THEOREM 5. *Given a regulated lattice* $(S, \wedge, \vee, \leqslant)$ *there exists a unique, locally compact, Hausdorff space* X *with a base* $\mathcal{S}$ *of interiors of compact sets such that* $(\mathcal{S}, \cap, \bigcup, \Subset)$ *is isomorphic to* $(S, \wedge, \vee, \leqslant)$. X *is compact if and only if* $(S, \wedge)$ *has an identity.*

Proof. Theorem 2 under Proposition II gives X and $\mathcal{S}$ with $(\mathcal{S}, \cap, \Subset)$ isomorphic to $(S, \wedge, \leqslant)$. Since S is a distributive lattice under $\leqslant$, the isomorphism implies that $\mathcal{S}$ is a distributive lattice under set inclusion. Hence $\vee$ corresponds to $\bigcup$ by Proposition III. Finally, the compactness criterion follows from that of Theorem 2 by Proposition I.

A study of lattices along the lines of section 5 would lead us to nothing more than Stone's theory [6]. For we have the following result whose trivial proof we omit.

PROPOSITION IV. *Let* $(S, \wedge, \vee)$ *be a lattice with partial ordering* $\leqslant$. *Then* $(S, \wedge, \vee, \leqslant)$ *is a regulated lattice if and only if* $(S, \wedge, \vee)$ *is a Boolean ring.*

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RUTGERS UNIVERSITY
THE STATE UNIVERSITY OF NEW JERSEY

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Cardinal multiplication of structures with a reflexive relation

by

Ralph McKenzie * (Berkeley, Calif.)

Introduction. This paper is a sequel to investigations of Chang, Jónsson and Tarski (reported in [1, 2, 3]) dealing with refinement properties for the operation of cardinal multiplication (direct product, cartesian product) of relational structures. The results presented here extend, and almost complete that theory, insofar as it applies to structures having a reflexive binary relation.

Our main result is a Lemma (3.1) whose formulation is rather technical, but roughly states that a structure has the "strict refinement property" defined in [3], provided that indistinguishable elements of the structure are identified. The lemma is proved for structures of the form $\mathfrak{A} = \langle A, S\rangle$ in which S is a binary relation over A and the relations $S|\breve{S}$ and $\breve{S}|S$ are connected over A; in particular it applies if S is reflexive and connected over A. The lemma yields for structures in this class a reduction of the ordinary refinement property to a purely set theoretic question, which is easily answered in every specific case if the general continuum hypothesis is assumed (Theorem 4.4). Independently of the GCH, it follows that every finite structure of this class has the refinement property.(1) Thus we obtain a useful description of the algebra of all finite reflexive isomorphism types—under operations of binary cardinal addition and multiplication—which has been suspected for some time: viz. this algebra is isomorphic to a "semi-ring" of polynomials, $Z^+[x]$ (Theorem 5.1).

Departing briefly from the main line of development, we prove in § 7 an interesting and unexpected form of the Cantor Bernstein theorem: (Corollary 7.2) *Let* $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}_0$ *and* $\mathfrak{C}_1$ *be similar relational structures of an arbitrary similarity type and assume that* $\mathfrak{C}_0 \times \mathfrak{A} \cong B$ *and* $\mathfrak{C}_1 \times \mathfrak{B} \cong \mathfrak{A}$. *If, in addition,* $\mathfrak{A}$ *is denumerable and* $\mathfrak{C}_0$ *is finite then* $\mathfrak{A} \cong \mathfrak{B}$.

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(1) This solves the central problem studied in [2].

Lemma 3.1 plays a fundamental role in the general theory of cardinal multiplication, considered as an abstract operation on reflexive binary types—i.e. on isomorphism types of structures $\langle A, R\rangle$ where R is a binary relation, reflexive over A. (This theory still presents some challenging unsolved problems, cf. § 9.) The later sections of our paper are devoted to this theory under the following headings: refinement theorems (§ 4), polynomial representations of structures (§ 5, § 6), cancellation theorems (§ 8). Among noteworthy new results we mention here the following: (Theorems 8.4 and 8.9) *Let* $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ *be reflexive binary structures, and suppose that* $\mathfrak{A}$ *is finite and that* $\mathfrak{A}\times\mathfrak{B}\cong\mathfrak{A}\times\mathfrak{C}$. *From each of the following suppositions it follows that* $\mathfrak{B}\cong\mathfrak{C}$: (1) $\mathfrak{A}$ *is connected*; (2) $\mathfrak{B}$ *is denumerable and is the cardinal sum of finitely many connected structures.*

In the ninth and final section we discuss several techniques, both old and new, by means of which the basic results can be applied to a wide variety of relational and algebraic systems. Some of the results thus obtained will be stated and some open problems connected with our work will be mentioned.

This paper is largely self contained although we frequently refer to Chang–Jónsson–Tarski [3]. We recommend that the reader consult also [1] and the first section of [6] for a history of results published prior to 1966. This opportunity is taken to give credit to Ralph Seifert and Alfred Tarski. Seifert's dissertation, [10], contained two important instances of Corollary 4.7 below, thus encouraging the author to find a more general theorem. Tarski's fertile questions led eventually to substantial improvements of the first results obtained.

1. Preliminaries. The basic concepts are to a large extent the same as in Chang–Jónsson–Tarski [3]. $B\times C$ and $\underset{i\in I}{P} A_i$ are the cartesian products, respectively, of the sets B and C and of the sets A_i $(i\in I)$, and ${}^{I}A$ is the set of all functions which map I into A. Since a natural number n is identified with the set $\{0, 1, \ldots, n-1\}$, ${}^{n}A$ is the set of all n-termed sequences $x = \langle x_0, x_1, \ldots, x_{n-1}\rangle$ all of whose terms belong to A. A relation R of rank n, or n-ary relation, over A is simply a subset of ${}^{n}A$. When R is binary we write xRy in place of $\langle x, y\rangle\in R$. The relative product of two binary relations F and G is the relation $H = F|G$ such that xHy iff for some z, $xFzGy$; $\breve{G}$ is the converse of G ($x\breve{G}y$ iff yGx); and id_A is the identity relation over A. A binary relation $R\subseteq{}^{2}A$ is reflexive over A iff $\mathrm{id}_A\subseteq R$; and is connected over A iff for every two distinct elements $x, y\in A$, there is a finite sequence $\langle z_0, \ldots, z_m\rangle$ with $z_0 = x$, $z_m = y$ and for every $k < m$, $\langle z_k, z_{k+1}\rangle\in R\cup\breve{R}$. The domain and the range of R will be denoted by $\operatorname{dom}R$ and $\operatorname{rng}R$, respectively.

The usual notation for equivalence relations will be employed so that, e.g. if E is an equivalence relation over A, then x/E is the E-class to which x belongs, and $B/E = \{x/E\colon x\in B\}$ when $B\subseteq A$. The kernel of a function $f\colon A\to B$, $\ker f$, is of course the equivalence relation E such that xEy iff $f(x) = f(y)$. The canonical map π_E from A to A/E maps x to x/E, and we have $\ker\pi_E = E$. In this paper the cardinal number of a set X will be represented by ${}^{\#}(X)$.

By a *relational structure* or, briefly, structure we mean a system $\mathfrak{A} = \langle A, R_t\rangle_{t\in T}$ consisting of a non-empty set A, and for each $t\in T$ a relation R_t of some finite positive rank $\varrho(t)$ over A. The function ϱ is called a similarity type and two structures are said to be similar if they have the same similarity type. To avoid certain trivialities we always implicitly assume that each of the fundamental relations R_t is non-empty. The notion of the isomorphism of two similar structures, written $\mathfrak{A}\cong\mathfrak{B}$, is the usual one. We follow the custom of denoting a structure, $\mathfrak{A}$, and its fundamental set, A, by a german letter and its roman equivalent. For brevity, we shall attribute to $\mathfrak{A}$ various properties of the set A—e.g. we may write that $\mathfrak{A}$ is finite, or infinite, or that some equivalence relation E is defined over $\mathfrak{A}$.

Four elementary operations on structures will be needed: cardinal product, cardinal sum, and the formation of substructures and quotient structures. If $\mathfrak{A}$ is a structure of type ϱ and B is a non-empty subset of A, then by $\mathfrak{A}(B)$, the restriction of $\mathfrak{A}$ to B, we mean the structure $\langle B, R_t\cap{}^{\varrho(t)}B\rangle_{t\in T}$. $\mathfrak{B}$ is called a *substructure* of $\mathfrak{A}$—written $\mathfrak{B}\subseteq\mathfrak{A}$—iff $\mathfrak{B} = \mathfrak{A}(B)$. Let there be given a system of similar structures

$$\mathfrak{B}_i = \langle B_i, S_{i,t}\rangle_{t\in T}, \quad i\in I. \tag{1}$$

By the *direct* (or *cardinal*) *product* of the system (1)—in symbols, $\underset{i\in I}{P}\mathfrak{B}_i$—we mean the structure $\mathfrak{B} = \langle B, S_t\rangle_{t\in T}$ where $B = \underset{i\in I}{P} B_i$ and, for each $t\in T$, S_t is the set of all $x\in{}^{\varrho(t)}B$ such that $\langle x_0(i), x_1(i), \ldots, x_{\varrho(t)-1}(i)\rangle\in S_{i,t}$ for every $i\in I$.

In order to define the sum of the system of structures (1) we first assume that the sets involved are pairwise disjoint, $B_i\cap B_j = \emptyset$ whenever $i\neq j$.(²) Then the *cardinal sum*—in symbols, $\sum_{i\in I}\mathfrak{B}_i$—is simply the union of the $\mathfrak{B}_i$, in other words the structure $\mathfrak{B} = \langle B, S_t\rangle_{t\in T}$ for which $B = \bigcup\{B_i\colon i\in I\}$ and $S_t = \bigcup\{S_{i,t}\colon i\in I\}$. If the sets B_i are not pairwise disjoint then, of course, we first replace the $\mathfrak{B}_i$ by isomorphic structures $\mathfrak{B}'_i$ whose fundamental sets are disjoint, and then construct the sum of the $\mathfrak{B}'_i$, which we again denote by $\sum_{i\in I}\mathfrak{B}_i$. Thus the cardinal sum is necessarily defined only up to isomorphism. The cardinal sum and product of two structures $\mathfrak{B}$ and $\mathfrak{C}$ will be denoted by $\mathfrak{B}+\mathfrak{C}$ and $\mathfrak{B}\times\mathfrak{C}$, respectively.

(²) Since we require that the fundamental set of a structure be non-empty, whenever the sum of a system such as (1) is discussed we implicitly assume that $I\neq\emptyset$.

We write $\mathfrak{m}\mathfrak{A}$ (respectively $\mathfrak{A}^{\mathfrak{m}}$) for the sum (product) of $\mathfrak{m}$ structures each isomorphic to $\mathfrak{A}$, where $\mathfrak{m}$ is an arbitrary cardinal number; and we express the relation "$\mathfrak{A}$ is a direct factor of $\mathfrak{C}$"—i.e. there exists a $\mathfrak{B}$ with $\mathfrak{A}\times\mathfrak{B}\cong\mathfrak{C}$—by writing $\mathfrak{A}\|\mathfrak{C}$.

The theory of cardinal addition, isolated from cardinal multiplication, is known to be rather trivial. One calls a structure $\mathfrak{A}$ *connected* if $\mathfrak{A}$ is not the sum of two of its substructures. [Note that if $\mathfrak{A} = \langle A, R\rangle$, R binary, then this is equivalent to R being connected over A as previously defined.] It is well known that every structure $\mathfrak{A}$ has a unique representation as the sum of connected substructures, called the *connected components* of $\mathfrak{A}$. From this fact the basic properties of cardinal addition are easily deduced. The importance of cardinal addition for the theory of cardinal multiplication stems from the following fact: *Let J_i $(i \in I)$ be a family of (non-void) sets, let $J = \bigcup\{J_i\colon i \in I\}$ and assume that $\mathfrak{B}_j$ $(j \in J)$ is a system of similar structures. Then*

$$\mathop{P}_{i\in I}\sum_{j\in J_i}\mathfrak{B}_j \cong \sum_{\substack{f\in P_{J_i}\\ i\in I}}\mathop{P}_{i\in I}\mathfrak{B}_{f(i)}\,.$$

Finally we define the related concepts of quotient structure and homomorphism, with the aid of some auxiliary notions. A function $f\colon A\to B$ induces a number of other maps for which we also use the symbol f. For example, if n is a natural number and $x \in {}^nA$ then we put $f(x) = \langle fx_0, fx_1, \ldots, fx_{n-1}\rangle$; if $X\subseteq A$, or $X \subseteq {}^nA$, then we put $f(X) = \{f(x)\colon x \in X\}$. Thus, if $\mathfrak{A} = \langle A, R_t\rangle_{t\in T}$ is a structure then we also have a structure $f(\mathfrak{A}) = \langle f(A), f(R_t)\rangle_{t\in T}$ which may be called the f-image of $\mathfrak{A}$. In these terms we say that f is a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, written $f\colon \mathfrak{A}\to\mathfrak{B}$, provided that $f(\mathfrak{A})\subseteq\mathfrak{B}$. Lastly, if E is an equivalence relation on $\mathfrak{A}$ and π is the canonical map $x \rightsquigarrow x/E$ then we put $\mathfrak{A}/E = \pi(\mathfrak{A}) = \langle A/E, R_t/E\rangle_{t\in T}$ and call it the *quotient structure* of $\mathfrak{A}$ mod E.

The refinement property. The formula $\mathfrak{A}\cong \mathop{P}_{i\in I}\mathfrak{B}_i$ is said to give a direct decomposition of $\mathfrak{A}$ into the factors $\mathfrak{B}_i$. Assume now that we have two isomorphic products

(1) $$\mathop{P}_{i\in I}\mathfrak{B}_i \cong \mathop{P}_{j\in J}\mathfrak{C}_j\,.$$

The assertion that the two products possess a common refinement, i.e. that (1) has a refinement, is taken to mean that there exist structures $\mathfrak{D}_{i,j}$ which satisfy

$$\mathfrak{B}_p \cong \mathop{P}_{j\in J}\mathfrak{D}_{p,j} \quad\text{and}\quad \mathfrak{C}_q \cong \mathop{P}_{i\in I}\mathfrak{D}_{i,q}$$

for all $p \in I$ and $q \in J$. As in [3], we say that a structure $\mathfrak{A}$ has the *refinement property* iff every pair of direct decompositions of $\mathfrak{A}$ possess a common refinement. We recall that, at least for finite structures, the refinement property is equivalent to the *unique factorization property*, i.e. to the proposition that a given structure has up to isomorphism exactly one decomposition into a product of indecomposable structures. (Where $\mathfrak{B}$ is indecomposable if $\#(\mathfrak{B}) > 1$, and $\mathfrak{B}\cong\mathfrak{C}\times\mathfrak{D}$ implies either $\#(\mathfrak{C}) = 1$ or $\#(\mathfrak{D}) = 1$.)

As was mentioned in the introduction, the principal object of this paper is to further extend the class of structures known to have the refinement property. We will now describe the methods developed for this purpose in [3].

Factor relations and decomposition functions. Every direct decomposition of a quotient structure $\mathfrak{A}/E$, when coupled with a specific isomorphism, say

$$h\colon \mathfrak{A}/E \cong \mathop{P}_{i\in I}\mathfrak{B}_i\,,$$

leads to a standard decomposition

(1) $$\bar{h}\colon \mathfrak{A}/E \cong \mathop{P}_{i\in I}\mathfrak{A}/F_i\,,$$

where $\mathfrak{A}/F_i \cong \mathfrak{B}_i$ for each i, and we have

(2) $$E\subseteq F_i \text{ and } \bar{h}(x/E)_i = x/F_i\,, \quad \text{for every } i\in I \text{ and } x\in A\,.$$

The relations F_q appearing in the standard decomposition are defined by the formula $F_q = \ker(\mathrm{pr}_q\circ h\circ\pi_E)$, where pr_q is the projection homomorphism from $\mathop{P}_{i\in I}\mathfrak{B}_i$ onto $\mathfrak{B}_q$.(³) In case $E = \mathrm{id}_A$ we replace $\mathfrak{A}/E$ by $\mathfrak{A}$ and x/E by x in the above and arrive at the important notion of a standard decomposition of $\mathfrak{A}$ associated with a given decomposition $\mathfrak{A}\cong\mathop{P}_{i\in I}\mathfrak{B}_i$. The relations F_i $(i\in I)$ are then said to form a (complete) system of *factor relations* for $\mathfrak{A}$. More precisely, we make the following definitions.

DEFINITION 1.1. Suppose that $E\subseteq F_i$ $(i\in I)$ where E and F_i are equivalence relations over $\mathfrak{A}$. We say that E is the direct product of the relations F_i—in symbols

$$E = \prod_{i\in I} F_i\,,$$

if (1) holds when the map $\bar{h}$ is defined by (2). This is equivalent to the satisfaction of the following conditions:

(i) $E = \bigcap\{F_i\colon i\in I\}$ (if $I = \emptyset$ this means $E = {}^2A$).

(ii) For each $x\in {}^IA$ there exists $u\in A$ such that x_iF_iu for all $i\in I$.

(³) The restriction that none of the fundamental relations of a structure is empty is necessary to make the projection map a homomorphism.

(iii) For all $t \in T$ and $x \in {}^{\varrho(t)}A$, if $x/F_i \in R_t/F_i$ for all $i \in I$, then $x/E \in R_t/E$.

If G and H are equivalence relations over $\mathfrak{A}$, then we put

$$G \times H = \prod_{i \in I} F_i$$

where $F_0 = G$ and $F_1 = H$—provided this product exists.

Definition 1.2. (i) Suppose that E and F are equivalence relations over $\mathfrak{A}$. F is said to be a factor relation of E over $\mathfrak{A}$ provided $E = F \times F'$ for some equivalence relation F' over $\mathfrak{A}$.

(ii) We denote by $\mathrm{FR}(\mathfrak{A}, E)$ the set of all factor relations of E over $\mathfrak{A}$, and we put $\mathrm{FR}(\mathfrak{A}) = \mathrm{FR}(\mathfrak{A}, \mathrm{id}_A)$—i.e. the set of all factor relations of $\mathfrak{A}$.

For fairly obvious reasons, virtually every positive result regarding the refinement property has involved a study of the factor relations of a structure, in some form or other. Typically, some weak and rather natural assumptions on $\mathfrak{A}$ are shown to imply special properties of $\mathrm{FR}(\mathfrak{A})$ from which the existence of refinements can be deduced. The concept of a decomposition function, which will be defined next, is a very convenient aid for formulating and proving these results. It is possible to associate with every standard decomposition of a structure a decomposition function which carries all the relevant information. However, for our purposes, it proves sufficient to work with decomposition functions for products of two factors only.

Definition 1.3. (i) Suppose that $F, F' \in \mathrm{FR}(\mathfrak{A})$ and $\mathrm{id}_A = F \times F'$. By the associated *decomposition function* of $\mathfrak{A}$ we mean the unique function $f\colon {}^2A \to A$ such that $xFf(x, y)$ and $f(x, y)F'y$ for all $x, y \in A$. The set of all decomposition functions of $\mathfrak{A}$ will be denoted by $\mathrm{DF}(\mathfrak{A})$.

(ii) Let $f \in \mathrm{DF}(\mathfrak{A})$. Then f^d is the decomposition function for which $f^d(x, y) = f(y, x)$. If $x, y \in A$ then we put $f_x(y) = f(y, x)$ and $f_x^d(y) = f(x, y)$.

Clearly, if $f \in \mathrm{DF}(\mathfrak{A})$ is the decomposition function for the decomposition $\mathrm{id}_A = F \times F'$, then we may recapture F and F' from f—in fact, $F = \ker f_x$ and $F' = \ker f_x^d$ for any $x \in A$. From this, one may easily derive the intrinsic characterization of the set $\mathrm{DF}(\mathfrak{A})$ [3; Definition 5.1 & Corollary 5.2]. The basic properties of factor relations and decomposition functions required in this paper are rather trivial and will be mentioned at the appropriate place in § 2.

Strong refinement properties. We briefly recall the two strong refinement properties which were defined and studied in [3]. A structure $\mathfrak{A}$ possesses the *strict refinement property* if the formula $\mathrm{id}_A = \prod_{i \in I} F_i = \prod_{j \in J} G_j$ always implies that $F_p = \prod_{j \in J} F_p | G_j$ and $G_q = \prod_{i \in I} F_i | G_q$, for every $p \in I$

and for every $q \in J$. A structure with this property obviously has the refinement property, as defined previously. In fact, if we replace two direct decompositions $\mathfrak{A} \cong \underset{i \in I}{P}\, \mathfrak{B}_i \cong \underset{j \in J}{P}\, \mathfrak{C}_j$ by corresponding standard decompositions $\mathfrak{A} \cong \underset{i \in I}{P}\, \mathfrak{A}/F_i \cong \underset{j \in J}{P}\, \mathfrak{A}/G_j$, then the structures $\mathfrak{D}_{i,j} = \mathfrak{A}/(F_i|G_j)$ will serve as the desired refinement. The following equivalent form of the strict refinement property proves very useful and suggestive.

Theorem 1.4. [3; Theorem 5.6] *The structure $\mathfrak{A}$ possesses the strict refinement property iff whenever $f, g \in \mathrm{DF}(\mathfrak{A})$, and $x, y \in A$, we have $f_x g_x(y) = g_x f_x(y)$.*

The second of the strong refinement properties can be regarded as a local form of the first. The pair $(\mathfrak{A}, u)$ has the *intermediate refinement property* (where $u \in A$) provided that the two formulas $f_u g_u(x) = u$ and $g_u f_u(x) = u$ are equivalent, for all $f, g \in \mathrm{DF}(\mathfrak{A})$ and for all $x \in A$. If the pair $(\mathfrak{A}, u)$ has the intermediate refinement property, we cannot immediately infer that $\mathfrak{A}$ has the (ordinary) refinement property. However, the implication does hold when u is a reflexive element in $\mathfrak{A}$—i.e. $\langle u, u, \ldots, u\rangle \in R_t$ for each of the fundamental relations R_t of $\mathfrak{A}$ [3; Theorem 6.6]. In that case, given that

$$\mathrm{id}_A = \prod_{i \in I} F_i = \prod_{j \in J} G_j,$$

the quotient structures $\mathfrak{A}/F_p$ and $\mathfrak{A}/G_q$ will be canonically isomorphic to the products of substructures, $\underset{j \in J}{P}\, \mathfrak{A}(u/H_{p,j})$ and $\underset{i \in I}{P}\, \mathfrak{A}(u/H_{i,q})$, respectively, where

$$H_{p,q} = \bigcap \{F_i\colon i \neq p\} \cap \bigcap \{G_j\colon j \neq q\}.$$

The refinement property for cardinal numbers. Cardinal multiplication applied to structures of the form $\mathfrak{A} = \langle A\rangle$, having no relations, is basically no more than the ordinary multiplication of cardinal numbers. We have to say a few words on this topic because some of our considerations reduce to this. By the product of a system, $\mathfrak{b}_i\ (i \in I)$, of cardinal numbers—in symbols, $\underset{i \in I}{P}\, \mathfrak{b}_i$—we mean, of course, the cardinal number $\mathfrak{b}$ which is equal to the cardinality of any cartesian product, $\underset{i \in I}{P}\, X_i$, in which $\#(X_i) = \mathfrak{b}_i$ for every $i \in I$. The theory of this operation is obviously strongly influenced by the fundamental assumptions of set theory, e.g. by the continuum hypothesis or its negation.

We say that a non-zero cardinal number $\mathfrak{n}$ has the *refinement property* iff the formula

$$\mathfrak{n} = \underset{i \in I}{P}\, \mathfrak{n}_i = \underset{j \in J}{P}\, \mathfrak{m}_j$$

always implies the existence of cardinals $\mathfrak{d}_{i,j}$ such that

$$\mathfrak{n}_p = \underset{j \in J}{P} \mathfrak{d}_{p,j} \quad \text{and} \quad \mathfrak{m}_q = \underset{i \in I}{P} \mathfrak{d}_{i,q}$$

for all $p \in I$ and $q \in J$. Let us observe that the above formula will have a refinement if for some $p \in I$ and $q \in J$, $\mathfrak{n}_p = \mathfrak{n} = \mathfrak{m}_q$; for we can then take $\mathfrak{d}_{i,q} = \mathfrak{n}_i$ and $\mathfrak{d}_{p,j} = \mathfrak{m}_j$, and $\mathfrak{d}_{i,j} = 1$ for all $i \neq p$ and $j \neq q$. Observe that every finite (non-zero) cardinal has the refinement property, and likewise the least infinite cardinal, $\aleph_0$. An infinite cardinal of the form $2^{\mathfrak{n}}$ does not have the refinement property ($2^{\mathfrak{n}} = 3^{\mathfrak{n}}$, and this cannot be refined).

If the general continuum hypothesis is assumed then it is an easy matter to decide which cardinals have the refinement property, viz. they are the finite (non-zero) numbers and the infinite limit cardinals. For example, the continuum hypothesis—$2^{\aleph_0} = \aleph_1$—is equivalent to the assertion that $\aleph_1$ does not have the refinement property. The *refinement problem* for cardinal numbers lies outside the scope of our paper and we will say no more about it.

We close this section with an elementary result which will be used as a lemma in the proof of Theorem 4.4. We give the proof now in order to avoid obscuring the principal features of the later argument; however, since it is rather involved and not very well motivated, the reader might profitably postpone reading what follows until the need for it arises in § 4.

DEFINITION 1.5. Suppose that $X_{i,j}$ ($i \in I$, $j \in J$) is a system of non-empty sets indexed by a product set $I \times J$. Let $\langle p, q\rangle \in I \times J$.

(i) We put $X_{p,*} = \underset{j \in J}{P} X_{p,j}$, and likewise, $X_{*,q} = \underset{i \in I}{P} X_{i,q}$.

(ii) For an arbitrary function f belonging to the product of the system $X_{i,j}$, we define $f_{p,*} \in X_{p,*}$ and $f_{*,q} \in X_{*,q}$ by the formulas $f_{p,*}(j) = f(p,j)$ and $f_{*,q}(i) = f(i,q)$.

THEOREM 1.6. *Let X be the cartesian product of a family of non-empty sets $X_{i,j}$ ($i \in I$, $j \in J$). Assume that correlated with all the functions $f \in X$, $g \in X_{p,*}$ and $h \in X_{*,q}$ ($\langle p, q\rangle \in I \times J$) there are given non-zero cardinal numbers $\mathfrak{n}(f)$, $\mathfrak{n}_p(g)$ and $\mathfrak{m}_q(h)$, respectively, which satisfy*

(i) $\mathfrak{n}(f) = \underset{i \in I}{P} \mathfrak{n}_i(f_{i,*}) = \underset{j \in J}{P} \mathfrak{m}_j(f_{*,j})$

for every $f \in X$. Assume also that the smallest of the numbers $\mathfrak{n}(f)$, call it $\bar{\mathfrak{n}}$, has the refinement property, and that either

(ii) *$\bar{\mathfrak{n}}$ is finite; or*

(ii′) *$\bar{\mathfrak{n}} = \mathfrak{n}(f)$ for precisely one element $f \in X$.*

Under these assumptions, there exists a system of cardinal numbers $\mathfrak{d}_{i,j}(x)$, for $x \in X_{i,j}$, which satisfies

(iii) *$\mathfrak{n}_p(g) = \underset{j \in J}{P} \mathfrak{d}_{p,j}\big(g(j)\big)$ and $\mathfrak{m}_q(h) = \underset{i \in I}{P} \mathfrak{d}_{i,q}\big(h(i)\big)$ whenever $\langle p, q\rangle \in I \times J$, $g \in X_{p,*}$ and $h \in X_{*,q}$.*

Proof. We derive the conclusion first under the assumption that the families $X_{i,j}$, $\mathfrak{n}$, $\mathfrak{n}_i$, $\mathfrak{m}_j$ satisfy (i) and (ii). We proceed by induction on the number of prime factors, counting repeated factors, of the finite number $\bar{\mathfrak{n}}$.

Let first $\bar{\mathfrak{n}} = 1$. Choose $\bar{f} \in X$ to satisfy $\bar{\mathfrak{n}} = \mathfrak{n}(\bar{f})$. For each $\langle k, l\rangle \in I \times J$ and $x \in X_{k,l}$ define $\bar{f}[k, l; x] \in X$ to be the function which has the value x at $\langle k, l\rangle$ and agrees with $\bar{f}$ everywhere else. We can see now that there is only one possible choice for $\mathfrak{d}_{k,l}(x)$. In fact, for every $f \in X$, (i) and (iii) imply $\mathfrak{n}(f) = \underset{i,j}{P} \mathfrak{d}_{i,j}\big(f(i,j)\big)$. Whence we must have $\mathfrak{d}_{i,j}\big(\bar{f}(i,j)\big) = 1$ and

$$\mathfrak{d}_{k,l}(x) = \mathfrak{n}(\bar{f}[k, l; x]) \,, \tag{1.1}$$

since $\bar{f}[k, l; x]$ agrees with $\bar{f}$ except at $\langle k, l\rangle$. We define $\mathfrak{d}_{k,l}(x)$ by (1.1) and verify that (iii) is then satisfied.

It follows from (i) that $\mathfrak{n}_i(\bar{f}[k, l; x]_{i,*}) = \mathfrak{n}_i(\bar{f}_{i,*}) = 1$ and $\mathfrak{m}_j(\bar{f}[k, l; x]_{*,j}) = \mathfrak{m}_j(\bar{f}_{*,j}) = 1$ for $i \neq k$ and $j \neq l$. Thus from (i), applied to $f = \bar{f}[k, l; x]$, we obtain (by (1.1))

$$\mathfrak{d}_{k,l}(x) = \mathfrak{n}_k(\bar{f}[k, l; x]_{k,*}) = \mathfrak{m}_l(\bar{f}[k, l; x]_{*,l}) \,. \tag{1.2}$$

Finally, let $p \in I$ and let $g \in X_{p,*}$. Defining $f \in X$ by the requirement that $f_{p,*} = g$ and $f_{i,*} = \bar{f}_{i,*}$ for all $i \in I - \{p\}$, we have that $f_{*,j} = \bar{f}[p, j; g(j)]_{*,j}$ for all $j \in J$. Thus, in view of (1.2), the condition (i) for this f, when factors equal to 1 are discarded, becomes

$$\mathfrak{n}(f) = \mathfrak{n}_p(g) = \underset{j \in J}{P} \mathfrak{d}_{p,j}\big(g(j)\big) \,. \tag{1.3}$$

The other half of (iii) may be verified in the same way, thus concluding the proof in the case $\bar{\mathfrak{n}} = 1$.

Let now $\bar{\mathfrak{n}}$ be a finite integer greater than 1, and assume that the theorem is valid in every case in which the least of the numbers $\mathfrak{n}(f)$ is a finite integer with fewer prime factors than $\bar{\mathfrak{n}}$ (counting repeated factors). Again we choose $\bar{f}$ such that $\bar{\mathfrak{n}} = \mathfrak{n}(\bar{f})$. Let $\mathfrak{p}$ be one of the prime divisors of $\bar{\mathfrak{n}}$, and let $I_0 = \{i \in I\colon \mathfrak{n}_i(\bar{f}_{i,*}) \neq 1\}$ and $J_0 = \{j \in J\colon \mathfrak{m}_j(\bar{f}_{*,j}) \neq 1\}$. From (i) and the assumption that $\bar{\mathfrak{n}}$ is finite we have

(2.1) *I_0 and J_0 are finite; $\mathfrak{n}_i(\bar{f}_{i,*}) = 1$ for $i \in I - I_0$ and $\mathfrak{m}_j(\bar{f}_{*,j}) = 1$ for $j \in J - J_0$.*

Our plan involves constructing a new system $\mathfrak{n}'$, $\mathfrak{n}'_p$, $\mathfrak{m}'_q$ satisfying (i) and for which $\mathfrak{n}'(\bar{f}) = \bar{\mathfrak{n}}/\mathfrak{p}$. The following well-known facts will be needed: Writing $\mathfrak{f}\cdot\mathfrak{g}$ for the product of two cardinals $\mathfrak{f}$, $\mathfrak{g}$, and $\mathfrak{g}|\mathfrak{h}$ ($\mathfrak{g}$ divides $\mathfrak{h}$) to denote the existence of an $\mathfrak{f}$ such that $\mathfrak{h} = \mathfrak{f}\cdot\mathfrak{g}$, we have

(a) if $\mathfrak{f}\cdot\mathfrak{g}=\mathfrak{f}\cdot\mathfrak{h}$, where $\mathfrak{f}$ is a finite non-zero cardinal then $\mathfrak{g}=\mathfrak{h}$;
(b) if $\mathfrak{p}\mid \mathop{P}\limits_{k\in K}\mathfrak{h}_k$, where $\mathfrak{p}$ is a (finite) prime number and $\mathfrak{h}_k=1$ for all but a finite number of $k\in K$, then $\mathfrak{p}\mid\mathfrak{h}_k$ for some $k\in K$.

The key statement to be proved is the following:

(2.2) *There exist $i_0\in I$ and $j_0\in J$ such that for every $g\in X_{i_0,*}$ and for every $h\in X_{*,j_0}$, if $g(j_0)=h(i_0)=\bar{f}(i_0,j_0)$ then $\mathfrak{p}\mid\mathfrak{n}_{i_0}(g)$ and $\mathfrak{p}\mid\mathfrak{m}_{j_0}(h)$.*

In the proof of (2.2) we will assume without loss of generality

(2.3) *Whenever $i\in I$ there is a function $g\in X_{i,*}$ such that $\mathfrak{p}$ does not divide $\mathfrak{n}_i(g)$.*

If (2.3) fails we can get (2.2) very quickly as follows. Let $i_0\in I$ such that $\mathfrak{p}\mid\mathfrak{n}_{i_0}(g)$ for all $g\in X_{i_0,*}$. By (i), $\mathfrak{p}\mid\mathfrak{n}(f)$ for all $f\in X$; and if $f\in X$ satisfies $f_{*,j}=\bar{f}_{*,j}$ for all $j\in J-J_0$, then by (i), (2.1) and fact (b) we must have $\mathfrak{p}\mid\mathfrak{m}_j(f_{*,j})$ for some $j\in J_0$. These facts imply that for some $j_0\in J_0$, $\mathfrak{p}\mid\mathfrak{m}_{j_0}(h)$ for every $h\in X_{*,j_0}$—simply because any system of functions $h^{(j)}\in X_{*,j}$ $(j\in J)$ defines a function $f\in X$ for which $f_{*,j}=h^{(j)}$ for every $j\in J$. Now the i_0, j_0 just obtained will fulfill (2.2).

Proceeding with the proof of (2.2) under the assumption (2.3), we let i_0 be any element of I for which $\mathfrak{p}\mid\mathfrak{n}_{i_0}(\bar{f}_{i_0,*})$ (existence of i_0 is ensured by (i), (2.1) and fact (b)). Now, by (i), (2.1) and (b), assuming $f\in X$ satisfies $f_{*,j}=\bar{f}_{*,j}$ for $j\in J-J_0$, and also $f_{i_0,*}=\bar{f}_{i_0,*}$—i.e. $f_{*,j}$ agrees with $\bar{f}_{*,j}$ at i_0, for every $j\in J$—we necessarily have that $\mathfrak{p}\mid\mathfrak{m}_j(f_{*,j})$ for some j. Hence by an argument entirely analogous to that in the last paragraph, we can choose $j_0\in J$ satisfying

(2.4) *If $h\in X_{*,j_0}$ and $h(i_0)=\bar{f}(i_0,j_0)$, then $\mathfrak{p}\mid\mathfrak{m}_{j_0}(h)$.*

It remains to verify that (2.3) and (2.4) imply that the pair i_0, j_0 fulfill (2.2); in otherwords, letting g be any element of $X_{i_0,*}$ which satisfies $g(j_0)=\bar{f}(i_0,j_0)$, we have to show that $\mathfrak{p}\mid\mathfrak{n}_{i_0}(g)$. By (2.1) and (2.3), there is an $f\in X$ for which $f_{i_0,*}=g$, $\mathfrak{n}_i(f_{i,*})=1$ if $i\in I-I_0$ and $i\neq i_0$, and $\mathfrak{p}$ does not divide $\mathfrak{n}_i(f_{i,*})$ if $i\in I_0$ and $i\neq i_0$. For such an f, (i) and (2.4) give $\mathfrak{p}\mid\mathfrak{n}(f)$ (since $f_{*,j_0}(i_0)=g(j_0)=\bar{f}(i_0,j_0)$); hence by (i) and (b), $\mathfrak{p}\mid\mathfrak{n}_i(f_{i,*})$ for some i. The only possibility is $i=i_0$—i.e. $\mathfrak{p}\mid\mathfrak{n}_{i_0}(g)$.

The argument for statement (2.2) is now complete and we continue with the inductive proof of our theorem for the given case $1<\bar{\mathfrak{n}}<\aleph_0$. Selecting i_0, j_0 to satisfy (2.2) we define, for each $i\in I$, $j\in J$ and for every $f\in X$, $g\in X_{i,*}$, $h\in X_{*,j}$

(2.5) (1) $\mathfrak{n}'(f)=\mathfrak{n}(f)/\mathfrak{p}$ *if* $f(i_0,j_0)=\bar{f}(i_0,j_0)$; *and* $\mathfrak{n}'(f)=\mathfrak{n}(f)$ *otherwise.*
(2) $\mathfrak{n}'_i(g)=\mathfrak{n}_i(g)/\mathfrak{p}$ *if* $i=i_0$ *and* $g(j_0)=\bar{f}(i_0,j_0)$; *while* $\mathfrak{n}'_i(g)=\mathfrak{n}_i(g)$ *otherwise*
(3) $\mathfrak{m}'_j(h)=\mathfrak{m}_j(h)/\mathfrak{p}$ *if* $j=j_0$ *and* $h(i_0)=\bar{f}(i_0,j_0)$; *while* $\mathfrak{m}'_j(h)=\mathfrak{m}_j(h)$ *otherwise.*

An easy application of fact (a) shows that the system $X_{i,j}$, $\mathfrak{n}'$, $\mathfrak{n}'_i$, $\mathfrak{m}'_j$ $(i\in I, j\in J)$ satisfies condition (i). Moreover, the smallest of the cardinal numbers $\mathfrak{n}'(f)$ is clearly $\bar{\mathfrak{n}}/\mathfrak{p}$. We may therefore employ the induction assumption to obtain a system of cardinals $\mathfrak{d}'_{i,j}(x)$ $(x\in X_{i,j})$ satisfying

(2.6) $\mathfrak{n}'_p(g)=\mathop{P}\limits_{j\in J}\mathfrak{d}'_{p,j}(g(j))$ *and* $\mathfrak{m}'_q(h)=\mathop{P}\limits_{i\in I}\mathfrak{d}'_{i,q}(h(i))$ *whenever* $\langle p, q\rangle\in I\times J$, $g\in X_{p,*}$ *and* $h\in X_{*,q}$.

The last step, of course, is to put

(2.7) $\mathfrak{d}_{i,j}(x)=\mathfrak{p}\cdot\mathfrak{d}'_{i,j}(x)$ *if* $\langle i,j\rangle=\langle i_0,j_0\rangle$ *and* $x=\bar{f}(i_0,j_0)$; *while* $\mathfrak{d}_{i,j}(x)=\mathfrak{d}'_{i,j}(x)$ *otherwise.*

It is an immediate consequence of (2.5), (2.6) and (2.7) that the system $\mathfrak{d}_{i,j}(x)$ satisfies statement (iii) of our theorem, and the inductive proof of theorem in the case of finite $\bar{\mathfrak{n}}$ is now complete.

To draw these tedious deliberations to a close, let us now assume that we are in the remaining case, where (i) and (ii′) are satisfied and $\bar{\mathfrak{n}}$ is an infinite cardinal having the refinement property. Let $\bar{f}$ be the unique member of X for which $\bar{\mathfrak{n}}=\mathfrak{n}(\bar{f})$. Since $\bar{\mathfrak{n}}$ has the refinement property the formula (i) with $f=\bar{f}$ has a refinement, and we have a system of cardinals $\mathfrak{z}_{i,j}$ $(i\in I, j\in J)$ for which

(3.1) $\mathfrak{n}_p(\bar{f}_{p,*})=\mathop{P}\limits_{j\in J}\mathfrak{z}_{p,j}$ *and* $\mathfrak{m}_q(\bar{f}_{*,q})=\mathop{P}\limits_{i\in I}\mathfrak{z}_{i,q}$ *whenever* $\langle p, q\rangle\in I\times J$.

Let us define, for $\langle i,j\rangle\in I\times J$, $f\in X$, $g\in X_{i,*}$, $h\in X_{*,j}$

(3.2) (1) $\mathfrak{n}'(f)=1$ *if* $f=\bar{f}$; $\mathfrak{n}'(f)=\mathfrak{n}(f)$ *otherwise.*
(2) $\mathfrak{n}'_i(g)=1$ *if* $g=\bar{f}_{i,*}$; $\mathfrak{n}'_i(g)=\mathfrak{n}_i(g)$ *otherwise.*
(3) $\mathfrak{m}'_j(h)=1$ *if* $h=\bar{f}_{*,j}$; $\mathfrak{m}'_j(h)=\mathfrak{m}_j(h)$ *otherwise.*

Using the familiar fact that the product $\mathfrak{f}\cdot\mathfrak{g}$ of two cardinals, at least one of which is infinite, is equal to the larger of the two we can demonstrate

(3.3) *The system $\mathfrak{n}'$, $\mathfrak{n}'_i$, $\mathfrak{m}'_j$ $(i\in I, j\in J)$ satisfies* (i). *Moreover*
(1) $\mathfrak{n}_i(g)=\mathfrak{n}'_i(g)\cdot\mathfrak{n}_i(\bar{f}_{i,*})$ *for all* $g\in X_{i,*}$;
(2) $\mathfrak{m}_j(h)=\mathfrak{m}'_j(h)\cdot\mathfrak{m}_j(\bar{f}_{*,j})$ *for all* $h\in X_{*,j}$.

The central fact used here is that whenever $g\in X_{i,*}$ and $g\neq\bar{f}_{i,*}$ then $\mathfrak{n}_i(g)>\bar{\mathfrak{n}}$—otherwise, defining $f\in X$ so that $f_{i,*}=g$ and $f_{k,*}=\bar{f}_{k,*}$ for $k\neq i$, we should have $f\neq\bar{f}$ but

$$\mathfrak{n}(f)=\mathfrak{n}_i(g)\cdot\big(\mathop{P}\limits_{k\neq i}\mathfrak{n}_k(\bar{f}_{k,*})\big)\leqslant\bar{\mathfrak{n}}\,.$$

Similarly, $\mathfrak{n}_j(h) > \bar{\mathfrak{n}}$ if $h \in X_{*,j}$ and $h \neq \bar{f}_{*,j}$. From these observations (3.3.1) and (3.3.2) follow readily. Then the argument to show that $\mathfrak{n}'$, $\mathfrak{n}'_i$, $\mathfrak{m}'_j$ satisfy (i) comes in two cases. Let $f \in X$. If $f = \bar{f}$ it's trivial. If $f \neq \bar{f}$, then letting $\mathfrak{n} = \underset{i \in I}{P}\, \mathfrak{n}'_i(f_{i,*})$ we have $\bar{\mathfrak{n}} < \mathfrak{n}$ and so

$$\mathfrak{n} = \mathfrak{n} \cdot \bar{\mathfrak{n}} = \underset{i \in I}{P}[\mathfrak{n}'_i(f_{i,*}) \cdot \mathfrak{n}_i(\bar{f}_{i,*})] = \mathfrak{n}'(f),$$

by (3.2.1) and (3.3.1). That $\mathfrak{n}'(f) = \underset{j \in J}{P}\, \mathfrak{m}'_j(f_{*,j})$ is obtained in a similar way.

To complete the proof of Theorem 1.6, we observe that (by (3.2) and (3.3)) the existence of a system, $\mathfrak{d}'_{i,j}(x)$, satisfying (2.6) with the new $\mathfrak{n}'$, $\mathfrak{n}'_i$, $\mathfrak{m}'_j$ is ensured by the half of the theorem already proved. Putting

$$\mathfrak{d}_{i,j}(x) = \mathfrak{z}_{i,j} \cdot \mathfrak{d}'_{i,j}(x) \quad (x \in X_{i,j}),$$

it is a trivial matter to derive (iii) from (2.6), (3.1) and (3.3.1 & 2).

2. Binary structures. Our principal interest is in the so-called binary structures, i.e. those of the form $\langle B, S\rangle$ where S has rank two. Although our first investigations were directed to the case where S is reflexive, it now appears that no extra efforts are required to establish the basic results for a certain class of connected structures which properly includes the class of all connected binary structures with a reflexive relation, viz. for the class $\boldsymbol{Q}$ introduced in Definition 2.1 below.

This section contains the definition and basic properties of special notions which we will apply in this paper only to binary structures. The next section gives the proof of the fundamental lemma regarding direct decompositions for structures in the class $\boldsymbol{Q}$, while the following sections develop the consequences of the lemma.

Applications of the results beyond the domain of binary structures will be discussed in some detail in § 9. As in [3], many interesting applications can be obtained in a trivial way through the observation that if $\mathfrak{A} = \langle A, R_t\rangle_{t \in T}$, and if either R is one of the relations R_t, or it can be obtained from the R_t by means of certain admissible operations, then $\mathrm{DF}(\mathfrak{A}) \subseteq \mathrm{DF}(\langle A, R\rangle)$. Thus if there can be constructed by such means a binary relation R satisfying some weak conditions then our fundamental lemma gives some significant properties of the decomposition functions of $\mathfrak{A}$. (The admissible operations on relations are, of course, simply the operations $\mathfrak{C}$ for which always

$$\mathrm{DF}(\langle A, R_1, \ldots, R_n\rangle) \subseteq \mathrm{DF}(\langle A, \mathfrak{C}(R_1, \ldots, R_n)\rangle);$$

for example, the operations of concatenation, intersection, permutation, and universal and existential projection are admissible.)

Definition 2.1. We define four classes of binary structures:

(1) $\boldsymbol{R} = \{\langle B, S\rangle\colon S$ is reflexive over $B\}$.

(2) $\boldsymbol{R}^c = \{\langle B, S\rangle\colon S$ is connected and reflexive over $B\}$.

(3) $\boldsymbol{U} = \{\langle B, S\rangle\colon S = {}^2B\}$.

(4) $\boldsymbol{Q} = \{\langle B, S\rangle\colon S|\breve{S}$ and $\breve{S}|S$ are connected over $B\}$.

One can easily verify that $\boldsymbol{U} \subset \boldsymbol{R}^c \subset \boldsymbol{Q}$.

Definition 2.2. Given an arbitrary binary structure $\mathfrak{B} = \langle B, S\rangle$, we define the "quasi-ordering" relations and the "quasi-identity" relation (4) of $\mathfrak{B}$:

$$x \leqslant_l y \iff \forall_z \in B(zSx \to zSy),$$

$$x \leqslant_r y \iff \forall_z \in B(xSz \to ySz),$$

$$x \leqslant y \iff x \leqslant_l y \;\&\; x \leqslant_r y,$$

$$x \approx y \iff x \leqslant y \;\&\; y \leqslant x;$$

if it becomes necessary to distinguish $\mathfrak{B}$ from some other structures then we write $\approx^{\mathfrak{B}}$ in place of $\approx$. We put

$$\mathfrak{SK}(\mathfrak{B}) = \mathfrak{B}/\approx,$$

and call it the *skeleton of* $\mathfrak{B}$. We call $\mathfrak{B}$ *thin* iff $\approx\, = \mathrm{id}_B$. (Elements of $\boldsymbol{U}$ might be called "obese".)

Obviously, the skeleton of $\mathfrak{B}$ is thin; an equivalent condition for $\mathfrak{B}$ to be thin is that $\mathfrak{B} \cong \mathfrak{SK}(\mathfrak{B})$.

Lemma 2.3. *Let* $\boldsymbol{K}$ *be any one of the classes* $\boldsymbol{R}$, $\boldsymbol{R}^c$, $\boldsymbol{U}$, $\boldsymbol{Q}$.

(i) *If* $\mathfrak{B} = \langle B, S\rangle \in \boldsymbol{K}$, *then* $\mathrm{dom}\, S = \mathrm{rng}\, S = B$.

(ii) $\mathfrak{B} \in \boldsymbol{K}$ *iff* $\mathfrak{SK}(\mathfrak{B}) \in \boldsymbol{K}$.

(iii) *If* $\underset{i \in I}{P}\mathfrak{B}_i \in \boldsymbol{K}$, *then* $\mathfrak{B}_i \in \boldsymbol{K}$ *for all* $i \in I$.

(iv) *If* $\boldsymbol{K} = \boldsymbol{R}$ *or* $\boldsymbol{U}$ *and we have* $\mathfrak{B}_i \in \boldsymbol{K}$ *for all* $i \in I$, *then* $\underset{i \in I}{P}\mathfrak{B}_i \in \boldsymbol{K}$

(v) *If* $\mathfrak{B} = \underset{i \in I}{P}\mathfrak{B}_i \in \boldsymbol{K}$, *then* $\mathfrak{SK}(\mathfrak{B}) \cong \underset{i \in I}{P}\mathfrak{SK}(\mathfrak{B}_i)$. *In fact, for each* $f \in B$ *we have* $f/\approx^{\mathfrak{B}} = \underset{i \in I}{P} f_i/\approx^{\mathfrak{B}_i}$. *A canonical isomorphism is defined by setting* $\sigma(f/\approx^{\mathfrak{B}}) = \langle f_i/\approx^{\mathfrak{B}_i}\colon i \in I\rangle$.

Proof. (i) is trivial. The only question arises when $\boldsymbol{K} = \boldsymbol{Q}$ and $*(B) = 1$, but then it follows from our tacit assumption that $S \neq \emptyset$. (ii)–(v) are very easy. The only requirement for the validity of (v) is the assumption that $\mathrm{dom}\, S = \mathrm{rng}\, S = B$, as in (i).

The procedure for constructing an arbitrary binary structure from

(4) Chang [2] called this the "maximal congruence relation" of $\mathfrak{B}$.

its skeleton is rather obvious and quite similar to cardinal addition. It will be convenient to have concise notation for this construction.

Definition 2.4. Suppose that $\mathfrak{C} = \langle C, T\rangle$ is a binary structure and that $Z_x\ (x \in C)$ is a system of non-empty, pairwise disjoint sets indexed by C.

(i) We put $\mathfrak{C}[Z] = \langle B, S\rangle$, where $B = \bigcup\{Z_x\colon x \in C\}$ and S is defined by requiring zSz' iff xTx', when $z \in Z_x$ and $z' \in Z_{x'}$.

(ii) By φ_Z we understand the unique homomorphism from $\mathfrak{C}[Z]$ onto $\mathfrak{C}$ which satisfies $\varphi_Z^{-1}(x) = Z_x$ for all $x \in C$.

Lemma 2.5. *Let $\mathfrak{B}$ be an arbitrary binary structure, let $\mathfrak{C}$ be a thin structure and let Z_x $(x \in C)$ be a system of sets as in Definition 2.4.*

(i) $\ker\varphi_Z = \approx^{\mathfrak{C}[Z]}$ *and* $\mathfrak{C} \cong \mathcal{SK}(\mathfrak{C}[Z])$.

(ii) *If* $\mathfrak{C} = \mathcal{SK}(\mathfrak{B})$, *and* $Z_{\bar{x}} = \bar{x}$ *for each element* $\bar{x} \in C$, *then* $\mathfrak{C}[Z] = \mathfrak{B}$.

Proof. Trivial.

Lemma 2.6. *Assume that $\mathfrak{C}$ and $\mathfrak{D}$ are thin structures, and that $Z_x (x \in C)$ and Z'_y $(y \in D)$ are systems of sets as in Definition 2.4. Then $\mathfrak{C}[Z] \cong \mathfrak{D}[Z']$ iff there is an isomorphism $\varphi\colon \mathfrak{C} \cong \mathfrak{D}$ such that $\#(Z_x) = \#(Z'_{\varphi(x)})$ for every $x \in C$.*

Proof. Let $\mathfrak{A} = \mathfrak{C}[Z]$, $\mathfrak{B} = \mathfrak{D}[Z']$. Assume that $\psi\colon \mathfrak{A} \cong \mathfrak{B}$. Then for $a \in A$, $\psi(a/\approx^{\mathfrak{A}}) = \psi(a)/\approx^{\mathfrak{B}}$. I.e., by 2.5(i) and 2.4, for $x \in C$, $\psi(Z_x) = Z'_{\varphi(x)}$ for some $\varphi(x) \in D$. Clearly, $\varphi\colon \mathfrak{C} \cong \mathfrak{D}$ and $\#(Z_x) = \#\big(\psi(Z_x)\big) = \#(Z'_{\varphi(x)})$ for all $x \in C$.

The other direction is even more obvious.

The next and final lemma formulates the basic properties of decomposition functions (Definition 1.3) which will be needed in the next section. 2.7 (i–vi) are taken from [3], while 2.7 (vii–ix) can be derived from Lemma 2.3 (v).

Lemma 2.7. *Assume that $\mathfrak{B} = \langle B, S\rangle \in \boldsymbol{Q}$, that $u, v, w, x, y, z \in B$, and that $f \in \mathrm{DF}(\mathfrak{B})$. Then*

(i) $f\big(f(x, y), z\big) = f(x, z) = f\big(x, f(y, z)\big)$.

(ii) $f(x, x) = x$.

(iii) $f_u x = f_u y$ *iff* $f_v x = f_v y$.

(iv) $x = y$ *iff* $f_u x = f_u y$ *and* $f^d_v x = f^d_v y$.

(v) xSu *and* ySv *jointly imply* $f(x, y)Sf(u, v)$.

(vi) $f_u xSf_v y$ *and* $f^d_w xSf^d_z y$ *jointly imply* xSy.

(vii) $x \approx u$ *and* $y \approx v$ *jointly imply* $f(x, y) \approx f(u, v)$.

(viii) $f_u x \approx f_u y$ *iff* $f_v x \approx f_v y$.

(ix) $x \approx y$ *iff* $f_u x \approx f_u y$ *and* $f^d_v x \approx f^d_v y$.

3. The fundamental lemma. Here it is.

Lemma 3.1. *Assume that $\mathfrak{B} = \langle B, S\rangle \in \boldsymbol{Q}$. We have then, for all $x, y \in B$ and for all $f, g \in \mathrm{DF}(\mathfrak{B})$, $f_x g_x y \approx g_x f_x y$.*

Our proof of this proposition will be completed directly following the proof of Lemma 3.3. We remark that although the next two lemmas are convenient steps in the proof of Lemma 3.1 they are also easy corollaries of 3.1.

Lemma 3.2. *Assume that $\mathfrak{B} = \langle B, S\rangle \in \boldsymbol{Q}$, that $x, y, z, u \in B$ and that $f, g \in \mathrm{DF}(\mathfrak{B})$.*

(i) *If $xSf_u y$ and $xSf_u z$ then $xSf_u g(y, z)$.*

(ii) *If $x\breve{S}f_u y$ and $x\breve{S}f_u z$ then $x\breve{S}f_u g(y, z)$.*

Proof. It suffices to prove (i), since $\mathfrak{B} \in \boldsymbol{Q}$ implies $\langle B, \breve{S}\rangle \in \boldsymbol{Q}$ and this structure has the same decompositions as $\mathfrak{B}$. Suppose that $xSf_u y$ and $xSf_u z$. If $y = z$ then $y = g(y, z)$ and the desired conclusion follows. Otherwise, we apply the connectedness of $\breve{S}|S$ and obtain a finite sequence $y = y_0, y_1, \ldots, y_n = z$ with $y_i\breve{S}|Sy_{i+1}$ for each $i < n$ (this is what the connectedness amounts to, since $\breve{S}|S$ is symmetric, $a\breve{S}|Sb$ iff $b\breve{S}|Sa$). Hence there are $w_0, \ldots, w_{n-1} \in B$ for which $w_i S y_i$ and $w_i S y_{i+1}$, for each $i < n$. We may ensure that $xSf_u y_i$ for each $i \leqslant n$, by replacing if necessary y_i by $f(z, y_i)$ for $1 \leqslant i \leqslant n$ and w_i by $f(x, w_i)$ for $0 \leqslant i < n$. We thus obtain

(1) $w_0, w_1, \ldots, w_{n-1};\ y_0, y_1, \ldots, y_n;\ x, u \in B$ *with* $xSf_u y_i$ $(i \leqslant n)$ *and* $w_i S y_i$, $w_i S y_{i+1}$ $(i < n)$.

Moreover we have $y = y_0$ and $z = y_n$.

Lemma 3.2 will therefore follow if we are able to prove that (1) implies $xSf_u g(y_0, y_n)$. We now prove this by induction on $n > 0$. For $n = 1$, we first note that $f_{w_0} xSy_i$ $(i = 0, 1)$ by Lemma 2.7 and therefore $f_{w_0} xSg(y_0, y_1)$ $\big($by 2.7 (ii), (v)$\big)$—i.e.

$$f_{w_0} xSf_t f_u g(y_0, y_1) \quad \text{where } t = g(y_0, y_1)\,. \tag{2}$$

As $f^d_{y_0} f_u g(y_0, y_1) = f(y_0, u) = f_u y_0$, the formula $xSf_u y_0$ may also be read as

$$f^d_x xSf^d_{y_0} f_u g(y_0, y_1)\,. \tag{3}$$

Combining (2) and (3) we obtain the desired conclusion, $xSf_u g(y_0, y_1)$ by 2.7 (vi).

To complete the proof we now assume that, in (1), $n > 1$ and that the corresponding statement with n replaced by $n-1$ implies $xSf_u g(y_0, y_{n-1})$. Two applications of this induction assumption give us that

$$xSf_u g(y_0, y_{n-1}) \quad \text{and} \quad xSf_u g(y_1, y_n)\,. \tag{4}$$

Setting $\overline{y}_0 = g(y_0, y_{n-1})$, $\overline{y}_1 = g(y_1, y_n)$ and $\overline{w}_0 = g(w_0, w_{n-1})$, we easily show that the system $\overline{w}_0;\ \overline{y}_0, \overline{y}_1;\ x, u$ *satisfies* (1). *Since* $g(\overline{y}_0, \overline{y}_1) = g(y_0, y_n)$;

the case $n = 1$ just handled gives the desired conclusion: $x S f_u g(y_0, y_n)$. This concludes the proof of Lemma 3.2.

LEMMA 3.3. *Assume that $\mathfrak{B} = \langle B, S\rangle \in \boldsymbol{Q}$, that $v, w, x, y \in B$ and that $f, g \in \mathrm{DF}(\mathfrak{B})$. If $f_v x \approx f_v y$ then also $f_v g_w x \approx f_v g_w y$.*

Proof. Suppose that $f_v x \approx f_v y$. We will show that this implies

(1) $$f_v g_w x \leqslant_l f_v g_w y \quad \text{(see Definition 2.2)}.$$

The argument for (1) contains the crux of the proof; indeed the remaining argument—which is omitted—can be summarized in a few words: Replacing S by $\breve{S}$ in the proof of (1) we obtain a proof that $f_v g_w x \leqslant_r f_v g_w y$; thus $f_v g_w x \leqslant f_v g_w y$. The symmetry of assumptions on x and y allows us to conclude that this formula remains valid upon exchanging x and y. We then infer that $f_v g_w x \approx f_v g_w y$ by Definition 2.2.

To begin the proof of (1), we apply 3.2 (i) to the formula $f_v x \approx f_v y$, and easily infer that $f_v x \leqslant_l f_v g(y, x)$, i.e. we have

(2) $$f_v g_x x \leqslant_l f_v g_x y.$$

Next we shall prove the following statement.

(3) *Let $a, b \in B$ such that $a\breve{S}|Sb$. If $f_v g_a x \leqslant_l f_v g_a y$ then $f_v g_b x \leqslant_l f_v g_b y$*

But first notice that (1) follows by a trivial inductive argument from (2), (3) and our assumption that $\breve{S}|S$ is connected over B. So everything hinges on the proof of (3).

In order to prove (3), we now assume

(α) $a\breve{S}|Sb$, $f_v g_a x \leqslant_l f_v g_a y$ and for a certain element $z \in B$, $z S f_v g_b x$;

and we have to get that $z S f_v g_b y$. Using (α) and 2.3 (i), we choose $c, d, e \in B$, and define two other elements u_0, u_1, to satisfy the following

(β) $$cSa,\ cSb,\ dSx,\ eSy \quad \text{and} \quad u_0 = g(d, c),\ u_1 = g(e, c).$$

From (β) and 2.7 (v), we then have

(γ) $$u_0 S g_a x,\ u_0 S g_b x \quad \text{and} \quad u_1 S g_b y.$$

Now we put

(δ) $$\bar{z} = f(z, u_0), \quad z_0 = g(\bar{z}, u_0), \quad z_1 = g(u_1, \bar{z});$$

and perform some calculations. We have

(ε₀) $\bar{z} S g_b x$, by (δ), (α), (γ), 2.7 (v) and 2.7 (i, ii), i.e. the fact that $g_b x = f(f_v g_b x, g_b x)$;

(ε₁) $z_0 S g_a x$ and $z_0 S g_b x$, by (δ), (γ), (ε₀) and 2.7 (v) and because $g_a x = g(g_b x, g_a x)$ and $g_b x = g(g_b x, g_b x)$;

(ε₂) $f_z z_0 S f_v g_a x$; by (α), (ε₁), 2.7 (v);

(ε₃) $f_z z_0 S f_v g_a y$, by (ε₂) and (α), i.e. the assumption that $f_v g_a x \leqslant_l f_v g_a y$;

(ε₄) $f_z z_0 S f_v g_b x$, by (α), (ε₁), 2.7 (v);

(ε₅) $f_z z_0 S f_v g_b y$, by (ε₃), (ε₄), 3.2 (i) and because $g_b y = g(g_a y, g_b x)$;

(ε₆) $z_1 S g_b y$, by (δ), (γ), (ε₀), 2.7 (v);

(ε₇) $f_z z_1 S f_v g_b y$ by (α), (ε₆), 2.7 (v);

(ε₈) $f_z \bar{z} S f_v g_b y$, by (ε₅), (ε₇), 3.2 (ii) and the fact that $\bar{z} = g(z_0, z_1)$.

Finally we note that by (δ), $f_z \bar{z} = z$ and hence (ε₈) already expresses the desired conclusion, $z S f_v g_b y$. This concludes the proof of (3), (1) and Lemma 3.3.

Proof of Lemma 3.1. Let $\mathfrak{B} \in \boldsymbol{Q}$, $x, y \in B$ and $f, g \in \mathrm{DF}(\mathfrak{B})$. We have $f_x f_x y \approx f_x y$, in fact the two sides are equal. Therefore by Lemma 3.3 $f_x g_x f_x y \approx f_x g_x y$, i.e.

(1) $$f_x(g_x f_x y) \approx f_x(f_x g_x y).$$

On the other hand, $f_x^d f_x y \approx x \approx f_x^d x$, and therefore by Lemma 3.3

(2) $$f_x^d(g_x f_x y) \approx f_x^d g_x x \approx x \approx f_x^d(f_x g_x y).$$

Now the desired conclusion, $g_x f_x y \approx f_x g_x y$, is clearly an immediate consequence of (1), (2) and Lemma 2.7 (ix), and so the proof is complete.

The following corollary is weaker than the fundamental lemma, i.e. some structures satisfy the corollary while failing to have the property expressed in Lemma 3.1. Nevertheless, it is the appropriate tool in one proof of a central theorem in the next section.

COROLLARY 3.4. *Suppose that $\mathfrak{B} \in \boldsymbol{Q}$ and that*

(1) $$\mathfrak{B} \cong \mathfrak{C} \times \mathfrak{D} \overset{\varphi}{\cong} \underset{i \in I}{P} \mathfrak{B}_i.$$

Let u be any element of $\mathfrak{C}$ such that uTu. Then $\mathfrak{C}' = \mathfrak{C}(u/\approx^{\mathfrak{C}}) \in \boldsymbol{U}$, and we have

(2) $$\varphi(\mathfrak{C}' \times \mathfrak{D}) = \underset{i \in I}{P} \mathfrak{B}'_i$$

for a certain system of structures $\mathfrak{B}'_i \subseteq \mathfrak{B}_i$.

Proof. The statement about $\mathfrak{C}(u/\approx^{\mathfrak{C}})$ is trivially equivalent to the assertion uTu.

Let $f \in \mathrm{DF}(\mathfrak{C} \times \mathfrak{D})$ be the decomposition function such that $f(\langle c, d\rangle, \langle c', d'\rangle) = \langle c, d'\rangle$. For $i \in I$, let $G_i = \ker(\mathrm{pr}_i \cdot \varphi)$, the kernel of the map $\mathfrak{C} \times \mathfrak{D} \to \mathfrak{B}_i$. For $i \in I$, let g^i be the decomposition function of $\mathfrak{C} \times \mathfrak{D}$ correlated with the decomposition

$$\mathrm{id}_{C \times D} = G_i \times \Big(\prod_{j \in I - \{i\}} G_j\Big).$$

Thus, $g^i(x, y) = \varphi^{-1}(h)$ where $h_i = \varphi(x)_i$ and $h_j = \varphi(y)_j$ for all $j \in I - \{i\}$.

Now it should be evident that (2) is equivalent to the following statement

(3) *If $y_i = \langle c_i, d_i\rangle \in C\times D$ and $c_i \approx u$ for all $i \in I$, and if $y = \langle c, d\rangle$ is the unique element of $C\times D$ such that $\varphi(y)_i = \varphi(y_i)_i$ for all $i \in I$, then $c \approx u$.*

To prove this we take $x = \langle u, d\rangle$ and note first that, by 2.3 (v), $x \approx f_x y_i$ (all $i \in I$) and so, by 2.7 (ix)

$$g_x^i x \approx g_x^i f_x y_i \quad (\text{all } i \in I)\,. \tag{4}$$

Next we have $g_x^i y_i = g_x^i y$ and hence, by 3.3, $g_x^i f_x y_i \approx g_x^i f_x y$ for all $i \in I$, which when added to (4) gives

$$g_x^i x \approx g_x^i f_x y \quad (\text{all } i \in I)\,. \tag{5}$$

Using 2.3 (v) again we find that (5) implies $x \approx f_x y$ and this means $u \approx c$, as desired.

4. Refinement theorems. One should observe at once the close resemblance that Lemma 3.1 bears to the necessary and sufficient condition for the strict refinement property expressed in Theorem 1.4. The lemma appears in its context to be the best conceivable substitute for the strict refinement property. According to it two elements $f_x g_x y$ and $g_x f_x y$ can not be distinguished by means of the fundamental relation of the structure; in general, there will be no further connection between two such elements.

The first two theorems below improve the results of [3] concerning binary structures. We get, in particular, the known result that the strict refinement property is possessed by every structure $\langle B, S\rangle$ in which S is a connected (reflexive) partial ordering relation over B. The proofs are immediate, by 3.1 and Definition 2.2.

THEOREM 4.1. *If $\mathfrak{B} \in \boldsymbol{Q}$ is a thin structure, then $\mathfrak{B}$ possesses the strict refinement property.*

THEOREM 4.2. *Suppose that $\mathfrak{B} \in \boldsymbol{Q}$, $u \in B$ and $u/\approx\, = \{u\}$. Then $(\mathfrak{B}, u)$ has the intermediate refinement property.*

Remark 4.3. [3; Theorem 7.3] says that if $\mathfrak{B}$ is reflexive and connected and if u is an anti-symmetric element of $\mathfrak{B}$ with respect to S, i.e. $xSuSx$ implies $x = u$, then $(\mathfrak{B}, u)$ has the intermediate refinement property. To infer this from Theorem 4.2 note that if uSu and $x \approx u$ then $xSuSx$.

With the observation that, by 2.3 (v) direct decompositions of $\mathfrak{B}$ lead to direct decompositions of the thin structure $\mathcal{SK}(\mathfrak{B})$, it is quite easy to get Lemma 3.1 back from Theorem 4.1. Thus we see that Theorem 4.1 expresses the full strength of the lemma.

We proceed immediately to a characterization of structures possessing the ordinary refinement property. (The previous discussion of the refinement property for cardinal numbers—at the end of § 1—now becomes relevant.) The characterization is the following:

THEOREM 4.4. *Suppose that $\mathfrak{B} \in \boldsymbol{Q}$. Define $\bar{\mathfrak{n}}$ to be the smallest cardinality of an $\approx$-coset of $\mathfrak{B}$. Then for $\mathfrak{B}$ to have the refinement property it is necessary and sufficient that $\bar{\mathfrak{n}}$ have the refinement property and that either* (i) *$\bar{\mathfrak{n}}$ is finite; or* (ii) *$\bar{\mathfrak{n}} = \#(u/\approx)$ for exactly one $\approx$-coset, $u/\approx$.*

Proof. For the sufficiency, assume that the condition is satisfied and let there be given two standard decompositions of $\mathfrak{B}$.

(1) $\underset{i\in I}{P}\,\mathfrak{A}_i \cong \mathfrak{B} \cong \underset{j\in J}{P}\,\mathfrak{B}_j$, *where* $\mathfrak{A}_i = \mathfrak{B}/F_i$ *and* $\mathfrak{B}_j = \mathfrak{B}/G_j$ *for all* $i \in I$ *and* $j \in J$.

From (1) with the aid of Lemma 2.3 (v) we obtain decompositions of the skeleton of $\mathfrak{B}$ which lead to two standard decompositions

(2) $\underset{i\in I}{P}\,\mathfrak{A}'_i \cong \mathcal{SK}(\mathfrak{B}) \cong \underset{j\in J}{P}\,\mathfrak{B}'_j$, *where* $\mathfrak{A}'_i = \mathcal{SK}(\mathfrak{B})/F'_i \cong \mathcal{SK}(\mathfrak{A}_i)$, $\mathfrak{B}'_j = \mathcal{SK}(\mathfrak{B})/G'_j \cong \mathcal{SK}(\mathfrak{B}_j)$.

Here of course F'_i is the kernel of the map $\mathcal{SK}(\mathfrak{B}) \to \mathcal{SK}(\mathfrak{A}_i)$ implicit in Lemma 2.3 (v), and analogously for G'_j. As a consequence of Theorem 4.1 applied to the thin structure $\mathcal{SK}(\mathfrak{B})$, the formula (2) has a canonical refinement

(3) $\mathfrak{A}'_p \cong \underset{j\in J}{P}\,\mathfrak{D}'_{p,j}$ *and* $\mathfrak{B}'_q \cong \underset{i\in I}{P}\,\mathfrak{D}'_{i,q}$ *for all* $p \in I$ *and* $q \in J$, *where* $\mathfrak{D}'_{i,j} = \mathcal{SK}(\mathfrak{B})/(F'_i|G'_j)$.

It is useful to reformulate this in the notation of § 1 for direct products of equivalence relations. Composing the canonical epimorphism $\mathfrak{B} \to \mathcal{SK}(\mathfrak{B})$ with the projections from $\mathcal{SK}(\mathfrak{B})$ onto its quotient structures we find

(4) $\mathfrak{A}'_i \cong \mathfrak{B}/\bar{F}_i$, $\mathfrak{B}'_j \cong \mathfrak{B}/\bar{G}_j$, $\mathfrak{D}'_{i,j} \cong \mathfrak{B}/\bar{H}_{i,j}$ *where* $\bar{F}_i = F_i/\approx^{\mathfrak{B}}$, $\bar{G}_j = G_j/\approx^{\mathfrak{B}}$ *and* $\bar{H}_{i,j} = F_i|G_j/\approx^{\mathfrak{B}}$.

The formulas (2) and (3) evidently become

(5) $\approx^{\mathfrak{B}} = \prod_{i,j} \bar{H}_{i,j}$, $\bar{F}_p = \prod_{j\in J} \bar{H}_{p,j}$ *and* $\bar{G}_q = \prod_{i\in I} \bar{H}_{i,q}$ *for every* $p \in I$, $q \in J$.

The basic idea for proving sufficiency in Theorem 4.4 is to expand the thin structures $\mathfrak{D}'_{i,j}$ appearing in (3), by the construction of Def. 2.4, to obtain new structures $\mathfrak{D}_{i,j} = \mathfrak{D}'_{i,j}[Z^{(i,j)}]$ which constitute a refinement of (1). Essentially this means finding a system of cardinal numbers $\mathfrak{d}_{i,j}(x) = \#(Z_x^{(i,j)})$, $x \in D'_{i,j}$, subject to certain restrictions. For this we are going to use Theorem 1.6. In the following argument we work directly with the structures $\mathfrak{B}/\bar{H}_{i,j} \cong \mathfrak{D}'_{i,j}$.

For the application of Theorem 1.6, let us define $X_{i,j} = B/\overline{H}_{i,j}$ $(\langle i, j\rangle \in I \times J)$ and $X = \underset{i,j}{P} X_{i,j}$. By (5) the functions $f \in X$ are in one-one correspondence with the $\approx^{\mathfrak{B}}$-cosets of $\mathfrak{B}$: for corresponding elements $f \in X$ and $u \in B/\approx^{\mathfrak{B}}$ we have $f(i, j) = x/\overline{H}_{i,j}$ whenever $x \in u$ and $\langle i, j\rangle \in I \times J$. To the function $f \in X$ we now attach the number $\mathfrak{n}(f) = \#(u)$ which is the cardinality of the corresponding $\approx^{\mathfrak{B}}$-coset. The assumption in Theorem 1.6 regarding the numbers $\mathfrak{n}(f)$ is clearly satisfied, in fact it is equivalent to the statement about $\overline{\mathfrak{n}}$ in Theorem 4.4 which we are assuming to be valid. Next, let $p \in I$ and, in the notation of Definition 1.5, let $g \in X_{p,*}$. By (5), there is a unique element $\overline{y} \in B/\overline{F}_p$ such that $g(j) = y/\overline{H}_{p,j}$ whenever $y \in \overline{y}$ and $j \in J$; in the canonical isomorphism ((2) and (4)) between $\mathfrak{B}/\overline{F}_p$ and $\mathfrak{SK}(\mathfrak{A}_p)$, $\overline{y}$ corresponds to an $\approx^{\mathfrak{A}_p}$-coset of $\mathfrak{A}_p$ which consists of all the elements y/F_p, $y \in \overline{y}$. Let $\mathfrak{n}_p(g)$ be the cardinal number of this coset. Define, in a similar manner, $\mathfrak{m}_q(h)$ for $h \in X_{*,q}$, $q \in J$.

Let us now verify condition (i) of Theorem 1.6 for the numbers introduced above. Let f be an arbitrary function belonging to X and let $x/\approx^{\mathfrak{B}} (x \in B)$ be the corresponding $\approx$-coset. Under the isomorphisms (1) the element x maps, on the one hand, to the function $\langle x_{i,*}\colon i \in I\rangle$ where $x_{i,*} = x/F_i$, and on the other hand, to $\langle x_{*,j}\colon j \in J\rangle$ where $x_{*,j} = x/G_j$. Whence by Lemma 2.3 (v) the set $x/\approx^{\mathfrak{B}}$ is carried on the one hand onto $\underset{i \in I}{P}(x_{i,*}/\approx^{\mathfrak{A}_i})$, and on the other hand onto $\underset{j \in J}{P}(x_{*,j}/\approx^{\mathfrak{B}_j})$. Consequently, we can infer that

$$\underset{i \in I}{P} \#(x_{i,*}/\approx^{\mathfrak{A}_i}) = \#(x/\approx^{\mathfrak{B}}) = \underset{j \in J}{P} \#(x_{*,j}/\approx^{\mathfrak{B}_j}) .$$

Now it follows from our definitions that the cardinal numbers in the above formula are, reading from left to right, none other than $\mathfrak{n}_i(f_{i,*})$, $\mathfrak{n}(f)$ and $\mathfrak{m}_j(f_{*,j})$ respectively—hence the formula simply expresses condition (i) of Theorem 1.6.

From Theorem 1.6 we obtain a system of cardinal numbers $\mathfrak{d}_{i,j}(u)$, $u \in B/\overline{H}_{i,j}$ satisfying the following for all $p \in I$ and $q \in J$:

(6) *If* $\overline{x} = x/F_p \in A_p$ *and* $\overline{y} = y/G_q \in B_q$ $(x, y \in B)$ *then*

$$\#(\overline{x}/\approx^{\mathfrak{A}_p}) = \underset{j \in J}{P} \mathfrak{d}_{p,j}(x/\overline{H}_{p,j}) ,$$

$$\#(\overline{y}/\approx^{\mathfrak{B}_q}) = \underset{i \in I}{P} \mathfrak{d}_{i,q}(y/\overline{H}_{i,q}) .$$

The proof of sufficiency is now to be concluded by using these numbers $\mathfrak{d}_{i,j}(u)$ to construct the desired refinement for (1). We put

$$\mathfrak{D}_{i,j} = (\mathfrak{B}/\overline{H}_{i,j})[Z^{(i,j)}]$$

defined as in Definition 2.4, where the disjointed system of sets $Z^{(i,j)}$ is selected so that we have

$$\#(Z_u^{(i,j)}) = \mathfrak{d}_{i,j}(u) \quad \text{for every } u \in B/\overline{H}_{i,j} .$$

We claim

(7) $\mathfrak{A}_p \cong \underset{j \in J}{P} \mathfrak{D}_{p,j}$ *and* $\mathfrak{B}_q \cong \underset{i \in I}{P} \mathfrak{D}_{i,q}$ *for* $p \in I$, $q \in J$;

and the proof is quite simple—based upon (1)–(6) above and Lemmas 2.5 and 2.6—but let us say a few words about it.

One can show very easily that if we define $\mathfrak{B}_{p,*} = \underset{j \in J}{P} \mathfrak{B}/\overline{H}_{p,j}$ then

$$\underset{j \in J}{P} \mathfrak{D}_{p,j} = \mathfrak{B}_{p,*}[Z^{(p,*)}]$$

where

$$Z_g^{(p,*)} = \underset{j \in J}{P} Z_{g(j)}^{(p,j)}$$

for every $g \in \mathfrak{B}_{p,*}$. By (2), (4), (5) there is a canonical isomorphism $\varphi\colon \mathfrak{SK}(\mathfrak{A}_p) \cong \mathfrak{B}_{p,*}$ which maps $\overline{x}/\approx^{\mathfrak{A}_p}$ onto $\langle x/\overline{H}_{p,j}\colon j \in J\rangle$, where $\overline{x} = x/F_p$. Now it follows from (6) and the definitions that $\#(t) = \#(Z_{\varphi(t)}^{(p,*)})$ for every $t \in A_p/\approx^{\mathfrak{A}_p}$. Consequently, it follows by Lemma 2.6 and Lemma 2.5 (ii) that $\mathfrak{A}_p \cong \mathfrak{B}_{p,*}[Z^{(p,*)}]$. The other half of (7) is proved similarly.

The following examples establish that the condition stated in Theorem 4.4 is necessary if the structure $\mathfrak{B}$ is to have refinement property.

Example 4.5. If the cardinal $\overline{\mathfrak{n}}$ defined in Theorem 4.4 does not have the refinement property then it is infinite and we have say $\overline{\mathfrak{n}} = \underset{i \in I}{P} \mathfrak{f}_i = \underset{j \in J}{P} \mathfrak{g}_j$ with no refinement. Let $\mathfrak{F}_i$ $(i \in I)$ and $\mathfrak{G}_j$ $(j \in J)$ be structures belonging to the class $\boldsymbol{U}$ (Definition 2.1.3), having cardinalities $\mathfrak{f}_i$, $\mathfrak{g}_j$ respectively. Let $\mathfrak{B}'$ be a substructure of $\mathfrak{B}$ obtained by throwing away all but one of the elements in some $\approx$-coset having $\overline{\mathfrak{n}}$ elements. Easy considerations based upon Lemma 2.6 show that $\mathfrak{B} \cong \mathfrak{B}' \times \underset{i \in I}{P} \mathfrak{F}_i \cong \mathfrak{B}' \times \underset{j \in J}{P} \mathfrak{G}_j$ and there is no refinement.

Example 4.6. Assume that the cardinal $\overline{\mathfrak{n}}$ defined in Theorem 4.4 is infinite and there are two elements $u, v \in B$ such that $\#(u/\approx) = \overline{\mathfrak{n}} = \#(v/\approx)$ and $u \not\approx v$. Deleting all the elements of $u/\approx$ except u we arrive at a substructure $\mathfrak{B}_0 \subseteq \mathfrak{B}$. Throwing away all the elements of $u/\approx \cup v/\approx$ other than u, v we obtain another substructure $\mathfrak{B}_1$. Let $\mathfrak{N}$ be any structure of the class $\boldsymbol{U}$ having cardinality $\overline{\mathfrak{n}}$. Now we have $\mathfrak{B} \cong \mathfrak{B}_0 \times \mathfrak{N} \cong \mathfrak{B}_1 \times \mathfrak{N}$ and there is no refinement.

Corollary 4.7. *Let* $\mathfrak{B} = \langle B, S\rangle$ *be a finite structure. If* $S|\breve{S}$ *and* $\breve{S}|S$ *are connected over* B *or, less generally,* S *is connected and reflexive over* B, *then* $\mathfrak{B}$ *has the refinement property.*

Proof. By Theorem 4.4.

The significance of our fundamental lemma (3.1) is perhaps most forcefully emphasized by the corollary of Theorem 4.4 asserting that every connected reflexive binary structure (member of $\boldsymbol{R}^c$) with a finite $\approx$ class has the refinement property. Another way of obtaining this result will be sketched below. It illustrates a method which we shall use again in § 9 to derive some applications for our results beyond the domain of binary structures. This method has not been fully formalized in the literature, although it was used implicitly by Chang [2] and independently, about the same time, by McKenzie [9; § 3]. It requires some preliminaries of a quite general character—terminated by Theorem 4.13—after which we shall return to the case at hand.

In the following definition we assume that $\boldsymbol{K}$ and $\boldsymbol{K}_0 \subseteq \boldsymbol{K}$ are non-void classes of relational structures of a fixed but arbitrary similarity type, each of which is closed under the operations of forming direct factors and direct products of finitely many structures. Let $\mathbf{1}$ denote any one element structure (with universal relations) similar to the structures in $\boldsymbol{K}$. We have, of course, $\mathbf{1} \in \boldsymbol{K}_0$. Let $\boldsymbol{K}_0'$ denote the class of all $\mathfrak{A} \in \boldsymbol{K}$ such that every direct factor of $\mathfrak{A}$ which belongs to $\boldsymbol{K}_0$ is isomorphic to $\mathbf{1}$.

DEFINITION 4.8. We say that $\boldsymbol{K}_0$ is a *characteristic subclass* of $\boldsymbol{K}$ if and only if the following conditions are satisfied:

(i) Assume that $\mathfrak{A} \cong \underset{i \in I}{P} \mathfrak{A}_i$ for some indexed system $\mathfrak{A}_i$ $(i \in I)$ and that $\mathfrak{A} \in \boldsymbol{K}$. Then $\mathfrak{A}_i \in \boldsymbol{K}_0$ for all $i \in I$ implies $\mathfrak{A} \in \boldsymbol{K}_0$; and, similarly, if $\mathfrak{A}_i \in \boldsymbol{K}_0'$ for all $i \in I$ then $\mathfrak{A} \in \boldsymbol{K}_0'$.

(ii) Whenever $\mathfrak{A} \in \boldsymbol{K}$, there are two structures $k_0(\mathfrak{A}) \in \boldsymbol{K}_0$ and $k_0'(\mathfrak{A}) \in \boldsymbol{K}_0'$ such that $\mathfrak{A} \cong k_0(\mathfrak{A}) \times k_0'(\mathfrak{A})$; and moreover the formulas $\mathfrak{A} \cong \mathfrak{A}_0 \times \mathfrak{A}_0'$, $\mathfrak{A}_0 \in \boldsymbol{K}_0$, $\mathfrak{A}_0' \in \boldsymbol{K}_0'$ imply that $\mathfrak{A}_0 \cong k_0(\mathfrak{A})$ and $\mathfrak{A}_0' \cong k_0'(\mathfrak{A})$. The structures $k_0(\mathfrak{A})$, $k_0'(\mathfrak{A})$ will be called respectively, the $\boldsymbol{K}_0$ and $\boldsymbol{K}_0'$ *sections* of $\mathfrak{A}$.

COROLLARY 4.9. *Suppose that* $\boldsymbol{K}_0$ *is characteristic in* $\boldsymbol{K}$. *If* $\mathfrak{A} \in \boldsymbol{K}$ *and* $\mathfrak{A} \cong \underset{i \in I}{P} \mathfrak{A}_i$ *then* $\mathfrak{A}_i \in \boldsymbol{K}$ *for all* $i \in I$ *and we have*

$$k_0(\mathfrak{A}) \cong \underset{i \in I}{P} k_0(\mathfrak{A}_i), \qquad k_0'(\mathfrak{A}) \cong \underset{i \in I}{P} k_0'(\mathfrak{A}_i).$$

Proof. Trivial by Definition 4.8.

The concept of characteristic subclass is certainly a natural one. For motivation we remark that if $\boldsymbol{K}$ contains exclusively finite structures then it just means that the semigroup $\hat{\boldsymbol{K}}$ consisting of isomorphism types of structures in $\boldsymbol{K}$, with the operation induced by binary cardinal multiplication (cf. § 5), decomposes into the direct product of the semigroups $\hat{\boldsymbol{K}}_0, \hat{\boldsymbol{K}}_0'$ with the projection maps induced by k_0, k_0'. In the more general case our definition requires more than this. Note that if $\boldsymbol{K}_0 \subseteq \boldsymbol{K}$ is characteristic then for every $\mathfrak{A} \in \boldsymbol{K}$, $\mathfrak{A}$ has the refinement property if and only if $k_0(\mathfrak{A})$ and $k_0'(\mathfrak{A})$ have the refinement property.

Some useful examples of characteristic classes further motivate the definition. (For each example below it is either well known or easily proved that all structures belonging to $\boldsymbol{K}_0$ have the refinement property.)

EXAMPLE 4.10. Let $\boldsymbol{K}$ be the subclass of $\boldsymbol{R}^c$ consisting of all $\mathfrak{A}$ with a finite $\approx$-coset. The class $\boldsymbol{K}_0 = \boldsymbol{U}_f$ of all finite members of $\boldsymbol{U}$ (Definition 2.1.3) is a characteristic subclass of $\boldsymbol{K}$. This is essentially well known (see [2]) and, of course, follows from Theorem 4.4; the proof is a small fragment of the argument for Theorem 1.6. The reader is invited to discover the intrinsic characterization of the complementary class $\boldsymbol{K}_0'$ and the functions k_0, k_0'.

EXAMPLE 4.11. The class $\boldsymbol{K}$ of all finite ternary structures $\mathfrak{A} = \langle A, R\rangle$, in which R is a binary operation over $\mathfrak{A}$, has as a characteristic subclass the class $\boldsymbol{K}_0$ of all finite groups. (The proof is to be found in [9; § 2]. This formulation of the result, however, is not found there.)

EXAMPLE 4.12. Let $\boldsymbol{K}$ be an arbitrary class of finite structures (having the same similarity type) closed under the formation of factors and finite products. As in [2], we say that a structure $\mathfrak{A} \in \boldsymbol{K}$ is *prime* in $\boldsymbol{K}$ if the formula $\mathfrak{A} \| \mathfrak{B} \times \mathfrak{C} (\mathfrak{B}, \mathfrak{C} \in \boldsymbol{K})$ always implies that either $\mathfrak{A} \| \mathfrak{B}$ or $\mathfrak{A} \| \mathfrak{C}$. We say that $\mathfrak{A}$ is *cancellable* in $\boldsymbol{K}$ if $\mathfrak{A} \times \mathfrak{B} \cong \mathfrak{A} \times \mathfrak{C} (\mathfrak{B}, \mathfrak{C} \in \boldsymbol{K})$ implies that $\mathfrak{B} \cong \mathfrak{C}$. Let $\boldsymbol{L}$ be any class of structures (including $\mathbf{1}$) which are prime and cancellable in $\boldsymbol{K}$. It was essentially shown in [2] that the class generated by $\boldsymbol{L}$, under binary cardinal multiplication and isomorphism, is a characteristic subclass of $\boldsymbol{K}$.

The introduction of the notion of a characteristic subclass is justified by the following elementary result, which isolates a crucial step in several existing proofs of the refinement property.

THEOREM 4.13. *Let* $\mathfrak{A} \in \boldsymbol{K}$ *and assume that* $\boldsymbol{K}_0 \subseteq \boldsymbol{K}$ *is characteristic in* $\boldsymbol{K}$. *Suppose that*

$$\text{(i)} \qquad \mathfrak{A} \cong \underset{i \in I}{P} \mathfrak{A}_i \cong \underset{j \in J}{P} \mathfrak{B}_j,$$

and also that there exist structures $\mathfrak{E}_i \in \boldsymbol{K}_0$ $(i \in I)$ *and* $\mathfrak{F}_j \in \boldsymbol{K}_0$ $(j \in J)$ *and structures* $\mathfrak{A}_{i,j}, \mathfrak{B}_{i,j}$ $(\langle i, j\rangle \in I \times J)$ *satisfying the following conditions whenever* $p \in I$, $q \in J$

$$\text{(ii)} \qquad \mathfrak{E}_p \times \mathfrak{A}_p \cong \underset{j \in J}{P} \mathfrak{B}_{p,j};$$

$$\text{(iii)} \qquad \mathfrak{F}_q \times \mathfrak{B}_q \cong \underset{i \in I}{P} \mathfrak{A}_{i,q};$$

$$\text{(iv)} \qquad \mathfrak{E}_p \times \mathfrak{A}_{p,q} \cong \mathfrak{F}_q \times \mathfrak{B}_{p,q}.$$

Suppose finally that $k_0(\mathfrak{A})$ has the refinement property. Under these assumptions (i) *has a refinement.*

Proof. We first apply Corollary 4.9 to (iv) and obtain $k_0'(\mathfrak{A}_{p,q}) \cong k_0'(\mathfrak{B}_{p,q}) \cong \mathfrak{D}_{p,q}$ say (because $k_0'(\mathfrak{F}) \cong \mathbf{1}$ if $\mathfrak{F} \in \boldsymbol{K}_0$ and $\mathbf{1}$ is cancellable). Then we apply the same reasoning to (ii), (iii) and compute

(1) *For all $p \in I$, $q \in J$ we have*

$$k_0'(\mathfrak{A}_p) \cong \underset{j\in J}{P}\mathfrak{D}_{p,j} \quad \text{and} \quad k_0'(\mathfrak{B}_q) \cong \underset{i\in I}{P}\mathfrak{D}_{i,q}.$$

Next, we apply Corollary 4.9 to (i) and obtain

(2) $$k_0(\mathfrak{A}) \cong \underset{i\in I}{P} k_0(\mathfrak{A}_i) \cong \underset{j\in J}{P} k_0(\mathfrak{B}_j).$$

Since $k_0(\mathfrak{A})$ has the refinement property, there exists a system of structures $\mathfrak{C}_{i,j}$ such that

(3) *For all $p \in I$, $q \in J$ we have*

$$k_0(\mathfrak{A}_p) \cong \underset{j\in J}{P}\mathfrak{C}_{p,j} \quad \text{and} \quad k_0(\mathfrak{B}_q) \cong \underset{i\in I}{P}\mathfrak{C}_{i,q}.$$

Finally, we put $\mathfrak{G}_{i,j} = \mathfrak{C}_{i,j}\times\mathfrak{D}_{i,j}$ and verify immediately, with the help of (1), (3) and Definition 4.8 (ii), that the $\mathfrak{G}_{i,j}$ constitute a refinement of (i). This concludes the proof.

The deduction of the refinement property of a structure $\mathfrak{A}$ by the method of characteristic classes—i.e., by Theorem 4.13—relies in the present instance on the following theorem, which here plays a role quite analogous to [2; Theorem 1.3] and [9; Lemma 3.2]. The new proof of the refinement property, for all structures in $\boldsymbol{R}^c$ which have a finite $\approx$-coset, is constructed without any difficulty by juxtaposing this theorem with Theorem 4.13 and Example 4.10 and we leave it to the reader to do this.

THEOREM 4.14. *Suppose that $\mathfrak{A} = \langle A, R\rangle \in \boldsymbol{Q}$ and that $u \in A$ is a reflexive element* (i.e. uRu). *Assume also that*

(i) $$\mathfrak{A} \cong \underset{i\in I}{P}\mathfrak{A}_i \overset{\varphi}{\cong} \underset{j\in J}{P}\mathfrak{B}_j.$$

Then there exists a system of structures $\mathfrak{E}_i, \mathfrak{F}_j, \mathfrak{A}_{i,j}, \mathfrak{B}_{i,j}$ (for all $i \in I$ and $j \in J$) satisfying the following conditions whenever $p \in I$, $q \in J$

(ii) $$\varphi(\mathfrak{E}_p\times\mathfrak{A}_p) = \underset{j\in J}{P}\mathfrak{B}_{p,j};$$

(iii) $$\varphi^{-1}(\mathfrak{F}_q\times\mathfrak{B}_q) = \underset{i\in I}{P}\mathfrak{A}_{i,q};$$

(iv) $$\varphi(\mathfrak{E}_p\times\mathfrak{A}_{p,q}) = \mathfrak{F}_q\times\mathfrak{B}_{p,q};$$

and moreover

(v) $$\mathfrak{E}_p\|\mathfrak{A}(u/\approx^{\mathfrak{A}}) \quad \text{and} \quad \mathfrak{F}_q\|\mathfrak{A}(u/\approx^{\mathfrak{A}}).$$

Proof. This is a strong result but the hard work of the proof was done in § 3. In fact, Theorem 4.14 is an easy consequence of Corollary 3.4. Admittedly, the suggestive notation employed in (ii), (iii) and (iv) is not quite accurate but it should be clear what is meant. Let h be the element of $P\mathfrak{A}_i$ which corresponds to $u \in A$ under (some) isomorphism (i). Then h_i is a reflexive element of $\mathfrak{A}_i$ whenever $i \in I$ and similarly $\varphi(h)_j$ is a reflexive element of $\mathfrak{B}_j$. We put

$$\mathfrak{E}_p = \underset{i\neq p}{P}\mathfrak{A}_i(h_i/\approx^{\mathfrak{A}_i}),$$

$$\mathfrak{F}_q = \underset{j\neq q}{P}\mathfrak{B}_j(\varphi(h)_j/\approx^{\mathfrak{B}_j}),$$

and 4.14(v) follows readily by 2.3(v).

Now there is only one possibility for $\mathfrak{A}_{p,q}$, $\mathfrak{B}_{p,q}$. Namely, let $X_{p,q}$ be the subset of A_p consisting of all elements x such that if f is the member of $P\mathfrak{A}_i$ with $f_i = h_i$ for $i \neq p$ and $f_p = x$, then $\varphi(f)_j \approx^{\mathfrak{B}_j}\varphi(h)_j$ for all $j \neq q$. We define $\mathfrak{A}_{p,q}$ to be the substructure of $\mathfrak{A}_p - \mathfrak{A}_p(X_{p,q})$. Similarly, let $Y_{p,q}$ be the set of all $y \in B_q$ s.t. $\varphi^{-1}(g)_i \approx^{\mathfrak{A}_i} h_i$ for all $i \neq p$, where $g_j = \varphi(h)_j$ for $j \neq q$ and $g_q = y$. We define $\mathfrak{B}_{p,q} = \mathfrak{B}_q(Y_{p,q})$.

In conclusion of the proof we only remark that 4.14 (iv) follows easily from the above definitions and the basic properties of $\approx$, and that 4.14 (ii), (iii) are obtained directly by appealing to Corollary 3.4.

To complete the work of this section we establish a theorem which will be useful subsequently.

LEMMA 4.15. *If, in Theorem 4.14, a certain structure $\mathfrak{A}_p$ is thin then all of the structure $\mathfrak{A}_{p,j}$ are thin. Similarly, if for some $q \in J$, $\mathfrak{B}_q$ is thin then all of the structures $\mathfrak{B}_{i,q}$ are thin.*

Proof. Assume that $\mathfrak{A}_p$ is thin and let $q \in J$. By 2.3 (v) and the definition of $\mathfrak{F}_q$, the $\approx$ relation of $\mathfrak{F}_q\times\mathfrak{B}_{p,q}$ is simply the restriction of the $\approx$ relation on $\underset{j\in J}{P}\mathfrak{B}_{p,j}$ to its substructure. Hence by 4.14 (ii), (iv) the same holds true for $\mathfrak{E}_p\times\mathfrak{A}_{p,q} \subseteq \mathfrak{E}_p\times\mathfrak{A}_p$. I.e., if $y \in A_{p,q}$ then $E_p\times(y/\approx^{\mathfrak{A}_{p,q}}) \subseteq E_p\times(y/\approx^{\mathfrak{A}_p})$. Since $\mathfrak{A}_p$ is thin we get that $y/\approx^{\mathfrak{A}_{p,q}} = \{y\}$ and it certainly follows that $\mathfrak{A}_{p,q}$ is thin.

THEOREM 4.16. *Suppose that $\mathfrak{A}_i, \mathfrak{B}_j \in \boldsymbol{R}^c$ $(i, j = 0, 1)$ and that*

(i) $$\mathfrak{A}_0\times\mathfrak{A}_1 \cong \mathfrak{B}_0\times\mathfrak{B}_1.$$

If $\mathfrak{A}_0$ and $\mathfrak{B}_0$ are thin then (i) *has a refinement.*

Proof. Assume that the suppositions of this theorem are verified and, in particular, $\mathfrak{A}_0$ and $\mathfrak{B}_0$ are thin. Let us choose any isomorphism φ for (i) and an element $u \in \mathfrak{A}_0\times\mathfrak{A}_1$. The requirements of Theorem 4.14, with $I = J = 2$, are met in this case and we therefore have the structures $\mathfrak{E}_i, \mathfrak{F}_j, \mathfrak{A}_{i,j}, \mathfrak{B}_{i,j}$ $(i, j = 0, 1)$ satisfying Theorem 4.14 (ii), (iii) and (iv), and of course $\mathfrak{E}_i, \mathfrak{F}_j \in \boldsymbol{U}$ by Theorem 4.14 (v). Moreover, because $\mathfrak{A}_0$ and $\mathfrak{B}_0$ are thin it follows, by Lemma 4.15 and by the definitions in the proof of Theorem 4.14, that

(1) $\mathfrak{E}_1 \cong \mathfrak{F}_1 \cong \mathbf{1}$ *and the structures* $\mathfrak{A}_{0,0}, \mathfrak{A}_{0,1}, \mathfrak{B}_{0,0}$ *and* $\mathfrak{B}_{1,0}$ *are thin.*

Let us make the definition

(2) $\mathfrak{C}_{0,1} = \mathfrak{A}_{0,1}$ *and* $\mathfrak{C}_{i,j} = \mathfrak{B}_{i,j}$ *for* $\langle i, j\rangle \neq \langle 0, 1\rangle$;

and verify that the structures $\mathfrak{C}_{p,q}$ constitute a refinement of (i).

Taking the skeletons of the isomorphic structures $\mathfrak{E}_0 \times \mathfrak{A}_{0,0}$ and $\mathfrak{F}_0 \times \mathfrak{B}_{0,0}$ (Theorem 4.14 (iv)) we deduce by (1) and (2.3 (v) that $\mathfrak{A}_{0,0} \cong \mathfrak{B}_{0,0}$. In view of (1), the formulas 4.14 (iv) thus yield

(3) $\mathfrak{A}_{0,0} \cong \mathfrak{B}_{0,0}$, $\mathfrak{E}_0 \times \mathfrak{A}_{0,1} \cong \mathfrak{B}_{0,1}$, $\mathfrak{A}_{1,0} \cong \mathfrak{F}_0 \times \mathfrak{B}_{1,0}$ *and* $\mathfrak{A}_{1,1} \cong \mathfrak{B}_{1,1}$.

Now, with (1), (2) and (3), the formulas 4.14 (ii, iii) for $p = q = 1$ yield $\mathfrak{A}_1 \cong \mathfrak{C}_{1,0} \times \mathfrak{C}_{1,1}$ and $\mathfrak{B}_1 \cong \mathfrak{C}_{0,1} \times \mathfrak{C}_{1,1}$. To complete the proof what remains is to infer that $\mathfrak{A}_0 \cong \mathfrak{C}_{0,0} \times \mathfrak{C}_{0,1}$ and $\mathfrak{B}_0 \cong \mathfrak{C}_{0,0} \times \mathfrak{C}_{1,0}$. This follows quite easily from formulas 4.14 (ii, iii) for $p = q = 0$, by again taking skeletons of both sides and noting that by (1) and (3) we have $\mathcal{SK}(\mathfrak{B}_{0,1}) \cong \mathfrak{A}_{0,1}$ and $\mathcal{SK}(\mathfrak{A}_{1,0}) \cong \mathfrak{B}_{1,0}$.

5. The algebra of finite reflexive relations. We restrict ourselves for a time to finite structures and at the same time discard the hypothesis of connectedness. Let $\boldsymbol{R}_f$ denote the class of finite structures $\mathfrak{B} = \langle B, S\rangle$ with a relation S that is binary and reflexive over B, and $\boldsymbol{R}_f^c$ denote the class of connected structures among them. We wish to define, and briefly to discuss, the algebra of isomorphism types associated with $\boldsymbol{R}_f$.

There are various ways to define this algebra but the end result is the same. Here we shall just assume that an object $\tau(\mathfrak{A})$—called the isomorphism type of $\mathfrak{A}$—has been correlated with every structure $\mathfrak{A} \in \boldsymbol{R}_f$, in such a way that the isomorphism types of two structures $\mathfrak{A}$ and $\mathfrak{B}$ are identical if, and only if, $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic. The set of all the types will be denoted $\hat{\boldsymbol{R}}_f$. We provide it with two binary operations, called cardinal sum and product of types, by requiring that for arbitrary $\mathfrak{A}, \mathfrak{B} \in \boldsymbol{R}_f$

$$\tau(\mathfrak{A}) + \tau(\mathfrak{B}) = \tau(\mathfrak{A} + B) \quad \text{and} \quad \tau(\mathfrak{A}) \cdot \tau(\mathfrak{B}) = \tau(\mathfrak{A} \times \mathfrak{B});$$

and thus arrive at an algebraic system $\mathfrak{R}_f = \langle \hat{\boldsymbol{R}}_f, +, \cdot\rangle$, which should be called the algebra of finite reflexive types.

(N.B.: By abstracting this way one arrives at many interesting algebras correlated with classes of structures. Usually, however, the set of types, $\hat{\boldsymbol{K}}$, will be rather a proper class—too big to be a set—and the operations induced on $\hat{\boldsymbol{K}}$ will be not all of finite rank; indeed, cardinal multiplication has been defined most generally as an operation without any definite rank. Loosely speaking, the only necessary requirements are that $\boldsymbol{K}$ be closed under all operations in question and that these operations preserve the relation of isomorphism between structures. The concepts and problems discussed in this paper are largely motivated by the hope of understanding these very general algebras. The difficulties involved in defining the algebras of isomorphism types within particular axiomatic systems of set theory are not a serious obstacle and need not concern us here.)

The object of this section is to describe the algebra $\langle \hat{\boldsymbol{R}}_f, +, \cdot\rangle$ in somewhat more classical and familiar terms.

We begin by observing that the set of additively indecomposable elements of $\hat{\boldsymbol{R}}_f$ is identical with $\hat{\boldsymbol{R}}_f^c$, the set of connected types. Clearly, $\alpha \cdot \beta$ is connected iff α and β are connected. One important connected type is the unit type 1, the type of any one element structure.

For $\cdot$, the useful notion of indecomposable element is a different one. We say that a type α is directly indecomposable (or simply, indecomposable) if $\alpha \neq 1$ and $\alpha = \beta \cdot \gamma$ implies $\beta = 1$ or $\gamma = 1$—in otherwords, if α is the isomorphism type of a directly indecomposable structure.

Now because we deal with finite types, or types of finite structures, every type α is a finite sum of connected types, and in turn every connected type other than 1 is a finite product of directly indecomposable connected types. Let $\xi_0, \xi_1, \ldots, \xi_n, \ldots$ $(n < \omega)$ be a list without repetitions of all the directly indecomposable connected types. Let $Z[x]$ denote the polynomial ring with integer coefficients constructed from an infinite list of indeterminates $x_0, x_1, \ldots, x_n, \ldots$ $(n < \omega)$. Let $Z^+[x]$ denote the subalgebra of $Z[x]$ composed of non-zero polynomials with positive coefficients. It follows then, since $\mathfrak{R}_f$ is generated by the ξ_k (together with 1) and, obviously, all of the equations which hold for $+$ and $\cdot$ in a commutative ring with unit are valid for the sum and product of types, that there is a unique homomorphism from $Z^+[x]$ onto $\mathfrak{R}_f$ which replaces x_k by ξ_k and maps the constant polynomial 1 to the unit type. For $P(x) \in Z^+[x]$, let $P(\xi) \in \hat{\boldsymbol{R}}_f$ denote its image.

To complete this train of thought we note that both the expression of an arbitrary type as the sum of connected types, and the expression of a connected type ($\neq 1$) as the product of indecomposable types ξ_k, are unique up to the order of the factors; the first is well known and the second is of course a consequence of Corollary 4.7. Thus it follows that the polynomial representing a given type is unique. And we have the following

THEOREM 5.1. *The algebra of finite reflexive types, $\mathfrak{R}_f$, is isomorphic to the polynomial semi-ring $Z^+[x]$.*

This theorem provides a new proof for some special corollaries of a celebrated theorem of L. Lovász [8]. We mention among them the following proposition, which will be strengthened in § 8.

COROLLARY 5.2. *If* $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in \boldsymbol{R}_f$ *and* $\mathfrak{A}\times\mathfrak{B} \cong \mathfrak{A}\times\mathfrak{C}$ *then* $\mathfrak{B}\cong\mathfrak{C}$.

We mention also the example discovered by Hashimoto and Nakayama [5] of a finite reflexive structure not having the refinement property. All such examples owe their existence to the fact that in some cases the product of two polynomials with negative coefficients has only positive coefficients. $Z[x]$ is a unique factorization domain, of course, but the refinement and unique factorization properties are not universally valid in $Z^+[x]$.

The correspondence between finite reflexive structures and polynomials will play a small but crucial part in § 8. Therefore, for later reference, we now make a definition which is justified by the proof of Theorem 5.1.

DEFINITION 5.3. $\mathfrak{X}_0, \mathfrak{X}_1, \ldots, \mathfrak{X}_n, \ldots$ $(n<\omega)$ are indecomposable structures belonging to $\boldsymbol{R}_f^c$, and to every indecomposable structure $\mathfrak{A}\in\boldsymbol{R}_f^c$ there corresponds exactly one $k<\omega$ such that $\mathfrak{A}\cong\mathfrak{X}_k$. For each $P\in Z^+[x]$ we let $P(\mathfrak{X})$—or $P(\mathfrak{X}_0, \ldots, \mathfrak{X}_n)$ if P depends only upon $x_0, \ldots, x_n$—be the structure which P denotes when we replace every symbol x_k by $\mathfrak{X}_k$, 1 by **1** (a one element structure) and then interpret $+$ and $\cdot$ as cardinal sum and product. Finally, if $\mathfrak{A}\in\boldsymbol{R}_f$, then we let $P_{\mathfrak{A}}$ denote the unique polynomial $P\in Z^+[x]$ for which $\mathfrak{A}\cong P(\mathfrak{X})$.

6. Discussion. This discussion is not essential for later sections, but may be instructive. The objective is to see how far the idea of polynomial representation can be extended to infinite reflexive structures, using the earlier results. Maximum generality is not attempted, although a further study might prove rewarding.

A first approximation to the most general concept of polynomial we shall use is the notion of *formal power series* in the symbols (indeterminates) x_n, $n<\omega$, *with cardinal number coefficients*. These may be thought of as expressions of the form

$$P=\sum_{\varphi}\mathfrak{p}_\varphi x^{(\varphi)}, \tag{6.1}$$

where $\mathfrak{p}_\varphi$ are arbitrary non-zero cardinal numbers, and $x^{(\varphi)}$ ranges over some non-void set of finite monomial combinations of the symbols x_n. (A more precise alternative is to identify P with its coefficient function $\mathfrak{p}$.) They form an algebra, $C^\star[x]$, under operations $+$ and $\cdot$, defined just as for ordinary power series. (Strictly speaking, this is not an algebra in the usual sense because the universe is a proper class.)

$C^\star[x]$ is evidently isomorphic to the algebra of isomorphism types of reflexive structures, all of whose connected components are finite: In the framework of Definition 5.3, the monomials $M^{(\varphi)}=x^{(\varphi)}$ represent the finite connected structures $M^{(\varphi)}(\mathfrak{X})$; the power series P in (6.1) represents, therefore, a structure $P(\mathfrak{X})$ having exactly $\mathfrak{p}_\varphi$ connected components isomorphic to $M^{(\varphi)}(\mathfrak{X})$.

Obviously, it is very useful to know that a structure can be expressed as a polynomial, or power series, combination of other, indecomposable structures, especially so if the expression is unique, as is the case above. Equally obviously, the theory of such expression meets serious obstacles when we attempt to extend it to structures much more general than those above, or to bring infinite cardinal multiplication—and therefore monomial terms of infinite length or having infinite exponents—into the picture. For example: for each n, $1<n<\omega$ let $\mathfrak{Z}_n=\langle n, R_n\rangle$ be the "cyclic" structure in which kR_nl iff $k=l$ or $k\equiv l+1\,(n)$. Clearly, each $\mathfrak{Z}_n$ is an isomorphic copy of one of the structures $\mathfrak{X}_k$ listed in Definition 5.3. One verifies that

$$\mathop{P}_{n}\mathfrak{Z}_n\cong 2\mathop{P}_{n}\mathfrak{Z}_n\cong 2^{\aleph_0}\mathfrak{B},$$

where $\mathfrak{B}$ is a certain connected structure which fails to be isomorphic to any product of directly indecomposable structures. So we see that neither uniqueness nor existence of a representing polynomial can be expected in the general case, nor is the one-one correspondence between the monomial constituents of the polynomial and the connected components of the structure preserved.

(Remark 6.2. The unpleasant features of the above example can be avoided in various ways. Let, for instance, $\boldsymbol{L}_m$ be the class of all thin, directly indecomposable members of $\boldsymbol{R}^c$, in which the fundamental relation connects every two elements by a path of length m or less. Let $\boldsymbol{K}_m$ be the class generated by $\boldsymbol{L}_m$ under arbitrary cardinal sum and product. Products of connected structures of $\boldsymbol{K}_m$ are connected and thin; and it follows easily from Theorem 4.1 that every structure in $\boldsymbol{K}_m$ can be expressed essentially uniquely in the form

$$\mathfrak{A}\cong\sum_{i\in I}\mathfrak{n}_i\mathop{P}_{j\in J_i}\mathfrak{A}_{i,j}, \tag{6.3}$$

where $\mathfrak{A}_{i,j}\in\boldsymbol{L}_m$ and the $\mathfrak{n}_i$ are non-zero. The precise meaning of "uniqueness" used here is explicated in Remark 6.5 below.)

We now formulate a theorem which implies that, in the above example, the system of factors $\langle\mathfrak{Z}_n\rangle$ at least is unique to within isomorphism.

THEOREM 6.4. *Suppose that* $\mathfrak{A}_i$ $(i\in I)$ *and* $\mathfrak{B}_j$ $(j\in J)$ *are thin, directly indecomposable members of* $\boldsymbol{R}^c$ *such that* (i) $\mathop{P}_{i\in I}\mathfrak{A}_i\cong\mathop{P}_{j\in J}\mathfrak{B}_j$. *Then there exists a one-one map* g *of* I *onto* J *such that* $\mathfrak{A}_i\cong\mathfrak{B}_{g(i)}$ *for every* $i\in I$.

Proof. (Sketch) The theorem follows immediately by Theorem 4.1 if the product (i) is connected, since it is thin in any case.

In the general case, we work with an isomorphism $\psi\colon \mathfrak{A} \cong \mathfrak{B}$ where $\mathfrak{A}$ and $\mathfrak{B}$ are connected components of the two products (i). One sees that $\mathfrak{A} = \mathfrak{A}_p \times \mathfrak{A}'_p$ for each $p \in I$, where $\mathfrak{A}'_p$ is a connected component of $\underset{i\neq p}{P}\mathfrak{A}_i$. This gives a system of factor relations over $\mathfrak{A}$, F_i and F'_i say, such that

(1) $$\mathrm{id}_A = F_p \times F'_p = \bigcap_{i\in I} F_i \quad (\text{for all } p \in I);$$

and moreover $\mathfrak{A}/F_p \cong \mathfrak{A}_p$. Working similarly with $\mathfrak{B}$ and translating back through ψ, we have also G_j, G'_j over $\mathfrak{A}$ such that

(2) $$\mathrm{id}_A = G_q \times G'_q = \bigcap_{j\in J} G_j \quad (\text{for all } q \in J);$$

and $\mathfrak{A}/G_q \cong \mathfrak{B}_q$.

Now $\mathfrak{A}$ has the strict refinement property (SRP), by Theorem 4.1, so we can argue as follows: For fixed $p \in I$, $F'_p \neq \mathrm{id}_A$ for otherwise $\mathfrak{A}_p$ would have only one element—by (2), pick $q = g(p)$ so that $F'_p \nsubseteq G_q$; by (SRP), $G_q = (G_q|F_p)\times(G_q|F'_p)$ and it follows since $\mathfrak{A}/G_q$ is indecomposable, that $F_p \subseteq G_q$. The same reasoning finds $p' \in I$ such that $G_q \subseteq F_{p'}$, and it evidently follows that $F_p = G_{g(p)}$. The rest is obvious.

Remark 6.5. The above developments may now be unified, and somewhat clarified, by introducing the general notion of polynomial which was earlier alluded to. For this, we take the indeterminates to be a sequence of symbols x_α, bi-uniquely corresponding to all the ordinal numbers α. A *monomial* is now to be construed as an arbitrary function φ, defined over any subset of the class of ordinals, and taking non-zero cardinal numbers as values. Of course it is more suggestive to think of φ as a "formal expression"

(6.6) $$x^{(\varphi)} = \prod_{\alpha\in S} x_\alpha^{\varphi(\alpha)},$$

in which S denotes the domain of φ. A *cardinal series* is defined to be an expression of the form

(6.7) $$P = \sum_{\varphi} \mathfrak{p}_\varphi x^{(\varphi)},$$

where the $\mathfrak{p}_\varphi$ are non-zero cardinal numbers, and where $x^{(\varphi)}$ ranges over some non-void sub*set* of this class of monomials. (It would be more precise to identify P with its coefficient function $\mathfrak{p}$, but of course this is not necessary.)

Operations of forming the product of any system, and the sum of any nonvoid system, of cardinal series can be defined simply by extending the usual definitions for power series. In fact, we identify the symbols x_α with the proper series, and then there is just one way to define these operations so that generalized associative, commutative and distributive laws are valid, and so that (6.6) and (6.7) serve to represent the series P as a sum-product of the x_α.

Attaching these operations to the class of cardinal series, we obtain an algebra $C^\star$. It is natural to call this the *free algebra for cardinal operations*, generated by the symbols x_α. It is evident that the gist of Remark 6.2 is just the fact that the algebra of types, $\hat{\boldsymbol{K}}_m$, is isomorphic to $C^\star$.

Finally, Theorem 6.4 and the example preceding it have led us to the following result. Let $\boldsymbol{L}$ be the class of all thin, directly indecomposable, reflexive connected structures, and let $\boldsymbol{K}$ be the class of (thin) structures generated by cardinal operation applied to $\boldsymbol{L}$. Let $\langle\mathfrak{X}_\alpha\rangle$ be a list of structures in $\boldsymbol{L}$, indexed by the ordinals, such that each structure in $\boldsymbol{L}$ appears isomorphically just once in the list. Then the map $x_\alpha \rightsquigarrow \mathfrak{X}_\alpha$ extends to represent each structure $\mathfrak{A} \in \boldsymbol{K}$ in the form $\mathfrak{A} \cong P(\mathfrak{X})$ where $P \in C^\star$. The cardinal operations in $\boldsymbol{K}$ are determined by the operations in $C^\star$. Thus the crucial question of when two cardinal series P and Q represent isomorphic structures—we say P and Q are "equivalent"—can now be interpreted as asking for the relations which define the algebra of isomorphism types, $\boldsymbol{K}$, as a homomorphic image of $C^\star$. The answer, in the current case, is:

Two cardinal series $P = \sum_{\varphi} \mathfrak{p}_\varphi x^{(\varphi)}$ and $Q = \sum_{\psi} \mathfrak{q}_\psi x^{(\psi)}$ are equivalent if, and only if, the same monomials appear in them and for each monomial $x^{(\varphi)}$ which appears, $\mathfrak{p}_\varphi x^{(\varphi)}$ and $\mathfrak{q}_\varphi x^{(\varphi)}$ are equivalent. Furthermore, two terms $\mathfrak{m}x^{(\varphi)}$ and $\mathfrak{n}x^{(\varphi)}$ are equivalent if, and only if, $\mathfrak{m}\cdot\mathfrak{c}(\varphi) = \mathfrak{n}\cdot\mathfrak{c}(\varphi)$, where $\mathfrak{c}(\varphi)$ is the least cardinality of a maximum set of isomorphic connected components in the structure $\underset{\alpha}{P}\,\mathfrak{X}^{\varphi(\alpha)}$.

(The proof is by an easy modification of the argument for Theorem 6.4. Some hope exists for arranging the sequence $\langle\mathfrak{X}_\alpha\rangle$ so that the function $\mathfrak{c}(\varphi)$ becomes calculable. We also expect that Lemma 3.1 might be used to obtain analogous results about the algebra of reflexive types generated by *all* indecomposable connected types. However, we shall let these matters rest here.)

Remark 6.6. Our proof of Theorem 6.4 seems to suggest that the connected components in a product of connected, reflexive structures deserve further study. Accordingly, it is natural to introduce a new operation, the *connected product*, to be applied to indexed families of connected reflexive structures having a distinguished element, say $\mathfrak{A}_i = \langle A_i, R_i, O_i\rangle$ where $O_i \in A_i$ for every $i \in I$. In fact, we would put $\overset{0}{\underset{i\in I}{P}}\mathfrak{A}_i = \langle C, T, O\rangle$, where $\langle C, T\rangle$ is defined as that connected component of $\underset{i\in I}{P}\langle A_i, R_i\rangle$ which contains the element $O = \langle O_i\colon i \in I\rangle$.

The two operations P and $\overset{0}{P}$ are closely related; for instance, when the full product is connected, they coincide. Their relationship is analogous in at least one respect to that between the full and the weak direct product of groups: we have $\overset{0}{\underset{i\in I}{P}}\mathfrak{A}_i = \mathfrak{A}_p \times \overset{0}{\underset{i\neq p}{P}}\mathfrak{A}_i$ for each $p \in I$. Thus Lemma 3.1 should prove useful in investigating the properties of $\overset{0}{P}$; and, in fact, the question arises which of the results of this paper can be formulated and proved for $\overset{0}{P}$. However, apart from noting that Theorem 6.4 has a valid version with P replaced by $\overset{0}{P}$ (the same proof works), we have not studied these problems.

7. Absorbing a finite factor. We prepare for the next section by proving, independently of the earlier work, a result of considerable intrinsic interest which concerns the finite factors of denumerable structures. By "denumerable" is meant finite or countably infinite. Apart from this limitation on the cardinality—an essential limitation, as shown by the example of W. Hanf, Example 8.7 below—the theorem of this section is valid for completely arbitrary relational structures. As regards the corollary to the theorem we mention that it solves a problem stated in Chang's thesis, and also that similar conclusions have been obtained in the literature under a variety of assumptions. These related results are discussed in [1; § 5].

THEOREM 7.1. *Let $\mathfrak{A}$ be any denumerable relational structure. Let $\mathfrak{B}$ be a finite structure. Assume that $\mathfrak{A}_n$ $(n<\omega)$ is a sequence of structures satisfying: $\mathfrak{A}\cong\mathfrak{B}\times\mathfrak{A}_0$ and $\mathfrak{A}_n\cong\mathfrak{B}\times\mathfrak{A}_{n+1}$ for each $n<\omega$. From these assumptions it follows that $\mathfrak{A}\cong\mathfrak{B}\times\mathfrak{A}$, and furthermore, $\mathfrak{A}\cong\mathfrak{A}_n$ for each n.*

COROLLARY 7.2. *If $\mathfrak{A}\cong\mathfrak{B}\times\mathfrak{C}\times\mathfrak{A}$, where $\mathfrak{A}$ is countable and $\mathfrak{B}$ finite, then $\mathfrak{A}\cong\mathfrak{B}\times\mathfrak{A}\cong\mathfrak{C}\times A$.*

Proof of the theorem. It surely suffices to show that $\mathfrak{A}\cong\mathfrak{B}\times\mathfrak{A}$ under the assumptions of the theorem, since we can then apply this result to $\mathfrak{A}_0$ in place of $\mathfrak{A}$, obtaining $\mathfrak{A}_0\cong\mathfrak{B}\times\mathfrak{A}_0\cong\mathfrak{A}$, and then inductively obtaining $\mathfrak{A}_n\cong\mathfrak{A}$ in a like manner.

Let then θ_0: $\mathfrak{A}\cong\mathfrak{B}\times\mathfrak{A}_0$, and θ_{n+1}: $\mathfrak{A}_n\cong\mathfrak{B}\times\mathfrak{A}_{n+1}$ for each $n<\omega$, where $\mathfrak{A}$ is countable and $\mathfrak{B}$ finite. We correlate with every element $a\in A$ two sequences:

(1) $\quad \zeta(a)=\langle b_0, b_1, \ldots\rangle \in {}^{\omega}B \quad$ and $\quad \eta(a)=\langle a_0, a_1, \ldots\rangle \in \underset{n<\omega}{P} A_n$,

defined by the formulas:

(2) $\quad \langle b_0, a_0\rangle=\theta_0(a); \quad \langle b_{n+1}, a_{n+1}\rangle=\theta_{n+1}(a_n), \quad$ for all $n<\omega$.

By writing that "the sequences $\boldsymbol{x}$ and $\boldsymbol{y}$ eventually agree" we mean, of course, that for some $m<\omega$, $x_n=y_n$ for all $n>m$. Now the following statement is fairly obvious:

(3) *Let $a\in A$ and $\boldsymbol{b}\in{}^{\omega}B$. If $\boldsymbol{b}$ and $\zeta(a)$ eventually agree, then there is a unique element $a'\in A$ satisfying: $\zeta(a')=\boldsymbol{b}$, and $\eta(a')$ eventually agrees with $\eta(a)$.*

The key to the construction of an isomorphism φ between $\mathfrak{A}$ and $\mathfrak{B}\times\mathfrak{A}$ is a simple combinatorial fact, Lemma 7.3 below. The lemma obviously provides us with a function f such that:

(4) *f: $\omega\to\omega-\{0\}$ (one-to-one and onto) and for every $a\in A$, the sequences $\zeta(a)$ and $\zeta(a)\circ f$ eventually agree.*

Given any function f satisfying (4), we make reference to (3) and define $\varphi(a)=\langle b_0, a'\rangle$, where $\zeta(a)=\langle b_0, b_1, \ldots\rangle$ and a' is the unique element of A for which $\zeta(a')=\langle b_{f_0}, b_{f_1}, \ldots\rangle$ and $\eta(a')$ agrees eventually with $\eta(a)$. One easily shows, using (1–4), that this function φ is a bijection from A onto $B\times A$, and that it preserves the finitary relations preserved by all the functions θ_n; i.e. φ: $\mathfrak{A}\cong\mathfrak{B}\times\mathfrak{A}$. We leave these concluding details to the reader, and go on to the auxiliary lemma.

LEMMA 7.3. *Let B be a finite set and let $\mathcal{A}\subseteq{}^{\omega}B$. Assuming $\mathcal{A}$ is countable, there exists a bijective mapping f: $\omega\to\omega-\{0\}$ such that the sequences $\boldsymbol{b}$ and $\boldsymbol{b}\circ f$ agree eventually whenever $\boldsymbol{b}\in\mathcal{A}$.*

Proof. Let $\boldsymbol{b}^{(0)}, \boldsymbol{b}^{(1)}, \ldots, \boldsymbol{b}^{(n)}, \ldots$ $(n<\omega)$ be elements of ${}^{\omega}B$ including all elements of $\mathcal{A}$. We define L_0 to be an infinite subset of ω such that $\boldsymbol{b}^{(0)}$ is a constant function on L_0. Such a set exists since B is finite. Next, let L_1 be an infinite set on which $\boldsymbol{b}^{(1)}$ is constant, with $L_1\subseteq L_0$. Continuing in this fashion (or, more rigorously, making a certain application of the axiom of choice which is usually omitted in informal proof) we arrive at a system of infinite subsets of ω, $\langle L_n\rangle_{n<\omega}$ such that $\boldsymbol{b}^{(n)}$ is constant on L_n, and $L_m\subseteq L_n$, whenever $n<m<\omega$.

With this system of sets, we can clearly correlate a strictly increasing sequence of integers, $i_0<i_1<i_2<\ldots$, such that $i_0=0$ and $i_p\in L_q$ whenever $q<p$.

Finally, we just define:

$$f(i_n)=i_{n+1} \text{ for all } n<\omega; \quad f(i)=i \text{ if } i\in\omega-\{i_n\colon n\in\omega\}.$$

The function f thus defined will have the desired properties, in fact $\boldsymbol{b}^{(q)}(j)=\boldsymbol{b}^{(q)}(f(j))$ for all $q<\omega$ and $j>i_q$. So the proof is complete.

8. Cancellation. Once again, we consider exclusively binary reflexive structures, i.e. those of the class $\boldsymbol{R}$. This section is addressed to the cancellation problem for such structures: Assume that (i) $\mathfrak{A}\times\mathfrak{B}\cong\mathfrak{A}\times\mathfrak{C}$, where $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}\in\boldsymbol{R}$. Under what additional hypotheses on the structures $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ can we conclude that $\mathfrak{B}\cong\mathfrak{C}$?

We are able to give much more complete results on this problem than those known at the time of Chang's survey [1; § 7]. According to Corollary 5.2, $\mathfrak{A}$ can be cancelled in (i) if both $\mathfrak{A}$ and $\mathfrak{B}$ are finite. More generally, we shall prove that any one of the following hypotheses is sufficient for the cancellation of $\mathfrak{A}$: (1) $\mathfrak{A}$ is finite and connected (no restriction on $\mathfrak{B}$ and $\mathfrak{C}$ except, of course, that they belong to $\boldsymbol{R}$); (2) $\mathfrak{A}$ is connected, thin and directly indecomposable; (3) $\mathfrak{A}$ is finite, $\mathfrak{B}$ is denumerable, and $\mathfrak{B}$ is the sum of finitely many connected components. Our entire discussion is founded upon Lemma 3.1 which will be applied, however, indirectly, through various results formulated in preceding sections.

Given a class $\boldsymbol{K}$ of structures, we denote by $\boldsymbol{K}^c$ the class of all connected structures in $\boldsymbol{K}$, by $\boldsymbol{K}_f$ the class of all finite structures in $\boldsymbol{K}$, by $\boldsymbol{K}_\omega$ the class of all denumerable (finite or at most countable) structures in $\boldsymbol{K}$. Thus are formed five subclasses of $\boldsymbol{R}$, viz. $\boldsymbol{R}_f$, $\boldsymbol{R}_\omega$, $\boldsymbol{R}^c$, $\boldsymbol{R}^c_f$ and $\boldsymbol{R}^c_\omega$. We shall use the fact that all of these classes are closed under binary cardinal multiplication and under the taking of direct factors. If for every $\mathfrak{B}, \mathfrak{C} \in \boldsymbol{K}$, $\mathfrak{A}\times\mathfrak{B} \cong \mathfrak{A}\times\mathfrak{C}$ implies $\mathfrak{B}\cong\mathfrak{C}$, then we say $\mathfrak{A}$ is *cancellable over the class* $\boldsymbol{K}$. The class consisting of all structures which (belong to $\boldsymbol{R}$ and) are cancellable over the class $\boldsymbol{R}$ is likewise closed under binary cardinal multiplication and under the taking of direct factors (obviously).

LEMMA 8.1. *Let $\mathfrak{A} \in \boldsymbol{R}^c$ and $\boldsymbol{K} \subseteq \boldsymbol{R}^c$. If $\mathfrak{A}$ is cancellable over $\boldsymbol{K}$, then $\mathfrak{A}$ is cancellable over the class $\boldsymbol{K}'$ which consists of all cardinal sums of structures belonging to $\boldsymbol{K}$.*

Proof. Assume that $\mathfrak{A}$ is cancellable over $\boldsymbol{K}$. Let then $\mathfrak{B}, \mathfrak{C} \in \boldsymbol{K}'$, say $\mathfrak{B} = \sum_{i\in I}\mathfrak{B}_i$ and $\mathfrak{C} = \sum_{j\in J}\mathfrak{C}_j$ with $\mathfrak{B}_i, \mathfrak{C}_j \in \boldsymbol{K}$ for all $i\in I$, $j \in J$. The connected components (maximal connected substructures) of $\mathfrak{A}\times\mathfrak{B}$ are $\mathfrak{A}\times\mathfrak{B}_i$, $i\in I$. The connected components of $\mathfrak{A}\times\mathfrak{C}$ are $\mathfrak{A}\times\mathfrak{C}_j$, $j\in J$. Thus, if φ: $\mathfrak{A}\times\mathfrak{B}\cong\mathfrak{A}\times\mathfrak{C}$ then φ must carry the one system of connected components onto the other, and we have a one-one map σ from I onto J such that $\varphi(\mathfrak{A}\times\mathfrak{B}_i) = \mathfrak{A}\times\mathfrak{C}_{\sigma i}$ for each $i \in I$. Therefore, since $\mathfrak{A}$ can be cancelled in $\boldsymbol{K}$, it follows from these assumptions that $\mathfrak{B}_i \cong \mathfrak{C}_{\sigma i}$ $(i\in I)$, and this clearly gives $\mathfrak{B}\cong\mathfrak{C}$.

LEMMA 8.2. *Let $\mathfrak{A}\in\boldsymbol{R}_f$, let $\mathfrak{B}, \mathfrak{C}\in\boldsymbol{R}$ and assume that $\mathfrak{A}\times\mathfrak{B}\cong\mathfrak{A}\times\mathfrak{C}$. This formula has a refinement if, and only if, $\mathfrak{B}\cong\mathfrak{C}$.*

Proof. One direction is obvious. For the other, assume we have a refinement: $\mathfrak{A}\cong\mathfrak{C}_{00}\times\mathfrak{C}_{01}$, $\mathfrak{B}\cong\mathfrak{C}_{10}\times\mathfrak{C}_{11}$, $\mathfrak{A}\cong\mathfrak{C}_{00}\times\mathfrak{C}_{10}$, $\mathfrak{C}\simeq\mathfrak{C}_{01}\times\mathfrak{C}_{11}$. By Corollary 5.2, we have $\mathfrak{C}_{01}\cong\mathfrak{C}_{10}$ (cancelling $\mathfrak{C}_{00}$ from the products isomorphic to $\mathfrak{A}$). Thus $\mathfrak{B}\cong\mathfrak{C}_{10}\times\mathfrak{C}_{11}\cong\mathfrak{C}_{01}\times\mathfrak{C}_{11}\cong\mathfrak{C}$.

LEMMA 8.3. (See Definition 2.2) *Let $\mathfrak{A}\in\boldsymbol{R}_f$ and $\mathfrak{B}\in\boldsymbol{R}$. If every $\approx$-coset of $\mathfrak{B}$ is infinite then we have*

$$\mathfrak{A}\times\mathfrak{B}\cong\mathcal{SK}(\mathfrak{A})\times\mathfrak{B}\,.$$

Proof. Straightforward, by Lemmas 2.3 (v), 2.5, 2.6.

THEOREM 8.4. *Every structure belonging to the class $\boldsymbol{R}^c_f$ is cancellable over the class $\boldsymbol{R}$.*

Proof. Assume that $\mathfrak{A}\in\boldsymbol{R}^c_f$. According to Lemma 8.1 it is enough to show that $\mathfrak{A}$ is cancellable over $\boldsymbol{R}^c$. Thus we assume that

$$(1)\qquad \mathfrak{A}\times\mathfrak{B}\cong\mathfrak{A}\times\mathfrak{C}\,,\quad \text{where } \mathfrak{B},\mathfrak{C}\in\boldsymbol{R}^c;$$

and attempt to infer that $\mathfrak{B}\cong\mathfrak{C}$.

In case $\mathfrak{B}$ has a finite $\approx$-coset then the same is true of the connected structure $\mathfrak{A}\times\mathfrak{B}$ (by 2.3 (v), since $\mathfrak{A}$ is finite), and we can employ Theorem 4.4 to deduce that (1) has a refinement. From this, it follows by Lemma 8.2 that $\mathfrak{B}\cong\mathfrak{C}$.

In the contrary case, (1) implies that neither $\mathfrak{B}$ nor $\mathfrak{C}$ has any finite $\approx$-coset, and we apply Lemma 8.3 to obtain

$$(2)\qquad \mathcal{SK}(\mathfrak{A})\times\mathfrak{B}\cong\mathcal{SK}(\mathfrak{A})\times\mathfrak{C}\,.$$

Now, $\mathcal{SK}(\mathfrak{A})$ is connected and thin, whence we deduce by Theorem 4.16 that (2) has a refinement. In this case, again, the argument is concluded by quoting Lemma 8.2.

THEOREM 8.5. *Assume that $\mathfrak{A}\in\boldsymbol{R}^c$ and that $\mathfrak{A}$ is thin and directly indecomposable. Then $\mathfrak{A}$ is cancellable over the class $\boldsymbol{R}$.*

Proof. Quite easy using Lemma 8.1 and Theorem 4.16.

Let us note that every structure $\langle A, R\rangle$, in which R is a linear ordering of A, is connected, thin and directly indecomposable and hence cancellable over $\boldsymbol{R}$, by Theorem 8.5. This result appeared in Chang's thesis. Some examples will now be given to motivate our next (and final) theorem, and also to convey an idea of where the boundaries lie which constrain further positive results.

EXAMPLE 8.6. If $\mathfrak{A}$ absorbs $\mathfrak{B}$—i.e. $\mathfrak{A}\times\mathfrak{B}\cong\mathfrak{A}$—then $\mathfrak{A}\times\mathfrak{B}\cong\mathfrak{A}\times\boldsymbol{1}$ and $\mathfrak{A}$ cannot be cancelled unless $\mathfrak{B}\cong\boldsymbol{1}$. This is a common phenomenon. For instance take $\mathfrak{A}=\mathfrak{B}^\omega$, or let $\mathfrak{A}=\mathfrak{B}$ equal a denumerable atomless boolean algebra (construed as a partly ordered set, and therefore as a member of $\boldsymbol{R}$).

EXAMPLE 8.7. In [4] W. Hanf exhibited a non-denumerable boolean algebra $\mathfrak{B}$ such that $\mathfrak{B}\cong\mathfrak{A}\times\mathfrak{A}\times\mathfrak{B}$ and $\mathfrak{B}\ncong\mathfrak{A}\times\mathfrak{B}$, where $\mathfrak{A}$ is a two-element boolean algebra. Chang noticed, in his doctoral dissertation, that it follows that $(\boldsymbol{1}+\mathfrak{A})\times\mathfrak{B}\cong(\boldsymbol{1}+\mathfrak{A})\times(\mathfrak{A}\times\mathfrak{B})$ and here the finite structure $\boldsymbol{1}+\mathfrak{A}$ cannot be cancelled. In this example, $\mathfrak{B}$ and $\mathfrak{A}\times\mathfrak{B}$ are connected and also thin.

EXAMPLE 8.8. Let $\mathfrak{A}$ be any member of $\boldsymbol{R}$, finite, connected and non-isomorphic to $\boldsymbol{1}$. Then we have $(\boldsymbol{1}+\mathfrak{A})\times\mathfrak{B}\cong(\boldsymbol{1}+\mathfrak{A})\times\mathfrak{C}$, where

$\mathfrak{B} = \omega\cdot\mathbf{1}+\omega\cdot\mathfrak{A}^2$ and $\mathfrak{C} = \omega\cdot\mathbf{1}+\omega\cdot\mathfrak{A}+\omega\cdot\mathfrak{A}^2$. Of course $\mathfrak{B} \not\cong \mathfrak{C}$. This example differs from the last in that $\mathfrak{B}$ and $\mathfrak{C}$ are now denumerable, but not connected. Essentially, of course, this is just a single example of non-cancellation in the algebra of formal power series with cardinal coefficients, $C^\star[x]$, which was defined in § 6; and there are many others.

In fact, we can easily concoct similar examples to show that a finite structure $\mathfrak{F} \in \boldsymbol{R}$ is cancellable over $\boldsymbol{R}$ iff it has the form $\mathfrak{F} = k\cdot\mathfrak{A}$ where $\mathfrak{A}$ is connected and k is a positive integer. ($k\cdot\mathfrak{A} \cong (k\cdot\mathbf{1})\times\mathfrak{A}$, $\mathfrak{A}$ is cancellable by Theorem 8.4 and it is a known and trivial fact that $k\cdot\mathbf{1}$ is cancellable [1; § 7], whence $k\cdot\mathfrak{A}$ is cancellable.)

THEOREM 8.9. *Assume that $\mathfrak{A} \in \boldsymbol{R}_f$ and that $\mathfrak{B}$ is the cardinal sum of a finite system of structures belonging to $\boldsymbol{R}_\omega^c$. Then $\mathfrak{A}\times\mathfrak{B} \cong \mathfrak{A}\times\mathfrak{C}$ implies $\mathfrak{B} \cong \mathfrak{C}$.*

This is our last theorem and the proof is more involved than the others in this section. The hypothesis on $\mathfrak{B}$ splits into two parts: that it be denumerable, and that it have only a finite number of connected components; the two preceding examples show that neither of these hypotheses can be dropped.

We need three more lemmas.

LEMMA 8.10. *Suppose that $\mathfrak{X}^n \| \mathfrak{Z}$ for every $n<\omega$, and that $\mathfrak{X} \in \boldsymbol{R}_f^c$ and $\mathfrak{Z} \in \boldsymbol{R}_\omega$. Then $\mathfrak{Z} \cong \mathfrak{X}\times\mathfrak{Z}$.*

Proof. Let say $\mathfrak{Z} \cong \mathfrak{X}^{n+1}\times\mathfrak{Z}_n$ for $n<\omega$. From the relation $\mathfrak{X}^{n+1}\times\mathfrak{Z}_n \cong \mathfrak{X}^{n+2}\times\mathfrak{Z}_{n+1}$, according to Theorem 8.4, we can cancel $\mathfrak{X}^{n+1}$ to obtain $\mathfrak{Z}_n \cong \mathfrak{X}\times\mathfrak{Z}_{n+1}$. Then the desired conclusion follows by Theorem 7.1.

LEMMA 8.11. *Suppose that $\mathfrak{F} \in \boldsymbol{R}_f^c$ and $\mathfrak{X}, \mathfrak{Z} \in \boldsymbol{R}$. Then $\mathfrak{Z}$ absorbs $\mathfrak{X}$ (i.e. $\mathfrak{Z} \cong \mathfrak{X}\times\mathfrak{Z}$) if and only if $\mathfrak{F}\times\mathfrak{Z}$ absorbs $\mathfrak{X}$.*

Proof. By Theorem 8.4.

LEMMA 8.12. *Suppose that $\boldsymbol{K}$ is a finite set of directly indecomposable members of $\boldsymbol{R}_f^c$. Assume that $\mathfrak{Z} \in \boldsymbol{R}_\omega^c$ and that $\mathfrak{Z}$ absorbs no member of $\boldsymbol{K}$. Then there exist two structures $\mathfrak{M}$ and $\mathfrak{D}$ such that $\mathfrak{Z} \cong \mathfrak{M}\times\mathfrak{D}$, where $\mathfrak{M}$ is a finite product of structures belonging to $\boldsymbol{K}$, and $\mathfrak{D}$ is "$\boldsymbol{K}$-free"—i.e. has no member of $\boldsymbol{K}$ as a direct factor.*

Furthermore, $\mathfrak{M}$ and $\mathfrak{D}$ are unique up to isomorphism provided that either

(i) *$\mathfrak{Z}$ contains a finite $\approx$-coset; or*

(ii) *every structure belonging to $\boldsymbol{K}$ is thin.*

Proof. Of course we allow that $\mathfrak{M}$ be the product of the empty system of structures, i.e. be isomorphic to $\mathbf{1}$. The existence of $\mathfrak{M}$ and $\mathfrak{D}$ follows in a quite obvious manner from Lemma 8.10.

Now assume that

$$\mathfrak{Z} \cong \mathfrak{M}\times\mathfrak{D} \cong \mathfrak{M}_1\times\mathfrak{D}_1 \tag{1}$$

are two such decompositions, and that either (i) or (ii) is true. Then (1) has a refinement (by Theorem 4.4 in case (i) holds; or by Theorem 4.16 in case (ii) holds, since $\mathfrak{M}$ and $\mathfrak{M}_1$ will then be thin); say $\mathfrak{M} \cong \mathfrak{Z}_{00}\times\mathfrak{Z}_{01}$, $\mathfrak{D} \cong \mathfrak{Z}_{10}\times\mathfrak{Z}_{11}$, $\mathfrak{M}_1 \cong \mathfrak{Z}_{00}\times\mathfrak{Z}_{10}$, $\mathfrak{D}_1 \cong \mathfrak{Z}_{01}\times\mathfrak{Z}_{11}$. From these formulas we infer that, because the finite connected structures $\mathfrak{M}$ and $\mathfrak{M}_1$ have the refinement property (Corollary 4.7), the structures $\mathfrak{Z}_{01}$ and $\mathfrak{Z}_{10}$ are isomorphic to products of structures from $\boldsymbol{K}$; and then we obtain $\mathfrak{Z}_{01} \cong \mathbf{1} \cong \mathfrak{Z}_{10}$ since they divide the $\boldsymbol{K}$-free structures $\mathfrak{D}_1, \mathfrak{D}$. Obviously, it follows that

$$\mathfrak{M} \cong \mathfrak{Z}_{00} \cong \mathfrak{M}_1 \quad \text{and} \quad \mathfrak{D} \cong \mathfrak{Z}_{11} \cong \mathfrak{D}_1 .$$

Proof of theorem 8.9. Let $\boldsymbol{R}_\omega^0$ denote the class of all structures $\mathfrak{Z} \in \boldsymbol{R}_\omega$ which have only finitely many connected components. If $\mathfrak{X} \in \boldsymbol{R}_f$ and $\mathfrak{Z} \in \boldsymbol{R}_\omega^0$, let $t(\mathfrak{X}, \mathfrak{Z})$ be the integer $t = m+n$, where $m = {}^\#(X)$ and n is the number of connected components of $\mathfrak{Z}$.

If our theorem fails then there will be a "minimal" counterexample. It is enough, therefore, to derive a contradiction from the following assumption:

(1) (a) *We have $\mathfrak{A} \in \boldsymbol{R}_f$, $\mathfrak{B}, \mathfrak{C} \in \boldsymbol{R}_\omega^0$ and $\mathfrak{B} \not\cong \mathfrak{C}$, but $\mathfrak{A}\times\mathfrak{B} \cong \mathfrak{A}\times\mathfrak{C}$. (Note that $\mathfrak{C} \in \boldsymbol{R}_\omega^0$ follows from the remaining assumptions.)*

(b) *Whenever $\mathfrak{A}_1 \in \boldsymbol{R}_f$, $\mathfrak{B}_1, \mathfrak{C}_1 \in \boldsymbol{R}_\omega^0$ and $t(\mathfrak{A}_1, \mathfrak{B}_1) < t(\mathfrak{A}, \mathfrak{B})$ then $\mathfrak{A}_1\times\mathfrak{B}_1 \cong \mathfrak{A}_1\times\mathfrak{C}_1$ implies $\mathfrak{B}_1 \cong \mathfrak{C}_1$.*

Now we proceed to work under the assumption (1) and to accumulate additional properties of $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ leading to a final contradiction. The following simple argument will be used several times. Let (Q) be an (isomorphism invariant) property of connected structures (of the class $\boldsymbol{R}$) such that whenever $\mathfrak{F}$ is finite and $\mathfrak{F}, \mathfrak{Z} \in \boldsymbol{R}_\omega^c$ then $\mathfrak{Z}$ has (Q) iff $\mathfrak{F}\times\mathfrak{Z}$ has (Q). We claim: *either every connected component of $\mathfrak{B}$ and of $\mathfrak{C}$ has the property* (Q), *or none of them have* (Q).

To see this, notice that the connected components of $\mathfrak{A}\times\mathfrak{B}$ are the products $\mathfrak{A}'\times\mathfrak{B}'$ where $\mathfrak{A}'$ and $\mathfrak{B}'$ are connected components of $\mathfrak{A}$ and $\mathfrak{B}$, respectively. Thus we easily see that if the italicized statement is not true then (because of the assumption about (Q), and our assumption that $\mathfrak{A}\times\mathfrak{B} \cong \mathfrak{A}\times\mathfrak{C}$) both $\mathfrak{B}$ and $\mathfrak{C}$ have components satisfying (Q) and components satisfying $\neg$(Q). In fact, we could then write $\mathfrak{B} = \mathfrak{B}_0+\mathfrak{B}_1$ and $\mathfrak{C} = \mathfrak{C}_0+\mathfrak{C}_1$ where all components of $\mathfrak{B}_0, \mathfrak{C}_0$ satisfy (Q) and all components of $\mathfrak{B}_1, \mathfrak{C}_1$ satisfy $\neg$(Q). Breaking up the isomorphic structures $\mathfrak{A}\times\mathfrak{B}$ and $\mathfrak{A}\times\mathfrak{C}$ in a similar way would then lead to the formulas $\mathfrak{A}\times\mathfrak{B}_i \cong \mathfrak{A}\times\mathfrak{C}_i$ $(i = 0, 1)$, since, for example, the assumption about (Q) implies that $\mathfrak{A}\times\mathfrak{B}_0$ is the sum of all the connected components of $\mathfrak{A}\times\mathfrak{B}$ which have the property (Q). These formulas imply that $\mathfrak{B}_i \cong \mathfrak{C}_i$ $(i = 0, 1)$, by (1b) (since $\mathfrak{B}_0$ and $\mathfrak{B}_1$ have fewer connected components than does $\mathfrak{B}$). However, this gives $\mathfrak{B} \cong \mathfrak{B}_0+\mathfrak{B}_1 \cong \mathfrak{C}_0+\mathfrak{C}_1 \cong \mathfrak{C}$, contradicting (1a).

By Lemma 8.11 we can take for (Q), in the above argument, the property of absorbing a given finite connected structure. We thus obtain

(2) *Let $\mathfrak{F} \in \boldsymbol{R}_f^c$. If one of the connected components of $\mathfrak{B}$ or $\mathfrak{C}$ absorbs $\mathfrak{F}$ then all of the components of $\mathfrak{B}$ and $\mathfrak{C}$ absorb $\mathfrak{F}$.*

With a little more work we infer

(3) *Either* (i) *every connected component of $\mathfrak{B}$ and $\mathfrak{C}$ has a finite $\approx$-coset; or* (ii) *$\mathfrak{A}$ is thin.*

For the property (Q) of having a finite $\approx$-coset is clearly of the type for which the above considerations apply. And in case no connected components of $\mathfrak{B}$ or $\mathfrak{C}$ have finite $\approx$-cosets (i.e. $\mathfrak{B}$ and $\mathfrak{C}$ have no such cosets) then we infer from Lemma 8.3 that $\mathcal{SK}(\mathfrak{A}) \times \mathfrak{B} \cong \mathcal{SK}(\mathfrak{A}) \times \mathfrak{C}$; we then infer from this formula and assumptions (1a), (1b) that $\mathfrak{A}$ and $\mathcal{SK}(\mathfrak{A})$ are of equal cardinality. This implies of course $\mathfrak{A}$ is thin (since it is finite).

At this point we bring in the machinery supplied by Definition 5.3. We have $\mathfrak{A} \cong P(\mathfrak{X})$, a finite polynomial combination of indecomposable, connected structures. Let $x_{i_1}, \ldots, x_{i_k}$ be the indeterminates which actually occur in the polynomial P ($k = 0$ is possible). Define

$$\boldsymbol{K} = \{\mathfrak{X}_{i_1}, \ldots, \mathfrak{X}_{i_k}\}$$

so that $\boldsymbol{K}$ consists (up to isomorphism) of the indecomposable connected structures which divide the connected components of $\mathfrak{A}$ (and every connected component of $\mathfrak{A}$ is a finite product of structures from $\boldsymbol{K}$). We claim

(4) *No connected component of $\mathfrak{B}$ or $\mathfrak{C}$ absorbs any structure belonging to $\boldsymbol{K}$.*

Suppose, for instance, that one component absorbs $\mathfrak{X}_{i_1}$. Then (by (2) above) all components do, and we have $\mathfrak{B} \cong \mathfrak{X}_{i_1} \times \mathfrak{B}$, $\mathfrak{C} \cong \mathfrak{X}_{i_1} \times \mathfrak{C}$. From this it clearly follows that, if we let P_1 be the polynomial obtained by replacing the exponents $x_{i_1}^l$ occurring in $P = P_{\mathfrak{A}}$ by the unit constant 1 and recollecting terms (in otherwords, evaluating P at $x_{i_1} = 1$), then the formula $\mathfrak{A} \times \mathfrak{B} \cong \mathfrak{A} \times \mathfrak{C}$ becomes $P_1(\mathfrak{X}) \times \mathfrak{B} \cong P_1(\mathfrak{X}) \times \mathfrak{C}$. However, $P_1(\mathfrak{X})$ is obtained from $\mathfrak{A}$ by factoring some powers of $\mathfrak{X}_{i_1}$ out of the connected parts; it definitely has a smaller cardinality than that of $\mathfrak{A}$. Thus we have a contradiction to the assumed minimality of the counterexample $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ (1b). Thus (4) is established.

Let us observe that the direct summands and factors of a thin structure (belonging to $\boldsymbol{R}$) are thin. Consequently, if $\mathfrak{A}$ is thin then the class $\boldsymbol{K}$ defined above consists entirely of thin structures.

Putting together (3), (4) and Lemma 8.12 we now find that every connected component $\mathfrak{Z}$ of $\mathfrak{B}$ or $\mathfrak{C}$ can be uniquely represented as $\mathfrak{Z} \cong \mathfrak{M} \times \mathfrak{D}$, where $\mathfrak{M}$ is a finite product of $\boldsymbol{K}$ structures and $\mathfrak{D}$ is $\boldsymbol{K}$-free. We call $\mathfrak{D}$ the $\boldsymbol{K}$-free part of $\mathfrak{Z}$. Then if $\mathfrak{M}'$ is any one of the connected components of $\mathfrak{A}$, it follows from the same facts that (the connected component of $\mathfrak{A} \times \mathfrak{B}$, or $\mathfrak{A} \times \mathfrak{C}$ respectively) $\mathfrak{M}' \times \mathfrak{Z}$ has a similarly unique representation—and of course $\mathfrak{M}' \times \mathfrak{Z} \cong (\mathfrak{M}' \times \mathfrak{M}) \times \mathfrak{D}$ is such a representation. In particular, $\mathfrak{Z}$ and $\mathfrak{M}' \times \mathfrak{Z}$ have isomorphic $\boldsymbol{K}$-free parts.

(5) *All of the $\boldsymbol{K}$-free parts of* (*connected components of*) *$\mathfrak{B}$ and $\mathfrak{C}$ are isomorphic to one another.*

For the proof of (5), let $\mathfrak{D}_0$ be the $\boldsymbol{K}$-free part of some component of $\mathfrak{B}$; define: $\mathfrak{Z}$ has the property (Q) iff $\mathfrak{Z} \cong \mathfrak{M} \times \mathfrak{D}_0$ where $\mathfrak{M}$ is a finite product of $\boldsymbol{K}$-structures; and argue exactly as in the paragraph preceding statement (2), making use of the last paragraph above. (It is not claimed that this property (Q) satisfies the full condition used before; but rather that it satisfies the condition in the cases required for the argument.)

Finally, let $\mathfrak{D}$ be the common $\boldsymbol{K}$-free part whose existence is ensured by (5). We factor $\mathfrak{D}$ out of all components of $\mathfrak{B}$ and $\mathfrak{C}$ simultaneously and obtain (via the distributive law, used throughout these pages):

$$\mathfrak{B} \cong \mathfrak{M}_0 \times \mathfrak{D} \quad \text{and} \quad \mathfrak{C} \cong \mathfrak{M}_1 \times \mathfrak{D},$$

where $\mathfrak{M}_0$ and $\mathfrak{M}_1$ are finite structures whose connected components are products of structures belonging to $\boldsymbol{K}$, and where $\mathfrak{D} \in \boldsymbol{R}_\omega^c$ and $\mathfrak{D}$ is $\boldsymbol{K}$-free. (Here is really the only crucial use of the assumption that $\mathfrak{B}$ and $\mathfrak{C}$ have finitely many connected components: to ensure that $\mathfrak{M}_0$ and $\mathfrak{M}_1$ are finite.) To these formulas we can add

$$(\mathfrak{A} \times \mathfrak{M}_0) \times \mathfrak{D} \cong (\mathfrak{A} \times \mathfrak{M}_1) \times \mathfrak{D},$$

by (1a). Moreover we claim that $\mathfrak{D}$ can be cancelled from this last formula. In fact, it is a consequence of (3) that either $\mathfrak{D}$ has a finite $\approx$-coset or every structure of $\boldsymbol{K}$ is thin. Thence it follows from Lemma 8.12 that $\mathfrak{D}$ is cancellable over the class of all finite products of structures of $\boldsymbol{K}$. Finally, our claim is a consequence of Lemma 8.1.

To conclude, we cancel $\mathfrak{D}$ in the last formula to obtain $\mathfrak{A} \times \mathfrak{M}_0 \cong \mathfrak{A} \times \mathfrak{M}_1$. Then by Corollary 5.2 we have $\mathfrak{M}_0 \cong \mathfrak{M}_1$, yielding $\mathfrak{B} \cong \mathfrak{C}$. This contradicts (1a) and shows, at last, that assumption (1) is untenable. Therefore the proof of Theorem 8.9 is now complete.

9. Concluding observations. The preceding sections give much support to the view that Lemma 3.1 is the fundamental result in the theory of cardinal multiplication of reflexive binary structures. However, this theory still presents some very basic problems which are probably inaccessible to the methods developed here. Among them we mention that, although all of the results of this paper have been known for some years for boolean algebras, it remains unknown whether given any integer $n > 2$, the formula $\mathfrak{A} \cong \mathfrak{A}^n$ implies $\mathfrak{A} \cong \mathfrak{A}^2$, for denumerable boolean algebras $\mathfrak{A}$. This problem is also unsolved for the whole class of denumerable, reflexive, binary structures; and apparently even among arbitrary denumerable structures no counter-example to the implication has been found. It is known, however, that for $n > 2$ there is a non-denumerable boolean algebra $\mathfrak{A}$ which satisfies $\mathfrak{A} \cong \mathfrak{A}^n$ and $\mathfrak{A} \ncong \mathfrak{A}^2$ [4; Theorem 5]. A few of the many open problems related to our work were mentioned in Remarks 6.5 and 6.6.

We now want to show how the ideas presented in this paper can be applied to relational structures of an arbitrary similarity type. There are basically two methods available for obtaining such applications, athough many combinations and variations are possible.

The first method is to attempt the proof of theorems analogous to Lemma 3.1 for other types of structures, or attempt to weaken the hypothesis in Lemma 3.1. By simply modifying the arguments of § 3 in the most obvious way, for example, one arrives at a nice generalization of Lemma 3.1—a result which may be guessed from its special case in the next theorem. Let S be any ternary relation over a non-void set B. Define S_0 to be the binary relation over B such that $x S_0 y$ iff $\exists z_0, z_1[\langle x, z_0, z_1\rangle \in S \;\&\; \langle y, z_0, z_1\rangle \in S]$. Define S_0' to be the binary relation over ${}^2B$ such that $x S_0' y$ (where $x = \langle x_0, x_1\rangle, y = \langle y_0, y_1\rangle$) iff $\exists z[\langle z, x_0, x_1\rangle \in S \;\&\; \langle z, y_0, y_1\rangle \in S]$.

THEOREM 9.1. *Suppose that S is a ternary relation over B, that f and g are decomposition functions of the structure $\langle B, S\rangle$, and that $x, y \in B$. Suppose further that S_0 is connected over B and that S_0' is connected over ${}^2B$. Then the two elements $f_x g_x y = a$ and $g_x f_x y = b$ are equivalent in the following sense: whenever $z_0, z_1 \in B$ we have $\langle a, z_0, z_1\rangle \in S$ iff $\langle b, z_0, z_1\rangle \in S$.*

The second method applies to structures with a definable binary relation. We begin with the following definition, a slight change from [3; Definition 7.1]: Given $\mathfrak{A} = \langle A, R_t\rangle_{t\in T}$, we let $\Delta(\mathfrak{A})$ be the set of all finitary relations, R, over A such that every decomposition function of $\mathfrak{A}$ is a decomposition function for the structure $\langle A, R\rangle$. Thus $\Delta(\mathfrak{A})$ includes all of the relations R_t, and moreover is closed under many of the natural set-theoretic operations on relations, such as intersection, directed union, concatenation, permutation and quantification, for instance. (Hence one can often deduce immediately that a relation R which has been defined in terms of the fundamental relations R_t belongs to $\Delta(\mathfrak{A})$, simply by inspecting the form of the definition.) With the above definition the following theorems, as in [3], are easy corollaries of the corresponding earlier results. They are listed in roughly increasing order of applicability.

THEOREM 9.2. (By Theorem 4.1) *Suppose that there exists in $\Delta(\mathfrak{A})$ a binary relation R such that the structure $\langle A, R\rangle$ is thin and belongs to $\boldsymbol{Q}$. Then $\mathfrak{A}$ has the strict refinement property.*

THEOREM 9.3. (By Theorem 4.2) *Suppose that $u \in A$, and that there exists in $\Delta(\mathfrak{A})$ a binary relation R such that the structure $\mathfrak{A}' = \langle A, R\rangle \in \boldsymbol{Q}$ and $u/\approx^{\mathfrak{A}'} = \{u\}$. Then the pair $(\mathfrak{A}, u)$ has the intermediate refinement property.*

THEOREM 9.4. (By Definition 4.8 and Theorems 4.13, 4.14) *Assume that $\boldsymbol{K}$ is a class of similar structures, that $\boldsymbol{K}_0$ is a characteristic subclass of $\boldsymbol{K}$ and that every member of $\boldsymbol{K}_0$ has the refinement property. Suppose then that $\mathfrak{A} \in \boldsymbol{K}$, and that there exist $u \in A$ and $R \in \Delta(\mathfrak{A})$ such that the structure $\mathfrak{A}' = \langle A, R\rangle$ belongs to $\boldsymbol{Q}$ and we have $\mathfrak{A}(u/\approx^{\mathfrak{A}'}) \in \boldsymbol{K}_0$. Under these assumptions it follows that $\mathfrak{A}$ has the refinement property.*

Finally, we should like to give three illustrative applications of the above theorems. The first result is new. Consider algebraic systems $\mathfrak{A} = \langle A, +\rangle$ with $+$ a binary operation defined on A. In the framework of this paper we naturally interpret this to mean that $+$ is a special kinb of ternary relation; but the usual notation for operations can be employed. *Assuming that $\mathfrak{A}$ has exactly one element u such that $u+x = x+u$ for all $x \in A$, then it follows that $(\mathfrak{A}, u)$ has the intermediate refinement property.* (Thus if this element u also satisfies $u+u = u$ then $\mathfrak{A}$ has the refinement property; cf. § 1, *Strong refinement properties*.) This assertion is immediate by Theorem 9.3 when we define xRy to mean $x+y = y+x$.

The second application is a new proof of an important classical result concerning binary algebras which first appeared in [7]. (We refer to a special case of Theorem 4.8 of that monograph. It is not clear to us whether the full theorem can be given a new proof by our methods.) Let $\boldsymbol{K}$ be the class of all finite algebraic systems $\mathfrak{A} = \langle A, +\rangle$, $+$ binary. Take for $\boldsymbol{K}_0$ the subclass where $\mathfrak{A}$ is a group in the usual sense. From Example 4.11, $\boldsymbol{K}_0$ is a characteristic subclass of $\boldsymbol{K}$. The result of Jónsson and Tarski asserts that if $\mathfrak{A} \in \boldsymbol{K}$ has a zero element u—i.e. $u+x = x+u = x$ for all $x \in A$—then $\mathfrak{A}$ has the refinement property. To infer this with the help of Theorem 9.4 let u be the zero element of $\mathfrak{A}$ and define R by the condition xRy iff

$$\forall z[x+(y+z) = y+(x+z) \;\&\; (z+x)+y = (z+y)+x] \;\&\; \\ \&\; \exists z[x = z+y] \,.$$

Now it follows from the form of this definition that $R \in \Delta(\mathfrak{A})$. From the obvious fact that xRu and xRx for all $x \in A$ it follows that $\mathfrak{A}' = \langle A, R\rangle$ belongs to the class $\boldsymbol{Q}$. We also easily see that the substructure $\mathfrak{A}(u/\approx^{\mathfrak{A}'})$ is identical with what was called in the Jónsson–Tarski monograph the "center" of $\mathfrak{A}$; it is an abelian group. The desired result thus reduces to its special case, which fulfills the remaining requirement of Theorem 9.4; viz. that all finite groups have the refinement property.

(Let us note that the above proof, when fully written out, would contain also the basic facts about direct decompositions of finite groups, and the elementary arguments from [9; § 2] which use these facts, as well as the considerations in this paper leading up to Theorem 9.4.)

Our final application exhibits a combination of the two basic approaches. Tarski has raised the question of whether the refinement property, or the strict refinement property, is possessed by every binary algebra $\mathfrak{A} = \langle A, +\rangle$ which satisfies $x+x = x$ for all x; or more broadly, what additional assumptions on such an algebra will ensure the validity of one of the refinement properties? Our result, which supplements some partial results in [3], is the following: *Suppose that* $x+x = x$ *for all* $x \in A$. *Suppose also that there exists in* A *an element* u *such that* $u+x = x+u$ *for all* $x \in A$, *and an element* v *such that* $x+(v+y) = (x+v)+y$ *for all* $x, y \in A$. *Then* $\mathfrak{A}$ *has the refinement property*; *in fact*, $(\mathfrak{A}, v)$ *possesses the intermediate refinement property*. (Corollary: Every idempotent semigroup having a commuting element possesses the strict refinement property.)

To prove that $(\mathfrak{A}, v)$ has the IRP under the given assumptions, suppose that $f, g \in \mathrm{DF}(\mathfrak{A})$. We define a binary relation and a ternary relation:

$$\langle x, y\rangle \in R \quad \text{iff} \quad \exists z(x = z+y = y+z);$$

$$\langle x, y, z\rangle \in S \quad \text{iff} \quad y+(x+z) = (y+x)+z .$$

Clearly we have $R, S \in \Delta(\mathfrak{A})$, so it follows that $f, g \in \mathrm{DF}(\langle A, R, S\rangle)$. R is obviously reflexive, and connected over A because, for instance, $\langle u+x, x\rangle$, $\langle u+x, u\rangle \in R$ for every element $x \in A$. Thus by Lemma 3.1 every pair of elements, $f_x g_x y = a$ and $g_x f_x y = b$, are quasi-identical in the structure $\langle A, R\rangle$; in particular we have aRb and bRa. Likewise, we have $\langle v, x, y\rangle$, $\langle x, x, x\rangle \in S$ whenever $x, y \in A$, and it readily follows that S_0 and S_0' are connected as required by Theorem 9.1. Thus every pair of elements a and b as above are also equivalent in the sense of Theorem 9.1. We can use these two facts to derive the desired conclusion, i.e. that $f_v g_v y = v$ implies $g_v f_v y = v$. In fact, assume that $f_v g_v y = v$ and let $b = g_v f_v y$. The equivalence of v and b in the sense of Theorem 9.1 means that $x+(b+z) = (x+b)+z$ for all $x, z \in A$. Since vRb and bRv we can pick z_0, z_1 so that $v = z_0+b$ and $b = v+z_1$. Then

$$v = z_0+b = z_0+(b+b) = (z_0+b)+b ,$$

i.e. $v = v+b$. Similarly, we get $b = v+b (= v)$, completing the proof.

Added in proof. A paper entitled *Über das starke Produkt von endlichen Graphen*, by W. Dörfler and W. Imrich, is to appear soon in the journal Sitzungsberichte der Österreichischen Akademie der Wissenschaften Mathem.=naturw. Klasse, II. In it, the authors give a totally different proof of the fact that every member of the class $\boldsymbol{R}_f^c$ has the refinement property.

Recently the author discovered a proof that every idempotent semigroup which has a finite equivalence class under the relation

$$xRy \Leftrightarrow \forall u, v(u+x+v = u+y+v)$$

possesses the refinement property. The essential point of the proof is a demonstration that the finite "rectangular" semigroups from a characteristic subclass of the mentioned class of semigroups. This proof will appear in a future publication.

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