On ω_1-categorical theories of fields

by

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0. Introduction. In this paper we prove that the only ω_1-categorical first-order theories of infinite fields are the theories of algebraically closed fields.

We adopt the usual formalization of the elementary theory of fields, in terms of an applied first-order predicate logic with equality, individual constants 0 and 1, and binary operation-symbols $+$ and $\cdot$. All these symbols get the usual interpretation. By considering any of the usual axioms for fields, we see easily that the class of fields is an EC class. We will be dealing more generally with EC_Δ classes of fields.

For n a prime, or 0, let ACF_n be the class of algebraically closed fields of characteristic n. It is well-known that ACF_n is an EC_Δ class. A basic property of ACF_n, due to Steinitz [19], is: If $\varkappa$ is an uncountable cardinal, and L_1 and L_2 are members of ACF_n of cardinality $\varkappa$, then $L_1 \cong L_2$.

From this one concludes by Vaught's Test that any two members of ACF_n are elementarily equivalent, i.e. satisfy exactly the same sentences of the language of field-theory. Let $Th(\mathrm{ACF}_n)$ be the set of all sentences of field-theory that hold in an arbitrary member of ACF_n. Then Steinitz's result may be formulated in model-theoretic terms as: $Th(\mathrm{ACF}_n)$ is $\varkappa$-categorical, for each uncountable $\varkappa$.

It is known, from Morley's proof of the Łoś Conjecture [14] that, for countable first-order logics, the ω_1-categoricity of a theory T implies the $\varkappa$-categoricity of T for each uncountable $\varkappa$.

With this background we investigated the possibility of extending the above result of Steinitz to other classes of fields. We have proved the following, which is the principal result of our paper:

THEOREM. *If $\mathcal{K}$ is an EC_Δ class of fields such that $Th(\mathcal{K})$ is ω_1-categorical, and $\mathcal{K}$ has no finite members, then $\mathcal{K}$ is one of the classes ACF_n.*

The above theorem is a corollary of a theorem about totally transcendental theories of fields. This theorem says that the only totally transcendental complete theories of infinite fields are the theories $Th(\mathrm{ACF}_n)$.

Though this is a stronger model-theoretic result than the theorem stated above, it has apparently no more algebraic interest than its corollary.

Let us consider briefly the EC_Δ classes of fields which are best understood at the present time.

A. *Real-closed fields.* One normally discusses the class of real-closed fields as a class of ordered fields, and in this formulation the class of real-closed fields is an EC_Δ class. But in a real-closed field the order $>$ is definable in terms of the field structure thus:

$$x > y \Leftrightarrow (\exists z)(x = y + z^2 \wedge z \neq 0) .$$

Thus the class of real-closed fields is an EC_Δ class.

B. *Algebraically complete fields with valuation.* Ax and Kochen ([4], [5], [6]) and Ersov [11] have given complete sets of axioms for various classes of fields with valuation. Ax [2] and Ersov [9] have shown that in certain of these classes we can define the valuation-rings in terms of the field structure. It follows that we can interpret in terms of the field structure all statements about the residue-class field and all statements about the ordered value-group. In this way many classes of Henselian valued fields can be construed as EC_Δ classes of fields.

Pseudo-finite fields. Ax [3] has classified all complete theories of pseudo-finite fields. More generally, he has discussed S-pseudo-finite fields, where S is a set of primes. There is an overlap with Ersov's paper [10], where absolutely algebraic fields of prime characteristic are discussed, for the latter fields are S-pseudo-finite for suitable S.

Separably closed fields. Ersov [10] has classified all complete theories of separably closed fields. This of course includes the case of algebraically closed fields.

If $\mathcal{K}$ is an EC_Δ class of fields of any of the above types, and $\mathcal{K}$ is not one of the classes ACF_n, then very little is known about the isomorphism types of members of $\mathcal{K}$.

For real-closed fields there is the result of Erdös, Gillman and Hendriksen [8] that if $\alpha > 0$ then any two η_α real-closed fields of cardinality $\aleph_\alpha$ are isomorphic, but this is vacuous unless $\aleph_\alpha = \sum_{\beta<\alpha} 2^{\aleph_\beta}$. In model-theoretic terms this result identifies the saturated uncountable real-closed fields, assuming the generalized continuum hypothesis. See, for example, [15]. Without using the generalized continuum hypothesis, one can prove the existence of special [15] real-closed fields of certain cardinalities, and using [15] deduce some isomorphism theorems.

The situation is completely analogous for algebraically complete fields with valuation, and rather similar for pseudo-finite fields. The papers [4], [5], [6], [11] identify many saturated fields with valuation. Ax's paper [3] identifies the saturated uncountable pseudo-finite fields.

Using a condition of Ehrenfeucht [7], and results of Morley [14], one can show that for any uncountable cardinal $\varkappa$ there exist non-isomorphic real-closed fields of cardinality $\varkappa$. This is because we can define a linear order in each real-closed field, and the Ehrenfeucht condition prohibits this for ω_1-categorical theories. The situation is similar for many classes of algebraically complete fields with valuation, since we can often interpret the theory of the ordered value-group in terms of the basic field-structure.

For the complete theories of pseudo-finite fields, and the complete theories of separably closed, but not algebraically closed, fields, we will point out, in the course of the proof of the main theorem, why these theories are not ω_1-categorical.

A consequence of the main theorem is that the ACF_n are the only EC_Δ classes of infinite fields allowing elimination of quantifiers. Of course, the theory of real-closed fields allows elimination of quantifiers, but only when $>$ is taken as a primitive notion. See [18]. The situation is analogous for certain valued fields. See [6].

In this paper we presuppose acquaintance with our paper [13] on the corresponding problems for abelian groups. We refer to that paper for explanation of notation and basic ideas.

The main idea taken from [13] is that of definable filtration. This idea is applied twice, firstly to the multiplicative group of a field, and secondly to the additive group of a field of finite characteristic.

A model-theoretic result which we did not use in [13], but which is very useful now, is Ehrenfeucht's condition [7].

As well as facts on abelian groups already utilized in [13], we use some results on field-extensions, to be found in [1], [12].

1. Outline of proof. We indicate the main steps of the proof.

(a) We prove that if K is a field with $Th(K)$ totally transcendental, and K_1 is a finite algebraic extension of K, then $Th(K_1)$ is totally transcendental.

(b) We prove that if $Th(K)$ is totally transcendental then K^* (the multiplicative group of non-zero elements of K) is the direct sum of a divisible group and a finite group. The proof uses a filtration on K^*.

(c) Using Ehrenfeucht's condition we refine (b) to prove that if K is infinite and $Th(K)$ is totally transcendental then K^* is divisible.

(d) From (a) and (c) we conclude that if K is infinite and $Th(K)$ is totally transcendental then $(K_1)^*$ is divisible for each finite algebraic extension K_1 of K. We then prove, by Galois theory, that if $(K_1)^*$ is divisible for every finite extension K_1 of a field K of characteristic 0, then K is algebraically closed. We conclude that if K is an infinite field of characteristic 0, and $Th(K)$ is totally transcendental, then K is algebraically closed.

(e) The characteristic p case is more involved. In this case we define a filtration of the additive group of K, using the endomorphism $x \mapsto x^p - x$. Using results from [13], we then show that if $Th(K)$ is totally transcendental then either

(i) for each finite extension K_1 of K, K_1 has no cyclic extension of dimension p, or

(ii) for each finite extension K_1 of K, K_1 has exactly one cyclic extension of dimension p.

In case (i), using (a) and (c) and some Galois theory, we show that K is algebraically closed.

The case (ii) includes the $\{p\}$-pseudo-finite fields of Ax. A special argument is used to prove that if K is infinite and satisfies (ii) then $Th(K)$ is not totally transcendental.

2. Model-theoretic preliminaries. For the various facts assumed in this paper, one should consult Section 1 of [13], where there are references to the literature.

We will be working with first-order predicate logics $\mathcal{L}$, with connectives $\neg$ and $\wedge$, quantifiers $\exists$ and $\forall$, identity symbol $=$, finitary relation-symbols and operation-symbols, and variables $v_0, v_1, \ldots, v_n, \ldots$

The basic semantic notions are assumed known. It $\mathcal{M}$ is an $\mathcal{L}$-structure, $|\mathcal{M}|$ will be the underlying set of $\mathcal{M}$.

If α is an ordinal, we form a logic $\mathcal{L}(\alpha)$ by adding to $\mathcal{L}$ distinct new individual constants c_η for $\eta < \alpha$. If $\mathcal{M}$ is an $\mathcal{L}$-structure and $s \in |\mathcal{M}|^\alpha$, then $(\mathcal{M}, s)$ is the obvious $\mathcal{L}(\alpha)$-structure where $s(\eta)$ corresponds to c_η for each $\eta < \alpha$.

If Σ is an $\mathcal{L}$-theory and $\varkappa$ is a cardinal, Σ is $\varkappa$-categorical if any two models of Σ of cardinality $\varkappa$ are isomorphic.

Suppose $\mathcal{L}$ is countable, and Σ is an $\mathcal{L}$-theory. Then Σ is totally transcendental if, for every model $\mathcal{M}$ of Σ and every $s \in |\mathcal{M}|^\omega$, $Th\big((\mathcal{M}, s)\big)$ has at most ω complete extensions in $\mathcal{L}(\omega+1)$.

Morley [14] proved that if Σ is an ω_1-categorical theory in a countable logic $\mathcal{L}$, then Σ is totally transcendental.

For proving that a theory is not totally transcendental, Ehrenfeucht's Condition is very useful. Suppose $\mathcal{M}$ is an $\mathcal{L}$-structure, $X \subseteq |\mathcal{M}|$, and $\varphi(v_0, \ldots, v_n)$ an $\mathcal{L}$-formula whose free variables occur in the list $v_0, \ldots, v_n$. φ is said to be *connected over* X (relative to $\mathcal{M}$) if, for all distinct elements $x_0, \ldots, x_n$ of X, there is a permutation π of $\{0, 1, \ldots, n\}$ such that $\langle x_{\pi(0)}, \ldots, x_{\pi(n)}\rangle$ satisfies $\varphi(v_0, \ldots, v_n)$ in $\mathcal{M}$. Then Theorem (Ehrenfeucht's Condition): Suppose $\mathcal{L}$ is countable, $\mathcal{M}$ is an $\mathcal{L}$-structure, X is an infinite subset of $|\mathcal{M}|$, and $\varphi(v_0, \ldots, v_n)$ is an $\mathcal{L}$-formula. Suppose both φ and $\neg\varphi$ are connected over X. Then $Th(\mathcal{M})$ is not totally transcendental.

For a proof, see [14].

3. Let $\mathcal{L}$ be the logic for field-theory, as described in the introduction. We construe fields as $\mathcal{L}$-structures. If K is a field, $Th(K)$ is the set of all $\mathcal{L}$-sentences that hold in K.

We come now to the first, and most tedious, step of the proof. We have to prove that if K and K_1 are fields, with $Th(K)$ totally transcendental, and K_1 a finite algebraic extension of K, then $Th(K_1)$ is totally transcendental.

The basic idea is simple. Let m be the dimension of K_1 over K. Let $x_0, \ldots, x_{m-1}$ be a basis for K_1 over K, where, without loss of generality, $x_0 = 1$. Each element of K_1 is uniquely of the form $\lambda_0 x_0 + \ldots + \lambda_{m-1} x_{m-1}$ where $\lambda_0, \ldots, \lambda_{m-1} \in K$. We define a map $\pi\colon K_1 \to K^m$ by $\pi(\lambda_0 x_0 + \ldots + \lambda_{m-1} x_{m-1}) = \langle \lambda_0, \ldots, \lambda_{m-1}\rangle$. π is 1-1 and onto. Under π, addition and multiplication on K_1 induce operations $\oplus$ and $\odot$ on K^m, as follows:

$$u \oplus v = \pi\big(\pi^{-1}(u) + \pi^{-1}(v)\big),$$

and

$$u \odot v = \pi\big(\pi^{-1}(u) \cdot \pi^{-1}(v)\big),$$

for all $u, v \in K^m$.

Clearly $\langle \lambda_0, \ldots, \lambda_{m-1}\rangle \oplus \langle \mu_0, \ldots, \mu_{m-1}\rangle = \langle \lambda_0 + \mu_0, \ldots, \lambda_{m-1} + \mu_{m-1}\rangle$. To give a corresponding definition for $\odot$, we first define elements τ_{ijk} $(0 \leqslant i, j, k \leqslant m-1)$ of K by:

$$x_i \cdot x_j = \sum_{k=0}^{m-1} \tau_{ijk} x_k .$$

Then $\langle \lambda_0, \ldots, \lambda_{m-1}\rangle \odot \langle \mu_0, \ldots, \mu_{m-1}\rangle = \langle \delta_0, \ldots, \delta_{m-1}\rangle$, where

$$\delta_k = \sum_{0 \leqslant i,j \leqslant m-1} \lambda_i \mu_j \tau_{ijk}$$

for $0 \leqslant k \leqslant m-1$.

In this way we can interpret every sentence about elements of K_1 as a sentence about m-tuples of elements of K.

We now give this a more precise metamathematical formulation. We preserve the notation of the preceding paragraph. Let t be some fixed map of the set of all ordered triples $\langle i, j, k\rangle$, where $0 \leqslant i, j, k \leqslant m-1$, 1-1 onto the set $\{0, 1, \ldots, m^3-1\}$. We are going to define, by induction, maps $\pi_0, \ldots, \pi_{m-1}$ from the set of terms of $\mathcal{L}(\omega+1)$ to the set of terms of $\mathcal{L}(\omega+m+1)$.

First we extend the Σ-notation to terms of $\mathcal{L}(\alpha)$ for any α. Suppose $\gamma_0, \ldots, \gamma_{n+1}$ are terms of $\mathcal{L}(\alpha)$. We define

$$\sum_{i\leqslant 0}\gamma_i = \gamma_0 \quad\text{and}\quad \sum_{i\leqslant n+1}\gamma_i = \Big(\sum_{i\leqslant n}\gamma_i\Big) + \gamma_{n+1}.$$

We need also a Σ-notation over pairs of subscripts. Thus, let $\gamma_{i,j}$ $(0 \leqslant i, j \leqslant n)$ be terms of $\mathcal{L}(\alpha)$. We define $\sum_{0\leqslant i,j\leqslant n}\gamma_{i,j}$ as $\sum_{0\leqslant i\leqslant n}\sum_{0\leqslant j\leqslant n}\gamma_{i,j}$.

Now we define $\pi_0, \ldots, \pi_{m-1}$ as follows:

(i) $\pi_k(0) = 0$ for $0 \leqslant k \leqslant m-1$;

(ii) $\pi_0(1) = 1$, $\pi_k(1) = 0$ for $1 \leqslant k \leqslant m-1$;

(iii) $\pi_k(v_n) = v_{nm+k}$ for $0 \leqslant k \leqslant m-1$ and $n < \omega$;

(iv) $\pi_k(c_n) = c_{m^3+nm+k}$ for $0 \leqslant k \leqslant m-1$ and $n < \omega$;

(v) $\pi_k(c_\omega) = c_{\omega+k}$ for $0 \leqslant k \leqslant m-1$;

(vi) $\pi_k(\gamma_1+\gamma_2) = \pi_k(\gamma_1)+\pi_k(\gamma_2)$ for $0 \leqslant k \leqslant m-1$;

(vii) $\pi_k(\gamma_1\cdot\gamma_2) = \sum_{\substack{0\leqslant i\leqslant m-1\\ 0\leqslant j\leqslant m-1}} \big(\pi_i(\gamma_1)\cdot\pi_j(\gamma_2)\big)\cdot c_{t(\langle i,j,k\rangle)}$.

We are using the constants $c_{t(\langle i,j,k\rangle)}$ to correspond to the field elements τ_{ijk}. The map $\gamma \mapsto \langle\pi_0(\gamma), \ldots, \pi_{m-1}(\gamma)\rangle$ should be thought of as a formal counterpart of the map π described earlier.

Next we define a map $\varphi \mapsto \hat\varphi$ from formulas of $\mathcal{L}(\omega+1)$, to formulas of $\mathcal{L}(\omega+m+1)$, by the following induction:

(i) if φ is $\gamma_1 = \gamma_2$, where γ_1 and γ_2 are terms, then $\hat\varphi$ is $\big(\pi_0(\gamma_1) = \pi_0(\gamma_2)\big)\wedge\ldots\wedge\big(\pi_{m-1}(\gamma_1) = \pi_{m-1}(\gamma_2)\big)$;

(ii) if φ is respectively $\neg\varphi_1, \varphi_1\wedge\varphi_2$ then $\hat\varphi$ is respectively $\neg\hat\varphi_1, \hat\varphi_1\wedge\hat\varphi_2$;

(iii) if φ is respectively $(\exists v_n)\psi$, $(\forall v_n)\psi$, then $\hat\varphi$ is respectively $(\exists v_{nm})(\exists v_{nm+1})\ldots(\exists v_{nm+(m-1)})\hat\psi$, $(\forall v_{nm})(\forall v_{nm+1})\ldots(\forall v_{nm+(m-1)})\hat\psi$.

If T is a set of formulas of $\mathcal{L}(\omega+1)$ then we define $\hat T$ as $\{\hat\varphi\colon \varphi \in T\}$.

Next we define a map $s \mapsto \hat s$ from $|K_1|^\omega$ to $|K|^\omega$, by:

(i) if $n = t(\langle i,j,k\rangle)$ then $\hat s(n) = \tau_{ijk}$;

(ii) if $n = m^3+n'm+k$, where $0 \leqslant k \leqslant m-1$, then n' and k are uniquely determined, and if $\pi\big(s(n')\big) = \langle\lambda_0, \ldots, \lambda_{m-1}\rangle$, we define $\hat s(n) = \lambda_k$.

Finally, we define a map $s \mapsto \hat s$ from $|K_1|^{\omega+1}$ to $|K|^{\omega+m+1}$, using conditions (i) and (ii) of the preceding paragraph, and in addition

(iii) if $0 \leqslant k \leqslant m-1$ and $\pi\big(s(\omega)\big) = \langle\lambda_0, \ldots, \lambda_{m-1}\rangle$ then $\hat s(\omega+k) = \lambda_k$.

The following basic lemma can now be proved by a simple induction, and we omit the proof.

LEMMA 1. *Let K, K_1 be as above, and let φ be a sentence of $\mathcal{L}(\omega+1)$. If $s \in |K_1|^\omega \cup |K_1|^{\omega+1}$, and $(K_1, s) \models \varphi$, then $(K, \hat s) \models \hat\varphi$.*

LEMMA 2. *Suppose K, K_1 are fields, with K_1 a finite algebraic extension of K, of dimension m. Suppose that $s_1 \in |K_1^\omega|$ and $Th\big((K_1, s_1)\big)$ has uncountably many complete extensions in $\mathcal{L}(\omega+1)$. Then there exists $s \in |K|^\omega$ such that $Th\big((K, s)\big)$ has uncountably many complete extensions in $\mathcal{L}(\omega+m)$.*

Proof. We adopt the notation of the discussion preceding Lemma 1.

Let $\langle T_\eta\rangle_{\eta<\lambda}$ be a well-ordering of the complete extensions of $Th\big((K_1, s_1)\big)$ in $\mathcal{L}(\omega+1)$, where λ is an uncountable cardinal.

Fix $\eta < \lambda$. Let

$$\{\varphi_0(c_\omega), \ldots, \varphi_l(c_\omega)\}$$

be a finite subset of T_η. Since T_η extends $Th\big((K_1, s_1)\big)$, it follows that

$$(K_1, s_1) \models (\exists v_0)[\varphi_0(v_0)\wedge\ldots\wedge\varphi_l(v_0)],$$

and therefore there exists $\overline{s_1} \in |K_1|^{\omega+1}$, extending s_1, such that

$$(K_1, \overline{s_1}) \models \varphi_0(c_\omega)\wedge\varphi_1(c_\omega)\wedge\ldots\wedge\varphi_l(c_\omega).$$

Therefore, by Lemma 1,

$$(K, \hat{\overline{s_1}}) \models \hat\varphi_0(c_\omega, \ldots, c_{\omega+m-1})\wedge\hat\varphi_1(c_\omega, \ldots, c_{\omega+m-1})\wedge\ldots\wedge\hat\varphi_l(c_\omega, \ldots, c_{\omega+m-1}).$$

Therefore every finite subset of $\hat T_\eta$ is consistent with $Th\big((K, \hat s_1)\big)$, since $\hat{\overline{s_1}}$ extends $\hat s_1$. Thus $\hat T_\eta$ can be embedded in a complete extension of $Th\big((K, \hat s_1)\big)$ in $\mathcal{L}(\omega+m)$.

Suppose $\eta, \delta < \lambda$ and $\eta \neq \delta$. Then there exists φ such that $\varphi \in T_\eta$ and $\neg\varphi \in T_\delta$. Thus $\hat\varphi \in \hat T_\eta$ and $\neg\hat\varphi \in \hat T_\delta$, so $\hat T_\eta$ and $\hat T_\delta$ have no common complete extension.

We conclude that $Th\big((K, \hat s_1)\big)$ has at least λ complete extensions in $\mathcal{L}(\omega+m)$. Put $s = \hat s_1$ and the lemma is proved.

The reason for the next lemma is that, in order to show that $Th(K_1)$ is totally transcendental, we have to look at arbitrary structures (K_1', s_1') where $K_1' \equiv K_1$ and $s_1' \in |K_1'|^\omega$.

LEMMA 3. *Suppose K, K_1 are fields, with K_1 a finite algebraic extension of K, of dimension m. Suppose $K_1' \equiv K_1$ and $s_1' \in |K_1'|^\omega$.*

Then there exist K'', K_1'' and $s_1'' \in |K_1''|^\omega$ such that

(i) $K'' \equiv K$, $K_1'' \equiv K_1$, and K_1'' is a finite algebraic extension of K'', of dimension m;

(ii) $(K_1'', s_1'') \equiv (K_1', s_1')$.

Proof. Select a basis $x_0, \ldots, x_{m-1}$ for K_1 over K, with $x_0 = 1$.

We augment the logic $\mathcal{L}$ by adjoining individual constants $b_0, \ldots, b_{m-1}$, and a 1-ary predicate-symbol L. Let $\mathcal{L}_1$ be the resulting logic. We construe $\mathcal{L}_1$-structures as $\mathcal{L}$-structures with distinguished elements corresponding to $b_0, \ldots, b_{m-1}$, and with a distinguished subset corresponding to L. We are particularly interested in those $\mathcal{L}_1$-structures where the underlying $\mathcal{L}$-structure is a field, the distinguished set forms a subfield, and the distinguished elements are a basis for the field over the subfield. It is obvious that this class of $\mathcal{L}_1$-structures is an EC_Δ. Our canonical example of an $\mathcal{L}_1$-structure has K_1 as its underlying field, $|K|$ corresponding to L, and x_i $(0 \leqslant i \leqslant m-1)$ corresponding to b_i $(0 \leqslant i \leqslant m-1)$. We denote this structure by $(K_1, K, x_0, \dots, x_{m-1})$.

Let Δ be the following set of $\mathcal{L}_1(\omega)$-sentences:

$$Th((K_1', s_1')) \cup Th((K_1, K, x_0, \dots, x_{m-1})) .$$

We claim Δ is satisfiable. By the Compactness Theorem it suffices to prove that every finite subset of Δ is satisfiable. In fact we show that if Δ_0 is a finite subset of $Th((K_1', s_1'))$ then there exists $s_{\Delta_0} \in |K_1|^\omega$ such that $(K_1, s_{\Delta_0}) \models \Delta_0$. From this it follows that every finite subset of Δ is satisfiable, since $(K_1, K, x_0, \dots, x_{m-1}) \models Th((K_1, K, x_0, \dots, x_{m-1}))$.

So, let Δ_0 be a finite subset of $Th((K_1', s_1'))$. Select $r < \omega$ such that if c_k occurs in a member of Δ_0 then $k \leqslant r$. Write Δ_0 as $\{\varphi_0(c_0, \dots, c_r), \dots \dots \varphi_l(c_0, \dots, c_r)\}$. Then $(K_1', s_1') \models \varphi_0(c_0, \dots, c_r) \wedge \dots \wedge \varphi_l(c_0, \dots, c_r)$, so

$$K_1' \models (\exists v_0) \dots (\exists v_r)[\varphi_0(v_0, \dots, v_r) \wedge \dots \wedge \varphi_l(v_0, \dots, v_r)] .$$

But $K_1' \equiv K_1$, so

$$K_1 \models (\exists v_0) \dots (\exists v_r)[\varphi_0(v_0, \dots, v_r) \wedge \dots \wedge \varphi_l(v_0, \dots, v_r)] .$$

Therefore there exists a function s from $\{0, \dots, r\}$ to $|K_1|$ such that $(K_1, s) \models \varphi_0(c_0, \dots, c_r) \wedge \dots \wedge \varphi_l(c_0, \dots, c_r)$, i.e. $(K_1, s) \models \Delta_0$.

Now let s_{Δ_0} be any extension of s to an element of $|K_1|^\omega$, and clearly $(K_1, s_{\Delta_0}) \models \Delta_0$.

We conclude that Δ is satisfiable. Let $\mathcal{M}$ be an $\mathcal{L}_1(\omega)$-structure satisfying Δ. Then $\mathcal{M}$ is of the form $(\mathcal{N}, s_1'')$, where $\mathcal{N}$ is an $\mathcal{L}_1$-structure and $s_1'' \in |\mathcal{N}|^\omega$. Since

$$\mathcal{N} \models Th((K_1, K, x_0, \dots, x_{m-1})),$$

we see that the underlying $\mathcal{L}$-structure of $\mathcal{N}$ is a field $K_1'' \equiv K_1$, and K_1'' has a subfield $K'' \equiv K$, and K_1'' is of dimension m over K''. Since $|\mathcal{N}| = |K_1''|$, $s_1'' \in |K_1''|^\omega$, and since $\mathcal{M} \models Th(K_1', s_1')$ we can conclude that $(K_1'', s_1'') \equiv (K_1', s_1')$. This proves the lemma.

LEMMA 4. *Suppose K, K_1 are fields, with K_1 a finite algebraic extension of K, of dimension m. Suppose $Th(K_1)$ is not totally transcendental. Then there exists $K'' \equiv K$ and $s'' \in |K''|^\omega$ such that $Th((K'', s''))$ has uncountably many complete extensions in $\mathcal{L}(\omega+m)$.*

Proof. Since $Th(K_1)$ is not totally transcendental, there exists $K_1' \equiv K_1$ and $s_1' \in |K_1'|^\omega$ such that $Th((K_1', s_1'))$ has uncountably many complete extensions in $\mathcal{L}(\omega+1)$. By Lemma 3, there exist K'', K_1'' and $s_1'' \in |K_1''|^\omega$ such that $K'' \equiv K$, $K_1'' \equiv K_1$, $(K_1'', s_1'') \equiv (K_1', s_1')$ and K_1'' is a finite algebraic extension of K'', of dimension m. Thus $Th((K_1'', s_1'')) = Th((K_1', s_1'))$, so $Th((K_1'', s_1''))$ has uncountably many complete extensions in $\mathcal{L}(\omega+1)$. Now we apply Lemma 2 to the data K'', K_1'', s_1''. We conclude that there exists $s'' \in |K''|^\omega$ such that $Th((K'', s''))$ has uncountably many complete extensions in $\mathcal{L}(\omega+m)$. This proves the lemma.

The next lemma is probably well-known (cf. [14], proof of 5.7).

LEMMA 5. *Suppose $\mathcal{L}$ is an arbitrary countable logic, Σ an $\mathcal{L}$-theory, and $1 \leqslant m < \omega$. Then Σ is totally transcendental if and only if for every model $\mathcal{M}$ of Σ and every $s \in |\mathcal{M}|^\omega$, $Th((\mathcal{M}, s))$ has at most ω complete extensions in $\mathcal{L}(\omega+m)$.*

Proof. Sufficiency is clear, since distinct complete extensions of $Th((\mathcal{M}, s))$ in $\mathcal{L}(\omega+1)$ extend to distinct complete extensions of $Th((\mathcal{M}, s))$ in $\mathcal{L}(\omega+m)$.

Necessity is proved by induction on m.

The result is trivial for $m = 1$. Suppose we have the result for $m \leqslant k$. Now suppose $\mathcal{M}$ is a model of Σ, $s \in |\mathcal{M}|^\omega$, and $Th((\mathcal{M}, s))$ has uncountably many complete extensions in $\mathcal{L}(\omega+k+1)$. Since Σ is totally transcendental, our induction hypothesis tells us that $Th((\mathcal{M}, s))$ has at most ω complete extensions in $\mathcal{L}(\omega+k)$. It follows that there exists Σ_1, a complete extension of $Th((\mathcal{M}, s))$ in $\mathcal{L}(\omega+k)$, such that Σ_1 has uncountably many complete extensions in $\mathcal{L}(\omega+k+1)$. Let $\mathcal{M}_1$ be a model of Σ_1, and for $\eta < \omega+k$ let $\overline{c_\eta}$ be the element of $|\mathcal{M}_1|$ corresponding to c_η. Let $\mathcal{M}_2$ be the $\mathcal{L}$-structure got from $\mathcal{M}_1$ by forgetting the structure corresponding to the constants c_η. Then $|\mathcal{M}_2| = |\mathcal{M}_1|$. Since Σ_1 extends $Th((\mathcal{M}, s))$ it is clear that $\mathcal{M}_2 \equiv \mathcal{M}$. We define $s_2 \in |\mathcal{M}_2|^\omega$ by:

$$s_2(n) = \overline{c}_{\omega+n} \quad \text{for} \quad 0 \leqslant n < k ,$$

and

$$s_2(k+n) = \overline{c}_n \quad \text{for} \quad n \geqslant 0 .$$

Since Σ_1 is complete, and has uncountably many complete extensions in $\mathcal{L}(\omega+k+1)$, it follows that $Th((\mathcal{M}_2, s_2))$ has uncountably many complete extensions in $\mathcal{L}(\omega+1)$, contradicting the assumption that Σ is totally transcendental. We conclude that $Th((\mathcal{M}, s))$ has at most ω complete extensions in $\mathcal{L}(\omega+m)$.

This completes the inductive step and the proof.

COROLLARY. *Suppose $\mathcal{L}$ is countable, Σ is a totally transcendental $\mathcal{L}$-theory, $n < \omega$, and Σ_1 is an extension of Σ in $\mathcal{L}(n)$. Then Σ_1 is totally transcendental.*

Proof. Assume the hypotheses. Let $\mathcal{L}_1 = \mathcal{L}(n)$. If Σ_1 has uncountably many complete extensions in $\mathcal{L}_1(\omega+1)$, then Σ has uncountably many complete extensions in $\mathcal{L}(\omega+n+1)$, contradicting Lemma 5.

The following lemma, the goal of this section is of basic importance for us because it enables us to use Galois theory on our problem.

LEMMA 6. *Suppose K, K_1 are fields, with K_1 a finite algebraic extension of K. Then if $Th(K)$ is totally transcendental, $Th(K_1)$ is totally transcendental.*

Proof. Let m be the dimension of K_1 over K. Suppose $Th(K_1)$ is not totally transcendental. Then, by Lemma 4, there exists $K'' \equiv K$ and $s'' \in |K''|^{\omega}$ such that $Th((K'', s''))$ has uncountably many complete extensions in $\mathcal{L}(\omega+m)$. Then, by Lemma 5, $Th(K)$ is not totally transcendental. This proves the lemma.

Remark. The following is an example of fields K and K_1, with K_1 a finite algebraic extension of K, $Th(K_1)$ totally transcendental and $Th(K)$ not totally transcendental. Take K as the field of real numbers and K_1 as the field of complex numbers. See [14] for proofs that $Th(K_1)$ is totally transcendental and $Th(K)$ is not totally transcendental. By Theorem 1 of this paper, and the celebrated theorem of Artin–Schreier that the only fields of finite codimension in their algebraic closure are real-closed or algebraically closed, the above is the only possible example (up to elementary equivalence).

4. The second step of the proof uses the notion of definable filtration, as in [13]. Our only modification of the treatment in [13] is that we use multiplicative notation rather than additive notation.

If K is a field, let K^* be the group of non-zero elements of K under multiplication. Then K^* is abelian. If n is a positive integer, let $(K^*)^n$ be the subgroup of K^* consisting of nth powers of elements of K^*. (Since we have no further use for the cartesian product notation no confusion should arise.) Clearly, when n divides m, $(K^*)^m \subseteq (K^*)^n$. Thus $\langle (K^*)^{n!}\rangle_{n<\omega}$ is a definable filtration of K. Let $(K^*)^{\infty} = \bigcap_{n<\omega} (K^*)^n$. Suppose K is ω_1-saturated. Then K^* is ω_1-saturated, and, by 3.3 of [13], $(K^*)^{\infty}$ is divisible, and so $(K^*)^{\infty}$ is a direct summand of K^*. Select a subgroup H of K^* such that $K^* = (K^*)^{\infty} \oplus H$. Then $(K^*)^{\infty} \cap H = \{1\}$. Now suppose $Th(K)$ is totally transcendental. By Lemma 3 of [13], there exists an integer n_0 such that $H \cap (K^*)^{n!} = H \cap (K^*)^{n_0!}$ for $n \geqslant n_0$. It follows that

$$H \cap (K^*)^{n_0!} = H \cap (K^*)^{\infty} = \{1\}.$$

Therefore for every x in H, $x^{n_0!} = 1$, so H is a group of $n_0!$th roots of unity, and so is finite, since K is a field.

We have proved that if K is an ω_1-saturated field and $Th(K)$ is totally transcendental then K^* is of the form $D \oplus H$ where D is divisible and H is finite. Since any field is elementarily equivalent to an ω_1-saturated field, we may apply Lemma 2 of [13] to conclude:

LEMMA 7. *Suppose K is a field and $Th(K)$ is totally transcendental. Then K^* is of the form $D \oplus H$, where D is divisible and H is finite.*

We now use the Ehrenfeucht Condition to get a refinement of Lemma 7.

LEMMA 8. *Suppose K is a field with $Th(K)$ totally transcendental. Suppose $K^* = D \oplus H$ where D is divisible and H is finite. Then $H = \{1\}$ or $D = \{1\}$.*

Proof. Assume the hypotheses of the lemma, and assume $H \neq \{1\}$ and $D \neq \{1\}$. Then D is infinite, so K is infinite.

Since H is a finite subgroup of K^*, H is cyclic [12]. Suppose H has n elements, and is generated by ζ. Then $n > 1$ and $x^n = 1$ for all x in H. Since D is divisible, $D \subseteq (K^*)^n$. Since $K^* = D \oplus H$ and $x^n = 1$ for all x in H, we conclude that $(K^*)^n \subseteq D$, so $(K^*)^n = D$. Thus $K^* = (K^*)^n \oplus H$. Therefore if $y \in K^*$, there exists k with $0 \leqslant k \leqslant n-1$ such that $y\zeta^k \in (K^*)^n$. On the other hand $\zeta \notin (K^*)^n$.

Now we define an n-ary relation R on $|K|$ by: $\langle x_0, \ldots, x_{n-1}\rangle \in R$ if and only if either

$$x_0 + x_1\zeta + \ldots + x_k\zeta^k + \ldots + x_{n-1}\zeta^{n-1} \in (K^*)^n$$

or

$$x_0 + x_1\zeta + \ldots + x_{n-1}\zeta^{n-1} = 0.$$

Suppose $x_0, \ldots, x_{n-1}$ are distinct elements of K. We look at two cases.

Case 1. $\langle x_0, \ldots, x_{n-1}\rangle \notin R$. Then $x_0 + x_1\zeta + \ldots + x_{n-1}\zeta^{n-1} \neq 0$ so

$$x_0 + x_1\zeta + \ldots + x_{n-1}\zeta^{n-1} \in K^*.$$

Therefore there exists k with $0 \leqslant k \leqslant n-1$ such that

$$\zeta^k \cdot (x_0 + x_1\zeta + \ldots + x_{n-1}\zeta^{n-1}) \in (K^*)^n.$$

Therefore

$$x_0\zeta^k + x_1\zeta^{k+1} + \ldots + x_{n-k-1}\zeta^{n-1} + x_{n-k} + x_{n-k+1}\zeta + \ldots + x_{n-1}\zeta^{k-1} \in (K^*)^n.$$

Therefore

$$\langle x_{n-k}, x_{n-k+1}, \ldots, x_{n-1}, x_0, \ldots, x_{n-k-1}\rangle \in R.$$

Therefore there exists a permutation π of $\{0, \ldots, n-1\}$ such that

$$\langle x_{\pi(0)}, \ldots, x_{\pi(n-1)}\rangle \in R.$$

Case 2. $\langle x_0, \ldots, x_{n-1}\rangle \in R$. We have two subcases.

Subcase 1. $x_0 + x_1\zeta + \ldots + x_{n-1}\zeta^{n-1} \in (K^*)^n$. Then

$$x_0 + x_1\zeta + \ldots + x_{n-1}\zeta^{n-1} \neq 0 \quad \text{and} \quad \zeta \cdot (x_0 + x_1\zeta + \ldots + x_{n-1}\zeta^{n-1}) \neq 0.$$

Also, since $\zeta \notin (K^*)^n$,

$$\zeta\cdot(x_0+x_1\zeta+\ldots+x_{n-1}\zeta^{n-1}) \notin (K^*)^n.$$

Now

$$\zeta\cdot(x_0+x_1\zeta+\ldots+x_{n-1}\zeta^{n-1}) = x_{n-1}+x_0\zeta+\ldots+x_{n-2}\zeta^{n-1}.$$

Therefore

$$x_{n-1}+x_0\zeta+\ldots+x_{n-2}\zeta^{n-1}\neq 0 \quad\text{and}\quad x_{n-1}+x_0\zeta+\ldots+x_{n-2}\zeta^{n-1}\notin (K^*)^n.$$

Therefore

$$\langle x_{n-1}, x_0, \ldots, x_{n-2}\rangle \notin R.$$

Therefore there exists a permutation π of $\{0, \ldots, n-1\}$ such that $\langle x_{\pi(0)}, \ldots, x_{\pi(n-1)}\rangle \notin R$.

Subcase 2. $x_0+x_1\zeta+\ldots+x_{n-1}\zeta^{n-1} = 0$. Now $x_0, \ldots, x_{n-1}$ are distinct, and $\zeta\neq 1$, so

$$x_0+x_1\zeta \neq x_1+x_0\zeta .$$

Therefore

$$x_1+x_0\zeta+x_2\zeta^2+\ldots+x_{n-1}\zeta^{n-1}\neq 0 .$$

Now, if $x_1+x_0\zeta+x_2\zeta^2+\ldots+x_{n-1}\zeta^{n-1}\notin (K^*)^n$, then

$$\langle x_1, x_0, x_2, \ldots, x_{n-1}\rangle \notin R .$$

On the other hand, if $x_1+x_0\zeta+x_2\zeta^2+\ldots+x_{n-1}\zeta^{n-1}\in (K^*)^n$, then $\langle x_1, x_0, x_2, \ldots, x_{n-1}\rangle \in R$, and the argument of Subcase 1 proves that $\langle x_{n-1}, x_1, x_0, x_2, \ldots, x_{n-2}\rangle \notin R$. In both cases there exists a permutation π of $\{0, \ldots, n-1\}$ such that

$$\langle x_{\pi(0)}, \ldots, x_{\pi(n-1)}\rangle \notin R .$$

It is clear that R is first-order definable using the constants ζ^k $(0\leqslant k\leqslant n-1)$. We define a function s from $\{0, \ldots, n-1\}$ to $|K|$ by:

$$s(k) = \zeta^k \quad \text{for} \quad 0\leqslant k\leqslant n-1 .$$

Then, by the corollary to Lemma 5, $Th\big((K, s)\big)$ is totally transcendental.

Let $\varphi(v_0, \ldots, v_{n-1})$ be a formula of $\mathcal{L}(n)$ defining the relation R in the structure (K, s). It is clear how to write down such a formula.

By what we have proved above about R it is clear that φ and $\neg\varphi$ are connected over $|K|$. But $|K|$ is infinite, since $D\neq\{1\}$. Now Ehrenfeucht's Condition implies that $Th\big((K, s)\big)$ is not totally transcendental. This gives a contradiction.

It follows that either $D = \{1\}$ or $H = \{1\}$.

Corollary. *Let K be a field and suppose $Th(K)$ is totally transcendental. Then either K is finite or K^* is divisible.*

Proof. By Lemma 7, K^* is of the form $D\oplus H$, where D is divisible and H finite. By Lemma 8, either $D = \{1\}$ or $H = \{1\}$. Thus K^* is either finite or divisible. Clearly if K^* is finite K is finite.

5. We can now proceed rapidly to a solution of our problem in the characteristic 0 case.

We use two well-known facts from Kummer theory. For n an integer, let $\boldsymbol{Z}(n)$ be the cyclic group of order n.

Fact 1. Suppose K is a field, p a prime, and K_1 a splitting field over K of x^p-1. Then the Galois group of K_1 over K is a subgroup of $\boldsymbol{Z}(p-1)$, and so is cyclic.

Fact 2. Suppose K is a field, p a prime not equal to the characteristic of K, and K contains p distinct pth roots of unity. Let K_1 be a Galois extension of K with Galois group $\boldsymbol{Z}(p)$. Then there exist ξ, a such that $K_1 = K(\xi)$, $\xi^p = a$, $a\in K$ and a has no pth root in K.

For a proof of these facts, see [12].

Lemma 9. *Suppose K is a field of characteristic 0 such that, for every finite algebraic extension K_2 of K, $(K_2)^*$ is divisible. Then K is algebraically closed.*

Proof. We prove first that, for each prime p, K has p distinct pth roots of unity. Suppose not, and let r be the least prime such that K does not have r distinct rth roots of unity. Clearly $r>2$. Let K_1 be a splitting field for x^r-1 over K. Then $K_1\neq K$. Let G be the Galois group of K_1 over K. Then by Fact 1 G is cyclic and of order $\leqslant r-1$. Let q be a prime dividing the order of G. Then $q<r$. By the converse of Lagrange's Theorem, G has a subgroup G_1 isomorphic to $\boldsymbol{Z}(q)$. Let K_2 be the fixed field of G_1. Then the Galois group of K_1 over K_2 is $\boldsymbol{Z}(q)$ by the Fundamental Theorem of Galois Theory [12]. Since $q<r$, K has q distinct qth roots of unity, so K_2 has q distinct qth roots of unity. Therefore by Fact 2, there exists ξ, a such that $K_1 = K_2(\xi)$, $\xi^q = a$, $a\in K_2$, and a has no qth root in K_2. Therefore $a\in K_2^*$, and K_2^* is not divisible. But K_2 is a finite algebraic extension of K, so K_2^* is divisible by hypothesis. This gives a contradiction. We conclude that, for each prime p, K has p distinct pth roots of unity.

Now suppose K is not algebraically closed. Then there exists a finite normal extension K_1 of K, with $K_1\neq K$. Let G be the Galois group of K_1 over K. Let p be a prime dividing the order of G. Then by the converse to Lagrange's Theorem G has a subgroup G_1 isomorphic to $\boldsymbol{Z}(p)$. Let K_2 be the fixed field of G_1. Then K_1 is a cyclic extension of K_2, with Galois group $\boldsymbol{Z}(p)$. But K_2 contains p distinct pth roots of unity. Then by Fact 2 there exist ξ, a such that $K_1 = K_2(\xi)$, $\xi^p = a$, $a\in K_2$ and a has no pth root in K_2. Therefore K_2^* is not divisible, although K_2 is a finite algebraic extension of K. This contradicts our assumption. We conclude that K is algebraically closed.

THEOREM 1 (Characteristic 0 case). *Suppose K is a field of characteristic 0. Then $Th(K)$ is totally transcendental if and only if K is algebraically closed.*

Proof. Sufficiency is proved in [14].

Suppose K is a field of characteristic 0 and $Th(K)$ is totally transcendental. Then K is infinite. By Lemma 6, $Th(K_2)$ is totally transcendental, for each finite extension K_2 of K. Applying the Corollary to Lemma 8, we conclude that $(K_2)^*$ is divisible for every finite extension K_2 of K. By Lemma 9, K is algebraically closed. This proves the theorem.

6. Before getting into the characteristic p case, we pause to show that some important theories of fields are not totally transcendental. The proofs use the Corollary to Lemma 8, but not Lemma 6, and so are significantly simpler than the proof of Theorem 1.

Separably closed fields. Let K be a field which is separably closed but not algebraically closed. Then K is of characteristic p for some prime p. Also, K is not perfect, so K is infinite and $K^* \neq (K^*)^p$. Thus K^* is not divisible, so $Th(K)$ is not totally transcendental.

Quasi-finite fields. Let K be a quasi-finite field, i.e. a perfect field with exactly one extension of each degree. (Actually, all we use is that K has an extension of degree 2.) We will assume that the characteristic of K is not 2.

Let K_1 be an extension of K of degree 2. Then obviously (or by Fact 2) there exist ξ, α such that $K_1 = K(\xi)$, $\xi^2 = \alpha$, $\alpha \in K$, and α has no square root in K. Thus $K^* \neq (K^*)^2$, so K^* is not divisible.

Therefore if K is infinite $Th(K)$ is not totally transcendental.

Remark. The case when the characteristic of K is 2 will have to wait till we prove the general case of Theorem 1.

7. For the characteristic p case of Theorem 1, the argument resembles that for the characteristic 0 case, but with some extra details. We use the Kummer theory as before, but also the Artin-Schreier theory. In particular we need:

Fact 3. Suppose K is a field of prime characteristic p, and suppose K_1 is a Galois extension of K with Galois group $\mathbf{Z}(p)$. Then there exist ξ, α such that $K_1 = K(\xi)$, $\xi^p - \xi = \alpha$, $\alpha \in K$, and there is no β in K such that $\beta^p - \beta = \alpha$.

LEMMA 10. *Suppose K is an infinite field of prime characteristic p, such that $Th(K)$ is totally transcendental. Then for each y in K there exists an x in K such that $x^p - x = y$.*

The proof of Lemma 10 takes some time, and we postpone it. Right now we show how to complete the proof of Theorem 1, modulo Lemma 10. We need an analogue of Lemma 9.

LEMMA 11. *Suppose K is a field of prime characteristic p such that for every finite algebraic extension K_2 of K we have:*

(i) *$(K_2)^*$ is divisible;*

(ii) *For each y in K_2 there exists an x in K_2 such $x^p - x = y$.*

Then K is algebraically closed.

Proof. We prove first that for each prime $q \neq p$ K has q distinct qth roots of unity. Suppose not, and let r be the least prime $\neq p$ such that K has fewer than r rth roots of unity. Let K_1 be a splitting field for $x^r - 1$ over K. Then $K_1 \neq K$. Let G be the Galois group of K_1 over K. Then by Fact 1 G is cyclic and of order $\leqslant r-1$. Let q be a prime dividing the order of G. Then $q < r$. By the converse of Lagrange's Theorem, G has a subgroup G_1 isomorphic to $\mathbf{Z}(q)$. Let K_2 be the fixed field of G_1. Then the Galois group of K_1 over K_2 is $\mathbf{Z}(q)$ by the Fundamental Theorem of Galois Theory.

Suppose first $q = p$. Then by Fact 3 there exist ξ, α such that $K_1 = K_2(\xi)$, $\xi^p - \xi = \alpha$, $\alpha \in K_2$, and there is no β in K_2 such that $\beta^p - \beta = \alpha$. But this contradicts assumption (ii). Therefore $q \neq p$.

Thus $q \neq p$ and $q < r$. By the minimality of r, K has q distinct qth roots of unity. Therefore by Fact 2 there exist ξ, α such that $K_1 = K_2(\xi)$, $\xi^q = \alpha$, $\alpha \in K_2$ and α has no qth root in K_2. Thus $(K_2^*)^q \neq (K_2)^*$, so $(K_2)^*$ is not divisible, contradicting assumption (i).

We conclude that for each prime $q \neq p$ K has q distinct qth roots of unity.

Now suppose K is not algebraically closed. Then, by the same argument as in the proof of Lemma 9, there exist a prime q, and finite extensions K_1, K_2 of K, such that K_1 is a Galois extension of K_2 with Galois group $\mathbf{Z}(q)$.

Suppose $q = p$. Then by Fact 3 there exist ξ, α such that $K_1 = K_2(\xi)$, $\xi^p - \xi = \alpha$, $\alpha \in K_2$, and there is no β in K_2 such that $\beta^p - \beta = \alpha$. This contradicts assumption (ii).

Suppose $q \neq p$. Then K has q distinct qth roots of unity. Then by Fact 2 there exist ξ, α such that $K_1 = K_2(\xi)$, $\xi^q = \alpha$, $\alpha \in K_2$ and α has no qth root in K_2. Thus $(K_2)^*$ is not divisible, contradicting assumption (i).

We conclude that K is algebraically closed.

THEOREM 1 (Characteristic p case). *Suppose K is a field of prime characteristic p. Then $Th(K)$ is totally transcendental if and only if K is finite or algebraically closed.*

Proof. Sufficiency for algebraically closed K is proved in [14], and is trivial for finite K.

Necessity. Suppose K is infinite and $Th(K)$ is totally transcendental. Then by Lemma 6 $Th(K_2)$ is totally transcendental for each finite extension K_2 of K. By the Corollary to Lemma 8, $(K_2)^*$ is divisible for each finite extension K_2 of K. Furthermore, by Lemma 10, for each y in K_2 there is an x in K_2 such that $x^p - x = y$. By Lemma 11, K is algebraically closed. This completes the proof.

We put together the parts of Theorem 1 to get:

THEOREM 1. *Suppose K is a field. Then $Th(K)$ is totally transcendental if and only if K is finite or algebraically closed.*

8. We have now to prove Lemma 10. We use the technique of definable filtrations from [13]. In the present application we work with additive notation.

In this section all fields are of prime characteristic p. Let F_p be the finite field of p elements. Any field of characteristic p can be construed as a vector-space over F_p. If V is any vector-space over F_p, we write $\dim V$ for the dimension of V.

If K is a field of characteristic p, let $\mathrm{Abs}(K)$ be the field of absolute numbers of K, i.e. the algebraic closure in K of F_p.

We define $\mathrm{Add}(K)$ as the underlying additive group of K. We will define a filtration of K consisting of subgroups of $\mathrm{Add}(K)$.

We define a map τ: $\mathrm{Add}(K) \to \mathrm{Add}(K)$ by:

$$\tau(x) = x^p - x \quad \text{for } x \text{ in } \mathrm{Add}(K).$$

Fact 4. τ is a homomorphism with kernel F_p.

For a proof, see [12].

We define subgroups H_m, $(m < \omega)$, of $\mathrm{Add}(K)$ by:

$$H_0 = \mathrm{Add}(K)$$
$$H_{m+1} = \tau[H_m] \quad \text{for} \quad m \geqslant 0.$$

Clearly $H_1 \subseteq H_0$, whence by induction $H_{m+1} \subseteq H_m$ for all m, so $\langle H_m \rangle_{m<\omega}$ is a filtration of K. It is clear that each H_m is definable, so $\langle H_m \rangle_{m<\omega}$ is a definable filtration. We note also that if $H_{m+1} = H_m$ for some m then $H_n = H_m$ for all $n \geqslant m$.

There are three possibilities.

(A) $H_0 = H_1 = H_2 = \ldots = H_m = \ldots$

(B) There exists a $k \geqslant 0$ such that $H_0 \neq H_1 \neq \ldots \neq H_{k+1} = H_{k+2} = \ldots$

(C) For all m, n with $m \neq n$, $H_m \neq H_n$.

We now analyse these possibilities, and show that if $Th(K)$ is totally transcendental then (B) and (C) cannot occur.

Case C. Suppose $Th(K)$ is totally transcendental and Case C holds.

Then we have the strictly descending chain

$$H_0 \supset H_1 \supset \ldots \supset H_n \supset H_{n+1} \supset \ldots$$

of subgroups of $\mathrm{Add}(K)$. We construe all subgroups of $\mathrm{Add}(K)$ as vector-spaces over F_p. Define H_∞ as $\bigcap_{n\geqslant 0} H_n$. Then H_∞ is a subspace of $\mathrm{Add}(K)$. Choose a subspace Δ complementary to H_∞ in $\mathrm{Add}(K)$. Then $\mathrm{Add}(K) = H_\infty \oplus \Delta$. Since $Th(K)$ is totally transcendental it follows by the corollary to Lemma 3 of [13] that there exists n such that

$$H_m \cap \Delta = H_n \cap \Delta \quad \text{for all } m \geqslant n.$$

Then $H_n \cap \Delta = H_\infty \cap \Delta = \{0\}$. It follows that $\mathrm{Add}(K) = H_n \oplus \Delta$, since $H_\infty \subseteq H_n$.

We claim that $H_n = H_\infty$. Suppose not. Then there exists an element x which is in H_n but not in H_∞. Since $\mathrm{Add}(K) = H_\infty \oplus \Delta$, there exist y, δ such that $y \in H_\infty$, $\delta \in \Delta$, and $x = y + \delta$. Then $y \in H_n$, so $x - y \in H_n$. But $x - y = \delta \in \Delta$. Since $H_n \cap \Delta = \{0\}$, $x - y = 0$, so $x = y \in H_\infty$, contrary to assumption. Therefore $H_n = H_\infty$.

Therefore $H_m = H_n$ for $m \geqslant n$. But this contradicts our assumption that $H_m \neq H_n$ if $m \neq n$.

Thus, if $Th(K)$ is totally transcendental, Case C cannot occur.

Case B. Suppose $Th(K)$ is totally transcendental and Case B holds. Then there exists k such that

$$H_0 \neq H_1 \neq \ldots \neq H_{k+1} = H_{k+2} = \ldots$$

First some notation. For $n \geqslant 1$ we define maps τ_n: $\mathrm{Add}(K) \to \mathrm{Add}(K)$ by:

$$\tau_1(x) = \tau(x) \qquad \text{for all } x;$$
$$\tau_{m+1}(x) = \tau(\tau_m(x)) \qquad \text{for all } x, \text{ and all } m \geqslant 1.$$

Then clearly τ_n is a homomorphism of $\mathrm{Add}(K)$ to $\mathrm{Add}(K)$, and $\tau_n[H_0] = H_n$.

For each integer $N \geqslant 1$ let F_{p^N} be the finite field of cardinality p^N. We know that $F_p \subseteq K$, but we cannot determine $F_{p^N} \cap K$ immediately, when $N > 1$.

Consider the equation (over F_p): $\tau_{k+1}(x) = 0$.

This equation has degree p^{k+1}. It is a simple consequence of the basic theory of finite fields [12] that the equation has p^{k+1} roots in $F_{p^{p^{k+1}}}$, i.e. that $\tau_{k+1}(x)$ splits into linear factors over $F_{p^{p^{k+1}}}$. Moreover, if F_{p^N} contains a root of the above equation, that root is a member of $F_{p^N} \cap F_{p^{p^{k+1}}}$.

It follows that

$$\{x|\ x \in K \wedge \tau_{k+1}(x) = 0\} \subseteq K \cap F_{p^{p^{k+1}}}.$$

We define the field K_0 by $K_0 = K \cap F_{p^{p^{k+1}}}$. Then K_0 is a finite subfield of K.

K_0 is of course closed under τ. We consider the quotient vector-space (over F_p) $K_0/\tau[K_0]$. We claim that $\dim_p K_0/\tau[K_0]=1$.

Firstly, since K_0 is a finite field it has a unique cyclic extension of degree p.

Next we have the following important fact, established in [1], pp. 203–4.

Fact 5. Suppose F is any field of characteristic p, and $\tau\colon F\to F$ is given by $\tau(x)=x^p-x$. Then the number of cyclic extensions of F of degree p is equal to $\dim_p F/\tau[F]$.

From Fact 5 and the preceding paragraph we conclude that $\dim_p K_0/\tau[K_0]=1$.

Now we claim $\dim_p K/\tau[K]=1$.

Firstly, since $H_0\neq H_1$, $K\neq\tau[K]$ so $\dim_p K/\tau[K]\geqslant 1$.

Next, let λ be an arbitrary member of K. Then $\tau_{k+1}(\lambda)\in H_{k+1}=H_{k+2}=\tau_{k+2}[H_0]$. Therefore there exists y in K such that $\tau_{k+1}(\lambda)=\tau_{k+2}(y)$. Therefore $\tau_{k+1}(\lambda)=\tau_{k+1}(\tau(y))$. Therefore $\tau_{k+1}(\lambda-\tau(y))=0$ Therefore $\lambda-\tau(y)$ is a root of the equation $\tau_{k+1}(x)=0$.

Therefore $\lambda-\tau(y)\in K\cap F_{p^{p^{k+1}}}=K_0$. Since $\dim_p K_0/\tau[K_0]=1$, we can select α such that $\alpha\in K_0$, $\alpha\notin\tau[K_0]$, and for all u in K_0 there exists r with $0\leqslant r\leqslant p-1$ such that $u-r\cdot\alpha\in\tau[K_0]$. Therefore there exists r with $0\leqslant r\leqslant p-1$ such that $\lambda-\tau(y)-r\cdot\alpha\in\tau[K_0]$. Therefore $\lambda-r\cdot\alpha\in\tau[K]$. Since λ was arbitrary it follows that $\dim_p K/\tau[K]\leqslant 1$. Therefore $\dim_p K/\tau[K]=1$. In addition, $\alpha\notin\tau[K]$, for otherwise $K=\tau[K]$.

By Fact 5, K has a unique cyclic extension of degree p. Furthermore we know that this extension is generated by a root of the equation $\tau(x)=\alpha$, where $\alpha\in K_0$ and $\alpha\notin\tau[K_0]$.

From the penultimate paragraph we see that, as vector-spaces over F_p, $K=\tau[K]\oplus\langle\alpha\rangle$, where $\langle\alpha\rangle$ is the 1-dimensional space generated by α. Observing that $\tau[\mathrm{Abs}(K)]\subseteq\mathrm{Abs}(K)$, that $\tau^{-1}[\mathrm{Abs}(K)]\subseteq\mathrm{Abs}(K)$, and that $\alpha\in\mathrm{Abs}(K)$, we conclude that

$$\mathrm{Abs}(K)=\tau[\mathrm{Abs}(K)]\oplus\langle\alpha\rangle\,.$$

So far in this analysis we have not used the assumption that $Th(K)$ is totally transcendental, but now we do. We will assume also that K is infinite.

By Lemma 6 and the Corollary to Lemma 8, $(K_1)^*$ is divisible for every finite algebraic extension K_1 of K. It follows easily that $(\mathrm{Abs}(K_1))^*$ is divisible for every finite algebraic extension K_1 of K. Since the only finite divisible group is the trivial one-element group, it follows that $\mathrm{Abs}(K)$ is infinite unless $(\mathrm{Abs}(K))^*=\{1\}$.

Suppose $(\mathrm{Abs}(K))^*=\{1\}$. Then $\mathrm{Abs}(K)=F_2$ and $p=2$. Let ζ be a root of the equation

$$x^2+x+1=0\,.$$

Then $K(\zeta)$ is an extension of K of degree 2, and $F_4\subseteq K(\zeta)$. Let $K_1=K(\zeta)$. Then $F_4\subseteq\mathrm{Abs}(K_1)$. We claim $\mathrm{Abs}(K_1)=F_4$. Suppose $u\in\mathrm{Abs}(K_1)$, and let f $(\in F_2[x])$ be the minimum polynomial of u over F_2. Since $u\in K(\zeta)$ there exist a, b in K such that $u=a+b\zeta$. If $b=0$, $u\in K$, so $u\in\mathrm{Abs}(K)=F_2$. So suppose $b\neq 0$. Consider the element g of $K[x]$ defined by $g(x)=f(a+bx)$. Since f is not the zero polynomial, and K is infinite, one verifies easily that g is not the zero polynomial over K. By observing that $f(x)=g(b^{-1}x-b^{-1}a)$, one proves easily that g is irreducible over K. But $g(\zeta)=0$, and $\zeta^2+\zeta+1=0$. It follows that for some constant c in K, with $c\neq 0$, $g(x)=c\cdot(x^2+x+1)$, i.e. $f(a+bx)=c\cdot(x^2+x+1)$. It follows that $f(x)=c_0+c_1x+x^2$ for some c_0, c_1 in F_2. Therefore

$$c_0+c_1(a+bx)+(a+bx)^2=c\cdot(x^2+x+1)\,.$$

Therefore

$$c_0+c_1a+c_1bx+a^2+b^2x^2=cx^2+cx+c\,.$$

Therefore

$$c_0+c_1a+a^2=c\,,$$
$$c_1b=c\,,$$
$$b^2=c\,.$$

Therefore $b^2=c_1b$, and since $b\neq 0$,

$$b=c_1\in F_2\,.$$

In fact, c_1 must be 1. Therefore $b=1$ and $c=1$. Therefore $c_0+a+a^2=1$. If $c_0=0$, $1+a+a^2=0$, and $a\in K$, although the equation $x^2+x+1=0$ has no root in K. Therefore $c_0=1$, so $a+a^2=0$, so $a\in F_2$. Since $a,b\in F_2$, $a+b\zeta\in F_2(\zeta)=F_4$, so $u\in F_4$. We have proved that $\mathrm{Abs}(K_1)=F_4$. But $(F_4)^*$ is not divisible. Therefore $(\mathrm{Abs}(K_1))^*$ is not divisible, contrary to what was proved earlier. This rules out the case where $(\mathrm{Abs}(K))^*=\{1\}$. We conclude that $\mathrm{Abs}(K)$ is infinite.

We now construe $\mathrm{Abs}(K)$ as an algebra over K_0. Since $\mathrm{Abs}(K)$ is infinite and K_0 is finite, the dimension of $\mathrm{Abs}(K)$ over K_0 is infinite. We select elements b_n $(n<\omega)$ of $\mathrm{Abs}(K)$ which form a linearly independent set over K_0. Define $s\in|K|^\omega$ by

$$s(0)=\alpha\,,$$
$$s(n+1)=b_n\quad\text{for}\quad n\geqslant 0\,.$$

We are going to show that $Th((K,s))$ has $2^{\aleph_0}$ complete extensions in $\mathcal{L}(\omega+1)$. This will establish that $Th(K)$ is not totally transcendental.

Suppose $K_0=F_{p^e}$. If N is such that $K_0\subseteq K\cap F_{p^N}$, we define T_N as the trace function from $K\cap F_N$ to K_0. Now $K\cap F_{p^N}$ is a cyclic extension of K_0, and it is well-known [1], [12] that the map $x \mapsto x^{p^e}$ generates the Galois group. We now apply the additive analogue of Hilbert's "Satz 90" [12], p. 77, to get immediately the following important fact:

Fact 6. If $y \in K \cap F_{p^N}$ then $T_N(y) = 0$ if and only if $y = u^{p^e} - u$ for some u in $K \cap F_{p^N}$.

Now we define a map δ: $\mathrm{Add}(K) \to \mathrm{Add}(K)$ by:

$$\delta(x) = x^{p^e} - x .$$

Then δ is an endomorphism of $\mathrm{Add}(K)$, and $\delta[\mathrm{Add}(K)]$ is a subgroup of $\mathrm{Add}(K)$.

We claim that $\alpha \notin \delta[\mathrm{Add}(K)]$. Since $\alpha \notin \tau[K]$, $\alpha \neq 0$. Since $\delta[K_0] = \delta[F_{p^e}] = \{0\}$, we see that the equation $x^{p^e} - x = \alpha$ has no root in K_0. If γ is any root of the above equation, and $\lambda \in F_{p^e}$, it is easily seen that $\gamma + \lambda$ is also a root of the equation. It follows easily that the polynomial $x^{p^e} - x - \alpha$ is irreducible over K_0, and $K_0(\gamma)$ is an extension of K_0 of degree p^e. Therefore $K_0(\gamma) = F_{p^{e \cdot p^e}}$. Therefore if $\alpha \in \delta[\mathrm{Add}(K)]$, $F_{p^{e \cdot p^e}} \subseteq K$, so $F_{p^{ep}} \subseteq K$. But $F_{p^{ep}}$ is the unique extension of F_{p^e} of degree p, and since $\alpha \notin \tau[F_{p^e}]$ it follows by Fact 5 that $\alpha \in \tau[F_{p^{ep}}]$. Since $\alpha \notin \tau[K]$, $F_{p^{ep}} \nsubseteq K$. Therefore $\alpha \notin \delta[\mathrm{Add}(K)]$.

Suppose $\sigma \in |F_p|^\omega$. We define a set $\mathrm{Cond}(\sigma)$ of conditions, involving the unknown x, as follows:
$\mathrm{Cond}(\sigma)$ consists of the conditions $b_n \cdot x - \sigma(n) \cdot \alpha \in \delta[\mathrm{Add}(K)]$, for $n < \omega$.

If $m < \omega$, we define $\mathrm{Cond}_m(\sigma)$ as the set of conditions

$$b_n \cdot x - \sigma(n) \cdot \alpha \in \delta[\mathrm{Add}(K)] \quad \text{for} \quad n \leqslant m .$$

We claim that the set $\mathrm{Cond}_m(\sigma)$ is satisfiable in K, i.e. that there exists an element $x_{m,\sigma}$ of K such that

$$b_n x_{m,\sigma} - \sigma(n) \cdot \alpha \in \delta[\mathrm{Add}(K)] \quad \text{for} \quad n \leqslant m .$$

Since $b_0, \ldots, b_m \in \mathrm{Abs}(K)$, it follows that $K_0(b_0, \ldots, b_m)$ is finite. Select N such that $K_0(b_0, \ldots, b_m) \subseteq K \cap F_{p^N}$. Suppose we can find $x_{m,\sigma}$ in $K \cap F_{p^N}$ such that $T_N(b_n x_{m,\sigma} - \sigma(n) \cdot \alpha) = 0$ for $n \leqslant m$. Then by Fact 6 there exist $y_{n,\sigma}$ $(n \leqslant m)$ in $K \cap F_{p^N}$ such that $b_n x_{m,\sigma} - \sigma(n) \cdot \alpha = (y_{n,\sigma})^{p^e} - y_{n,\sigma}$ for $n \leqslant m$. But then $b_n x_{m,\sigma} - \sigma(n) \cdot \alpha \in \delta[\mathrm{Add}(K)]$ for $n \leqslant m$. Then $x_{m,\sigma}$ satisfies $\mathrm{Cond}_m(\sigma)$.

Therefore we want to solve the system of equations

$$T_N(b_n x - \sigma(n) \cdot \alpha) = 0 \qquad (n \leqslant m) \text{ in } K \cap F_{p^N} .$$

This is equivalent to solving

$$T_N(b_n x) = T_N(\sigma(n) \cdot \alpha) \qquad (n \leqslant m) \text{ in } K \cap F_{p^N}.$$

Now, $K \cap F_{p^N}$ is a separable extension of K_0, and $b_0, \ldots, b_m$ are linearly independent over K_0. It follows directly from [1], p. 89 that the system $T_N(b_n x) = T_N(\sigma(n) \cdot \alpha)$ $(n \leqslant m)$ has a solution x_m, in $K \cap F_{p^N}$.

We conclude that $\mathrm{Cond}_m(\sigma)$ is satisfiable in K. Since m was arbitrary, every finite subsystem of $\mathrm{Cond}(\sigma)$ is satisfiable in K.

Now, corresponding to $\mathrm{Cond}(\sigma)$, we define a set $C(\sigma)$ of sentences of $\mathcal{L}(\omega+1)$ thus: $C(\sigma)$ consists of all sentences

$$(\exists v_0)\left(v_0^{p^e} + \lambda(n) c_0 = v_0 + c_{n+1} \cdot c_\omega\right)$$

for $n < \omega$, where $\lambda(n)$ is a term defining the element $\sigma(n)$ of F_p. [Think of the elements of $C(\sigma)$ as the "informal" sentences

$$(\exists v_0)\left(v_0^{p^e} - v_0 = c_{n+1} \cdot c_\omega - \lambda(n) c_0\right),$$

and recall that for the $s \in |K|^\omega$ we defined earlier, $s(0) = \alpha$, and $s(n+1) = b_n$].

By what we proved above, $c(\sigma) \cup Th((K, s))$ is finitely satisfiable, and so satisfiable. Select a complete extension $\bar{c}(\sigma)$ of $c(\sigma) \cup Th((K, s))$ in $\mathcal{L}(\omega+1)$.

Suppose $\sigma_1, \sigma_2 \in |F_p|^\omega$ and $\sigma_1 \neq \sigma_2$. We claim that $\bar{c}(\sigma_1) \neq \bar{c}(\sigma_2)$. Suppose $\sigma_1(m) \neq \sigma_2(m)$. It will suffice to show that there is no x in K such that $b_m x - \sigma_1(m) \alpha \in \delta[\mathrm{Add}(K)]$ and $b_m x - \sigma_2(m) \alpha \in \delta[\mathrm{Add}(K)]$. If there exists such an x, it follows by subtraction that

$$\left(\sigma_2(m) - \sigma_1(m)\right) \alpha \in \delta[\mathrm{Add}(K)] .$$

Now $\sigma_2(m) - \sigma_1(m) \neq 0$, so if $\left(\sigma_2(m) - \sigma_1(m)\right) \alpha \in \delta[\mathrm{Add}(K)]$ then $\alpha \in \delta[\mathrm{Add}(K)]$. But we proved earlier that $\alpha \notin \delta[\mathrm{Add}(K)]$. Thus there is no x in K such that $b_m x - \sigma_1(m) \cdot \alpha \in \delta[\mathrm{Add}(K)]$ and $b_m x - \sigma_2(m) \cdot \alpha \in \delta[\mathrm{Add}(K)]$. Therefore $\mathrm{Cond}_m(\sigma_1)$ and $\mathrm{Cond}_m(\sigma_2)$ have no common solution in K. It follows that $c(\sigma_1) \cup c(\sigma_2) \cup Th((K, s))$ is not satisfiable, so $\bar{c}(\sigma_1) \neq \bar{c}(\sigma_2)$.

We conclude that $Th((K, s))$ has at least $2^{\aleph_0}$ complete extensions in $\mathcal{L}(\omega+1)$, since the map $\sigma \mapsto \bar{c}(\sigma)$ is 1-1 on $|F_p|^\omega$, and $|F_p|^\omega$ has cardinality $2^{\aleph_0}$. It follows that $Th(K)$ is not totally transcendental.

We have proved that if Case B holds and K is infinite then $Th(K)$ is not totally transcendental.

Under the assumption that K is infinite and $Th(K)$ is totally transcendental, we have shown that neither Case B nor Case C can occur. This leaves only Case A, when $K = \tau[K]$. But if $K = \tau[K]$ then for each y in K there exists an x in K such that $x^p - x = y$. Thus if K is infinite and $Th(K)$ is totally transcendental then for each y in K there exists an x in K such that $x^p - x = y$. But this is Lemma 10, at last. Theorem 1 is proved.

Remark. Because of the lengthy analysis involved in Case B, it is worthwhile giving an example of an infinite field K such that:

(i) For every finite algebraic extension K_1 of K, $(K_1)^*$ is divisible;

(ii) For every finite algebraic extension K_1 of K, Case B holds, i.e. there exists $k \geqslant 0$ such that

$$K_1 \neq \tau[K_1] \neq \ldots \neq \tau_{k+1}[K_1] = \tau_{k+2}[K_1] = \ldots$$

Let p be a prime, and let K be the closure of F_p under algebraic extensions of degree prime to p. In terms of the so-called supernatural numbers ([3], [12]), K is the unique extension of F_p of degree s, where

$$s = \prod_{q \text{ prime}} q^{n(q)}$$

and $n(q) = \infty$ if $q \neq p$, and $n(p) = 1$. In the terminology of Ax [3], K is $\{p\}$-pseudo-finite. (See Ax's Proposition 9.)

Since K has a cyclic extension of degree p, $K \neq \tau[K]$ by Fact 3. We claim $\tau[K] = \tau_2[K]$. K has a unique extension of degree p, generated by a root of the equation $x^p - x - 1 = 0$. By Fact 5, for every u in K there is an r in F_p such that $u - r \in \tau[K]$. Therefore $\tau(u) - \tau(r) \in \tau_2[K]$. But $\tau(r) = 0$. Therefore $\tau(u) \in \tau_2[K]$. Since u was arbitrary, $\tau[K] \subseteq \tau_2[K]$, whence $\tau[K] = \tau_2[K]$.

Therefore Case B holds for K. A similar argument will prove that Case B holds for each finite extension K_1 of K.

Now we prove that K^* is divisible. Clearly it suffices to prove that, for each prime q, $K^* = (K^*)^q$. Since K is perfect the result is clear for $q = p$.

Suppose q is a prime $\neq p$, and that q divides $p^N - 1$ for some N which is relatively prime to p. For such N, $F_{p^N} \subseteq K$, so K contains a primitive $(p^N - 1)$th root of unity. Since q divides $p^N - 1$, K contains a primitive qth root of unity, so K contains q distinct qth roots of unity. Suppose now $K^* \neq (K^*)^q$. Then for some a in K^*, $a \notin (K^*)^q$. The polynomial $x^q - a$ is irreducible over K, since K has q distinct qth roots of unity. Therefore K has an extension of degree q. But K has no extension of degree q. Therefore $K^* = (K^*)^q$ if q divides $p^N - 1$ for some N which is relatively prime to p.

Finally suppose that q is a prime $\neq p$, and for all N which are relatively prime to p, q does not divide $p^N - 1$. Then q is relatively prime to $p^N - 1$, whenever N is relatively prime to p. Now suppose $a \in K^*$. Then $a \in (F_{p^N})^*$ for some N which is relatively prime to p. Then $a^{p^N - 1} = 1$. Since q is relatively prime to $p^N = 1$ there exist integers m, n such that $mq + n\cdot(p^N - 1) = 1$. Then $a = a^1 = a^{mq + n\cdot(p^N-1)} = a^{mq} = (a^m)^q$. Therefore $a \in (K^*)^q$. Since a was arbitrary, $K^* = (K^*)^q$.

This concludes our proof that K^* is divisible. A similar argument shows that $(K_1)^*$ is divisible for all finite extensions K_1 of K.

Thus K has all the required properties.

9. ω_1-categoricity. We now get, as a corollary of Theorem 1, the principal result of our paper.

THEOREM 2. *If $\mathcal{K}$ is an EC_Δ class of fields such that $Th(\mathcal{K})$ is ω_1-categorical, and $\mathcal{K}$ has no finite members, then $\mathcal{K}$ is one of the classes ACF_n.*

Proof. Assume the hypothesis. Let K be a member of $\mathcal{K}$. Then $Th(K)$ is ω_1-categorical, so by [14], 3.8, $Th(K)$ is totally transcendental. K is infinite. We conclude by Theorem 1 that K is algebraically closed. Thus all members of $\mathcal{K}$ are algebraically closed. By ω_1-categoricity all members of $\mathcal{K}$ have the same characteristic, so $\mathcal{K} \subseteq \mathrm{ACF}_n$ for some n. Since any two members of ACF_n are elementarily equivalent [16], we conclude that $\mathcal{K} = \mathrm{ACF}_n$.

10. Elimination of quantifiers. Tarski [17] proved that the theory of algebraically closed fields of specified characteristic admits elimination of quantifiers. We will prove a converse of this result.

DEFINITION. Let Σ be an $\mathcal{L}$-theory. Then Σ *admits elimination of quantifiers* if and only if the following condition holds: if $\varphi(v_0, \ldots, v_n)$ is an $\mathcal{L}$-formula with all its free variables in the list $v_0, \ldots, v_n$, then there exists a quantifierf-ree formula $\psi(v_0, \ldots, v_n)$ such that

$$\Sigma \models (\forall v_0) \ldots (\forall v_n)\big(\varphi(v_0, \ldots, v_n) \leftrightarrow \psi(v_0, \ldots, v_n)\big).$$

(Here $\leftrightarrow$ is material equivalence, definable from $\neg$ and $\wedge$.)

It is easily seen that if Σ admits elimination of quantifiers then Σ is model-complete.

We want to know which theories of fields admit elimination of quantifiers. We emphasize that we are talking about elimination of quantifiers in the basic logic for fields. It is well-known [17] that the theory of real-closed fields admits elimination of quantifiers when we use the auxiliary predicate $>$ for order. The situation is analogous for valued fields [6].

Suppose K is a field such that $Th(K)$ admits elimination of quantifiers. Then for each $n \geqslant 1$ there is a predicate $p_n(x_0, \ldots, x_n)$ which is a Boolean combination of polynomial equations with integral coefficients such that for each $K_1 \equiv K$, and $x_0, \ldots, x_n$ in K_1, $P_n(x_0, \ldots, x_n)$ holds in K_1 if and only if there is a y in K_1 such that $x_0 + x_1\cdot y + x_2\cdot y^2 + \ldots + x_n\cdot y^n = 0$. The obvious example is when K is algebraically closed, and $P_n(x_0, \ldots, x_n)$ is $(x_0 \neq 0) \to \neg(x_1 = x_2 = \ldots = x_n = 0)$.

Lemma 12. *Suppose K is a field such that $Th(K)$ admits elimination of quantifiers. Then $Th(K)$ is ω_1-categorical.*

Proof. Assume the hypothesis. Suppose K_1 and K_2 are models of $Th(K)$ of cardinality ω_1. Then K_1 and K_2 have the same characteristic. Clearly K_1 and K_2 both have transcendence degree ω_1 over their respective ground fields. Let L_1, L_2 be respectively the ground fields of K_1, K_2. Let $\{b_{1,\lambda}\colon \lambda < \omega_1\}$ and $\{b_{2,\lambda}\colon \lambda < \omega_1\}$ be respectively transcendence bases for K_1 over L_1 and K_2 over L_2. Let $J_1 = L_1(b_{1,0}, \ldots, b_{1,\lambda}, \ldots\colon \lambda < \omega_1)$, and $J_2 = L_2(b_{2,0}, \ldots, b_{2,\lambda}, \ldots\colon \lambda < \omega_1)$. Then K_1 is algebraic over J_1 and K_2 is algebraic over J_2. Also by the Steinitz theory there exists an isomorphism $\sigma\colon J_1 \cong J_2$. We claim σ extends to an isomorphism $\bar{\sigma}\colon K_1 \cong K_2$.

σ extends to an isomorphism $\sigma^*\colon J_1[x] \cong J_2[x]$ such that $\sigma^*(x) = x$. We claim that for each f in $J_1[x]$ f has a root in K_1 if and only if $\sigma^*(f)$ has a root in K_2. Suppose f is $x_0 + x_1\cdot x + \ldots + x_n \cdot x^n$, where $x_0, x_1, \ldots, x_n \in J_1$. Then f has a root in K_1 if and only if there exists y in K_1 such that $x_0 + x_1\cdot y + \ldots + x_n \cdot y^n = 0$. Now we use the predicate P_n introduced earlier. f has a root in K_1 if and only if $P_n(x_0, x_1, \ldots, x_n)$ holds in K_1. But P_n is quantifier-free, so $P_n(x_0, x_1, \ldots, x_n)$ holds in K_1 if and only if $P_n\big(\sigma(x_0), \sigma(x_1), \ldots, \sigma(x_n)\big)$ holds in K_2. But $P_n\big(\sigma(x_0), \sigma(x_1), \ldots, \sigma(x_n)\big)$ holds in K_2 if and only if there exists Z in K_2 such that $\sigma(x_0) + \sigma(x_1)\cdot Z + \ldots + \sigma(x_n)Z^n = 0$, i.e. if and only if $\sigma^*(f)$ has a root in K_2. This proves the claim that f has a root in K_1 if and only if $\sigma^*(f)$ has a root in K_2.

Since for $i = 1, 2$ K_i is algebraic over J_i, Lemma 1 on p. 255 of [3] (with an obvious modification) implies that σ extends to the required $\bar{\sigma}$: $K_1 \cong K_2$.

Since K_1 and K_2 were arbitrary models of $Th(K)$ of cardinality ω_1, we conclude that $Th(K)$ is ω_1-categorical.

Theorem 3. *Suppose K is an infinite field such that $Th(K)$ admits elimination of quantifiers. Then K is algebraically closed.*

Proof. Assume the hypothesis. Then $Th(K)$ is ω_1-categorical, and so by Theorem 2 K is algebraically closed.

Theorem 3 is the promised converse to Tarski's result.

11. Concluding remarks. It would be interesting to classify ω_1-categorical theories of division rings. By the methods of this paper one can show that if K is a division ring with $Th(K)$ totally transcendental then the centre of K is either finite or algebraically closed, but this is all we know at the moment.

More generally one would like categoricity results for wider classes of rings. A specimen result is that if K is an algebraically closed field and $M_n(K)$ is the ring of $n \times n$ matrices over K then $Th\big(M_n(K)\big)$ is ω_1-categorical. Here also our knowledge is fragmentary.

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