresponds exactly to that of Lemma 10 of [11], and therefore I omit the details. A suitable value of γ is

$$\gamma = \max_i 2m_i,$$

where $p^{m_i} \| k\lambda_i$ for $i = 1, \ldots, n$. Since the λ_i are kth power free, m_i is certainly less than $2k$ for all i.

We can now obtain a lower bound for $\mathfrak{S}$.

LEMMA 12. (i) *There is an absolute constant $c_1 > 0$ such that if $c > c_1$, then*

$$\prod_p \chi(p) > \tfrac{1}{2} \qquad (p\nmid \Pi,\ p > c).$$

(ii) *There is a constant $c_2 = c_2(\varepsilon) > 0$ such that if $c > c_2$, then*

$$\prod_p \chi(p) \geqslant \Pi^{-\varepsilon} \qquad (p|\Pi,\ p > c).$$

(iii) *We have*

$$\mathfrak{S} \geqslant c_3 \Pi^{-\varepsilon},$$

where $c_3 = c_3(\varepsilon) > 0$.

(*The products in* (i) *and* (ii) *are over all primes which satisfy the condition in parentheses.*)

Proof. (i) Suppose $p\nmid \Pi$. If $p \nmid a$, then, by Lemma 1 (ii),

$$S(\lambda_i a, p^s) \ll p^{(1-\nu)s}.$$

Thus

$$\sum_{\substack{a=1\\ p\nmid a}}^{p^s} p^{-ns} \prod_{i=1}^{n} S(\lambda_i a, p^s) \ll p^{s(1-\nu n)} \ll p^{-2s}$$

(since $n \geqslant 3k$ for all k). Therefore, by (46),

$$|\chi(p)-1| \ll \sum_{s=1}^{\infty} p^{-2s} \ll p^{-2},$$

from which (i) follows easily.

(ii) Let p be a prime such that $p > k$ (and therefore $p \nmid k$). Since at most $n-3$ of the λ_i are divisible by p, we may assume that $\lambda_1, \ldots, \lambda_t$ are not divisible by p, while the remaining λ_i are divisible by p; where $t \geqslant 3$. Using the notation of Lemma 11 (i), we have

$$p\big(N(p)+1\big) = \sum_{a=0}^{p-1} S(\lambda_1 a, p) \ldots S(\lambda_t a, p)$$
$$= p^t + \sum_{a=1}^{p-1} S(\lambda_1 a, p) \ldots S(\lambda_t a, p).$$

Now by Lemma 1 (i), each of the $S(\lambda_i a, p)$ in the above formula has absolute value at most $(k-1)p^{1/2}$, and therefore

$$|N(p)+1-p^{t-1}| \leqslant p^{-1}(p-1)(k-1)^t p^{t/2} < (k-1)^t p^{t/2}.$$

Since $t \geqslant 3$, and therefore $t-1 > t/2$, for any $\varepsilon > 0$ there is a constant $c_2(\varepsilon, t)$ such that

$$N(p) \geqslant p^{t-1-\varepsilon}$$

provided that $p > c_2(\varepsilon, t)$.

It now follows from Lemma 11 (i) and (47) that for any $\varepsilon > 0$ there is a constant $c_2 = c_2(\varepsilon)$ (the maximum of $k, c_2(\varepsilon, 3), \ldots, c_2(\varepsilon, n)$) such that if p is any prime greater than c_2, then $M(p^s) \geqslant p^{(n-1)s-\varepsilon}$ for all s and so

$$\chi(p) \geqslant p^{-\varepsilon}.$$

The conclusion of (ii) follows from this.

(iii) Since $n \geqslant k^2+1$ for all $k \geqslant 4$, it follows from the results of Davenport and Lewis [7] which were discussed in § 1 that, in the notation of Lemma 11 (ii), $N(p^\gamma) \geqslant 1$ for all primes p; hence, by that lemma,

$$M(p^s) \geqslant p^{(n-1)(s-4k)}$$

for all primes p and all s, and so, by (47),

$$\chi(p) \geqslant p^{-4k(n-1)}$$

for all primes p.

For any $\varepsilon > 0$, let $c = \max(c_1, c_2)$, where c_1, c_2 are as in (i) and (ii). Then

$$\prod_{p\leqslant c} \chi(p) \geqslant \prod_{p \leqslant c} p^{-4k(n-1)} = c_3,$$

say, and (iii) now follows from (45), (i) and (ii).

7. Completion of the proof of Theorem 1. Assuming that (37) (and hence (8)) holds, we deduce from the corollary to Lemma 8 together with Lemmas 10 and 12 that

$$\mathcal{N}(P) \geqslant c_\varepsilon \Pi^{-\nu-\varepsilon} P^{n-k} + \Pi^{-\nu} P^{n-k} E,$$

where $c_\varepsilon > 0$ and

$$E \ll \Pi^{1/(n-1)} P^{-\sigma+2\delta} + \Pi^{\nu} P^{-1+(n-1)\delta} + (\Pi^{1/(n-1)} P^{-\sigma+\varepsilon})^{n-2k},$$

σ being defined by (17). We choose ε, δ in such a way that $0 < (n-1)\delta = \varepsilon < \frac{1}{3}\sigma$ and

$$\frac{\psi}{1-\theta} > \frac{1}{(n-1)(\sigma-3\varepsilon)};$$

this is possible since $\psi \geqslant 1/\{(n-1)\sigma\}$ by (1) and (4). It follows that if $P^{1-\theta} > 2\Pi^{\nu}$, so that (37) holds, then

$$E \ll \Pi^{1/(n-1)} P^{-\sigma+\varepsilon} \ll P^{-2\varepsilon}.$$

Hence there is a constant $D_\theta \geqslant 2$ such that if $P^{1-\theta} > D_\theta \Pi^{\nu}$, then $\mathcal{N}(P) > 0$. By the remarks at the beginning of § 4, this completes the proof of the theorem.

§ 8. Use of diminishing ranges for equation (2). Davenport and Lewis [7] obtained their results on $G^*(k)$ for $k \geqslant 12$ (see § 1) by using diminishing ranges for the variables together with Vinogradov's estimates. I shall now indicate how their method can be combined with the ideas of § 2 to 7 to obtain an analogue of Theorem 1 with smaller n; I shall omit the details, which are messy but straightforward. I shall refer to Davenport and Lewis [7] as DL.

As before, we start with a given number θ such that $0 < \theta < 1$, and we assume that $\lambda_1, \ldots, \lambda_n$ are non-zero integers satisfying (21). We also assume that the congruence condition (see § 1) corresponding to equation (2) is satisfied; the size of n will emerge from our argument. In our final result we shall have an inequality like (3) in which the exponent of $|\lambda_1\lambda_2 \ldots \lambda_n|$ is at least 1 and we shall have $n \geqslant 2k+1$; hence, by Lemma 7, we assume without loss of generality that (22) and (23) hold. For ease of reference we follow the notation of DL as much as possible; we use the same notational conventions as in § 2 above.

We define Π, Λ by (7) as before, and we re-write (2) as

$$c_1x_1^k + \ldots + c_{2r}x_{2r}^k + b_1y_1^k + \ldots + b_ty_t^k + b_1'y_1'^k + \ldots + b_t'y_t'^k = 0, \tag{48}$$

and assume, without loss of generality, that

$$\Lambda = c_1 > 0, \quad c_2 < 0.$$

We denote a typical coefficient of (48), that is, of (2), by λ. We write

$$B = \max\{|b_i|, |b_j'|;\ i, j = 2, \ldots, t\}$$

and note that

$$\Pi = |c_1 \ldots c_{2r} b_1 \ldots b_t b_1' \ldots b_t'| \geqslant \Lambda B \geqslant B^2.$$

For any large positive integer P, we write

$$P_i = P^{(1-\nu)^{i-1}} \quad (i = 1, \ldots, t),$$

where, as before, $\nu = 1/k$; we assume from the outset that

$$P_t \geqslant \Lambda^\nu, \quad P \geqslant 2\Lambda. \tag{49}$$

This time we define $\mathcal{N}(P)$ as the number of solutions of (48) such that

$$\begin{cases} P \leqslant |c_i|^\nu x_i \leqslant (2r+2)^\nu P \ (i = 1, \ldots, 2r), \\ P_i \leqslant |b_i|^\nu y_i \leqslant (2r+2)^\nu P_i,\ P_i \leqslant |b_i'|^\nu y_i' \leqslant (2r+2)^\nu P_i \ (i = 1, \ldots, t), \end{cases} \tag{50}$$

and we seek to show that $\mathcal{N}(P) > 0$.

The relevant trigonometric sums are of the form

$$T(\alpha, \lambda, Q) = \sum_x e(\lambda\alpha x^k),$$

where x runs through all integers in the range

$$Q \leqslant |\lambda|^\nu x \leqslant (2r+2)^\nu Q,$$

λ is a non-zero integer and $Q = P_i$ for some i. (The $S_i(\alpha)$ of (9) are thus $T(\alpha, \lambda_i, P)$.) We define $S(a, q)$ by (12) as before. We now define

$$I(\beta, Q) = \sum_m \nu^{-1} m^{-1+\nu} e(\beta m),$$

where m runs through all integers in the closed interval

$$[Q^k, (2r+2)Q^k];$$

we note that $I(\beta, Q)$ satisfies the result corresponding to Lemma 2. We now define

$$V(\alpha) = \prod_{i=1}^{2r} T(\alpha, c_i, P) \prod_{i=1}^{t} T(\alpha, b_i, P_i) \prod_{i=1}^{t} T(\alpha, b_i', P_i),$$

and as before we have

$$\mathcal{N}(P) = \int_J V(\alpha)\, d\alpha$$

where $J = [0, 1]$.

For each $\alpha \in J$, Davenport and Lewis considered a rational approximation a/q to α and then estimated the $T(\alpha, \lambda, Q)$ by using the fact that $\lambda a/q$ is a rational approximation to $\lambda\alpha$ since λ is integral. However, this means that the bound on the error in the approximation $\lambda a/q$ is large for large λ, which causes difficulties when we want to keep track of the contribution from λ to our error terms. One way out would be to consider separate approximations to the $\lambda\alpha$ as we did for Theorem 1; but this would prevent us from using the approach of Davenport and Lewis to the "semi-major" arcs (see below) and so would lead to a weaker result. Instead, we shall use the following lemma, which is obtained by modifying the proofs of Lemmas 7, 8, and 9 of Davenport [5].

LEMMA 3′. *Suppose that λ, γ are non-zero integers, $Q \geqslant |\lambda|^\nu > 0$, and $\alpha = (a/q) + \beta$, where*

$$(a, q) = 1, \quad 0 < q < (|\lambda|^{-\nu}Q)^{1-\delta}, \quad |\lambda\beta| \leqslant \gamma q^{-1}(|\lambda|^{-\nu}Q)^{1-k-\delta}.$$

If $(|\lambda|^{-\nu}Q)^{\delta/2} \geqslant \gamma$, then

$$T(\alpha, \lambda, Q) = |\lambda|^{-\nu} q^{-1} S(\lambda a, q) I(\pm\beta, Q) + E, \tag{51}$$

where $\pm$ is the sign of λ and $|E| \leqslant c(\varepsilon, \delta, k) q^{3/4+\varepsilon}$; and each of the two expressions on the right-hand side of (51) has absolute value

$$\leqslant c(\delta, k) q^{-\nu} Q \min(1, |\beta|^{-1} Q^{-k}).$$

Since we must apply this lemma with $\gamma = \Lambda$, $Q = P$, we cannot take δ to be small without forcing k to be very large. As the form in which we have given Lemma 4 is not satisfactory unless δ is small, it is convenient to use instead the following lemma, which is obtained from Vinogradov [12], Chapter VI, Theorem 1.

LEMMA 4′. *Suppose that* $k \geqslant 12$ *and* $|\alpha-(a/q)| \leqslant q^{-2}$, *where*

$$(a, q) = 1, \quad (|\lambda|^{-\nu}Q)^{1-\delta} \leqslant q \leqslant (|\lambda|^{-\nu}Q)^{k-1+\delta}.$$

Then

$$|T(\alpha, \lambda, Q)| \leqslant c(k, \delta)\,|\lambda|^{\mu}(|\lambda|^{-\nu}Q)^{1-\varrho(\delta)},$$

where

$$\varrho(\delta) = \left(1+\frac{1}{30k}\right)\frac{1-\delta}{3k^2\log\{12k(k+1)/(1-\delta)\}}, \tag{52}$$

$$\mu = \frac{2}{3k^2\log\big(12k(k+1)\big)}.$$

In order to use this lemma, we assume from now on that

$$k \geqslant 12.$$

Instead of using Vinogradov's mean-value theorem we shall use the following mean-value theorem, which is essentially Lemma 10 of DL.

LEMMA 5′. *We have*

$$\int_0^1 \prod_{i=1}^{t} |T(\alpha, b_i, P_i)T(\alpha, b_i', P_i)|\,d\alpha \ll |b_1\ldots b_t b_1'\ldots b_t'|^{-\nu/2}P^{k-k(1-\nu)^t}.$$

For each $\alpha \in J$ there is a rational a/q such that

$$(a, q) = 1, \quad 0 < q < (\Lambda^{-\nu}P)^{k-1/2}, \tag{53}$$

$$\left|\alpha-\frac{a}{q}\right| < q^{-1}(\Lambda^{-\nu}P)^{1/2-k}; \tag{54}$$

we shall subdivide J into major arcs, semi-major arcs, and minor arcs according to the nature of the approximation a/q. (Note that the subdivision here is slightly different from that used in DL.)

The set of all α in J such that (54) holds is called a *major arc* if and only if a and q satisfy both (53) and the condition

$$\begin{cases} q < (|c_i|^{-\nu}P)^{1/2} & (i = 1, \ldots, 2r), \\ q < (|b_i|^{-\nu}P_i)^{1/2},\ q < (|b_i'|^{-\nu}P_i)^{1/2} & (i = 1, \ldots, t). \end{cases} \tag{55}$$

We can estimate the contribution to $\mathcal{N}(P)$ from the major arcs by using Lemma 3′ with $\delta = \frac{1}{2}$ and arguing as in § 5 above; the details are very similar to those of DL, Lemma 16. It turns out that if

$$r+t \geqslant k+1, \quad r \geqslant 1, \quad P \geqslant \Lambda^5, \tag{56}$$

then the contribution from the major arcs is

$$\Pi^{-\nu}\mathfrak{S}R(P)+P^{\Phi}E, \tag{57}$$

where $\mathfrak{S}$ is precisely the singular series considered in § 6,

$$R(P) = \int_{-1/2}^{1/2} \prod_{i=1}^{2r} I(\pm\beta, P) \prod_{i=1}^{t} \{I(\pm\beta, P_i)I(\pm\beta, P_i)\}\,d\beta,$$

the $\pm$ being the signs of c_i, b_i, b_i' respectively,

$$\Phi = 2r+k-2k(1-\nu)^t, \tag{58}$$

and

$$E \ll P_t^{-1}+(P/P_t)^{-(r-1/2)}+(B^{-\nu}P_t)^{1-r\nu-t\nu}. \tag{59}$$

By a similar argument to that in DL, Lemma 15, it is easily shown that

$$R(P) \gg P^{\Phi}.$$

Our argument in § 6 really depended only on the conditions (21), (22), and (23), the inequality $n \geqslant 2k+1$ and the fact that $N(p^{\gamma}) \geqslant 1$ for all primes p. Since these conditions are all satisfied here, it follows that

$$\mathfrak{S} \geqslant c_{\varepsilon}\Pi^{-\varepsilon}, \tag{60}$$

where $c\varepsilon > 0$.

Now we consider pairs a, q such that (55) does not hold. For such a pair, the set of all α in J such that (54) holds is called a *semi-major arc* if and only if a and q satisfy both (53) and the condition

$$q < (|c_i|^{-\nu}P)^{1/2} \quad \text{for at least } r \text{ values of } i. \tag{61}$$

We now assume further that

$$t \geqslant 2; \tag{62}$$

by (49), this ensures that $\Lambda^{-\nu}P \geqslant B^{-\nu}P_t$ and hence that

$$q \geqslant (B^{-\nu}P_t)^{1/2} \tag{63}$$

for any semi-major arc. To estimate $V(\alpha)$ when α belongs to a semi-major arc, we apply the bound in Lemma 3′ to the r sums $T(\alpha, c_i, P)$ to which, by (61), it applies, and use the trivial upper bound for the remaining sums. We must then integrate with respect to α and sum with respect to a and q, using (63), to find the contribution from the semi-major arcs, which turns out to be

$$\ll P^{\Phi}(B^{-\nu}P_t)^{1-r\nu/2} \tag{64}$$

provided that

$$P \geqslant \Lambda^5, \quad r > 2k. \tag{65}$$

(Cf. DL, Lemma 14.)

We say that an element α of J *belongs to a minor arc* if and only if α belongs neither to a major arc nor to a semi-major arc. Clearly if α belongs to a minor arc, then

$$q \geqslant (|c_i|^{-\nu}P)^{1/2} \text{ for at least } r \text{ values of } i,$$

and hence we can apply the bound of Lemma 4′ (with $\delta = \frac{1}{2}$) to r of the sums $T(\alpha, c_i, P)$. The contribution from the minor arcs can then be estimated by using Lemma 5′ as in DL, Lemma 11; this contribution is

$$\ll \Pi^{-\nu/2} P^{\Phi+k(1-\nu)^t - r\varrho}, \tag{66}$$

where $\varrho = \varrho(\frac{1}{2})$ is defined by (52) with $\delta = \frac{1}{2}$.

By the discussion of major arcs we shall need at least $P_t \geqslant \Pi^\nu$ to ensure that $\mathcal{N}(P) > 0$; hence we assume that

$$P^k \geqslant (2\Pi)^{(1-\nu)^{1-t}}. \tag{67}$$

We make the further assumptions

$$r \geqslant 6k, \qquad (1-\nu)^{t-1} \leqslant \min\left\{\frac{2r\varrho}{2k-1}, \frac{1}{5k}\right\}. \tag{68}$$

Inequalities (67) and (68) imply that our previous assumptions (49), (56), (62), (65) all hold, and also that $n \geqslant 2k+1$ and

$$B^\nu \leqslant P_t^{1/2}, \qquad (1-\nu)^{t-1} < 2\left(r\varrho - k(1-\nu)^t\right).$$

Hence the following estimate of $\mathcal{N}(P)$ can be obtained from the relations (57) to (60), (64), and (66):

$$\mathcal{N}(P) \geqslant c_\varepsilon \Pi^{-\nu-\varepsilon} P^\Phi + O\{P^{\Phi-(1-\nu)^{t-1}}\} + O\{\Pi^{-\frac{1}{2}\nu} P^{\Phi-\frac{1}{2}(1-\nu)^{t-1}}\}.$$

By arguing as in § 7, we finally obtain the following

THEOREM 2. *Suppose that* $k \geqslant 12$, $n = 2r+2t$, *and write*

$$\frac{1}{\psi} = \left(1 - \frac{1}{k}\right)^{t-1},$$

$$\varrho = \left(1 + \frac{1}{30k}\right)\frac{1}{6k^2 \log\{24k(k+1)\}}.$$

Suppose also that

$$r \geqslant 6k, \qquad \frac{1}{\psi} \leqslant \min\left\{\frac{2r\varrho}{2k-1}, \frac{1}{5k}\right\}. \tag{69}$$

Then for any $\theta > 0$ *there is a constant* c_0' *with the following property. If* $\lambda_1, \ldots, \lambda_n$ *are non-zero integers which are not all of the same sign if* k *is even and which satisfy the congruence condition for* (2), *then* (2) *has a solution in non-zero integers such that*

$$|\lambda_1 x_1^k| + \ldots + |\lambda_n x_n^k| < c_0' |\lambda_1 \ldots \lambda_n|^{\psi+\theta}.$$

We note that the hypotheses of the theorem imply that $t \gg k\log k$. If $r \gg k\log k$, we can choose $t \ll k\log k$ so that (69) holds and $\psi \ll k^2$. Hence it follows from Theorem 2 and the results of Chowla and Shimura mentioned in § 1 that if k is odd and sufficiently large and $n > ck\log k$, then (2) has a solution with

$$|\lambda_1 x_1^k| + \ldots + |\lambda_n x_n^k| < |\lambda_1 \ldots \lambda_n|^{\Lambda k^2},$$

where c and Λ are suitable constants.

It is interesting that the bounds in terms of Λ given by Theorems 1 and 2 are roughly the same, since both yield a solution of (2) such that

$$|x_i| \ll \Lambda^{\Lambda k^2 \log k} \qquad (i = 1, \ldots, n). \tag{70}$$

§ 9. Use of diminishing ranges for inequality (5). In [10] I prove a theorem corresponding exactly to Theorem 1 on solutions of inequality (5) such that

$$|\lambda_1 x_1^k| + \ldots + |\lambda_n x_n^k| < K_\theta |\lambda_1 \ldots \lambda_n|^{k\psi+\theta},$$

where n and ψ are defined by (1) and (4), and $\lambda_1, \ldots, \lambda_n$ are real numbers which satisfy $|\lambda_i| \geqslant 1$ for all i and which are not all of the same sign if k is even; I shall call this theorem *Theorem* Λ. The proof in [10] is by analytic methods using equal ranges, and it depends in an essential way on Theorem 1; I now discuss briefly the possibility of improving Theorem Λ by using diminishing ranges and Theorem 2.

We assume that $k \geqslant 12$ and re-write (5) as

$$|c_1 x_1^k + \ldots + c_r x_r^k + b_1 y_1^k + \ldots b_t y_t^k + b_1' y_1'^k + \ldots + b_t' y_t'^k| < 1, \tag{71}$$

and use the notation introduced in § 8. We let $\mathcal{N}(P)$ denote the number of solutions of (71) such that (50) holds with r in place of $2r$, and we estimate $\mathcal{N}(P)$ by using the fact that

$$\mathcal{N}(P) \geqslant \mathcal{J}(P) = k \int_J V(\alpha) f(\alpha)\, d\alpha,$$

where $J = [0, \infty)$ and f is a suitable kernel function with the property that $|f(\alpha)| \leqslant 1$ for all $\alpha > 0$ (see [10], Lemma 2 or [11], Lemma 12).

Since the λ are not necessarily rational, we cannot obtain rational approximations for the $\lambda\alpha$ from those for α. Hence for $i = 1, \ldots, r$ we consider approximations a_i/q_i to the $\lambda_i \alpha = c_i \alpha$ such that (27) holds. If, for some α, $q_1 \ldots q_r$ is small enough, we can deal with (71) by applying Theorem 2 to a suitable equation in r variables (cf. [10], Lemma 4 or [11], Lemma 15); this argument requires that $r \geqslant \Gamma^*(k)$, which implies $r \geqslant k^2+1$ for certain even k.

If $q_1 \ldots q_r$ is always reasonably large, we estimate $\mathscr{J}(P)$ as follows. The main term, which comes from the small values of a, is $c\Pi^{-\nu}P^{\Phi}$, where $c > 0$ and Φ is given by (58). To estimate the contribution from those a for which at least one q_i is large, the only possibility seems to be to use the same method as for the minor arcs in § 8, except that we can now use Lemma 4; this yields an error term which is

$$\ll \Pi^{-\nu}P^{\Phi}\Lambda^{r\sigma}|b_1 \ldots b_t b_1' \ldots b_t'|^{\frac{1}{2}\nu}P^{-\sigma+k(1-\nu)^t+2\delta}, \tag{72}$$

where σ is given by (17). Similar arguments to those in [10], Lemma 5, and [11], Lemma 16, yield a reasonable upper bound for the contribution from the remaining a, provided that $r \geqslant k^2+1$, but complications arise if $r \leqslant k^2$.

It is clear from (72) that in order to show that $\mathscr{J}(P) > 0$ we need $t \gg k\log k$ and $P^{\sigma} \gg \Lambda^{t\nu}$; and hence at best we obtain a solution of (71) with

$$|x_i| \ll \Lambda^{Ak^2(\log k)^2},$$

which is slightly worse than (70). Moreover, the result obtained would be unsymmetrical in the λ_i, and would involve more than k^2+1 variables. Thus this approach does not seem to yield a marked improvement on Theorem A; this was one of the main reasons for giving a detailed proof of Theorem 1 rather than of Theorem 2.

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