[8] A. Lévy, *Definability in axiomatic set theory, I*, in: Logic, Methodology and Philosophy of Science, Proceedings of the 1964 — Congress at Jerusalem, Amsterdam 1965, pp. 127–151.
[9] — *Axiom schemata of strong infinity in axiomatic set theory*, Pacific J. Math. 10 (1960), pp. 223–228.
[10] D. Hilbert and W. Ackermann, *Grundzüge der theoretischen Logik*, 5th ed., Berlin–Heidelberg–New York 1967.
[11] J. C. Sheperdson, *Inner models for set theory, I*, Journ. Symb. Logic 16 (1951), pp. 161–190.

Reçu par la Rédaction le 22. 12. 1969

Proximity approach to extension problems

by

M. S. Gagrat and **S. A. Naimpally** (Kanpur, India)

1. Introduction. Let X and Y be dense subspaces of topological spaces αX and αY respectively. An important class of problems in Topology deals with necessary and/or sufficient conditions under which a continuous function $f\colon X\to Y$ has a continuous extension $\bar{f}\colon \alpha X\to\alpha Y$ (or Y). Among several known results in this class, the following result, due to Taimanov [12], has many applications:

(1.1) *A necessary and sufficient condition that a continuous function $f\colon X\to Y$, where X is dense in a T_1-space αX and Y compact Hausdorff, has a continuous extension $\bar{f}\colon \alpha X\to Y$ is that for every pair of disjoint closed sets F_1, F_2* of Y,

$$\mathrm{Cl}_{\alpha X} f^{-1}(F_1)\cap \mathrm{Cl}_{\alpha X} f^{-1}(F_2)=\emptyset .$$

Lodato [7] has shown that a generalized proximity δ_0 (called LO-*proximity* in this paper) can be introduced in αX as follows: $A\,\delta_0 B$ iff $A^-\cap B^-\neq\emptyset$ (we use the bar to denote closure when no confusion is possible). It is well known that in the case of a compact Hausdorff space, δ_0, as defined above, is a unique compatible Efremovič proximity (called EF-*proximity* in this paper) (see Efremovič [3]). Taimanov's Theorem can now be interpreted as follows: If αX and Y are assigned the LO-proximity δ_0 and the EF-proximity δ_0 respectively, then f has a continuous extension if and only if f is proximally continuous. It is interesting to note that whereas X has the subspace LO-proximity induced by δ_0 on αX, Y has an EF-proximity.

This investigation began with an attempt to prove Taimanov's Theorem by the use of bunches and clusters (see Lodato [7] and Leader [6]). However, we found a general theorem which includes several results, including Taimanov's result mentioned above, as special cases.

The 2nd Section gives preliminary results needed to prove our theorems. For a survey of EF-proximity spaces see for example [10]. An uptodate account of LO-proximity is written by Mozzochi [9].

Section 3 contains our fundamental results on extensions of functions. In Section 4 we use our results to get, for LO-proximity spaces, a satisfactory generalization of the well-known Smirnov compactification theorem for EF-proximity spaces. We believe that no such generalization has so far appeared in the literature.

In Section 5 we prove a general result on extensions of continuous functions. This generalization of Taimanov's result yields the extension results of McDowell [8], Blefko [2] and Engelking [4]. In the final section we show that our results also include an Extension Theorem due to Ponomarev [11] concerning Wallman compactifications of T_1-spaces.

2. Preliminaries. The purpose of this section is merely to recall the known results concerning LO-proximity spaces and the reader familiar with them need read only (2.9) through (2.13), which are new results.

(2.1) DEFINITION. A binary relation δ on the power set of X is a LO-*proximity* iff

(i) $A\delta B$ implies $B\delta A$,

(ii) $(A\cup B)\delta C$ iff $A\delta C$ or $B\delta C$,

(iii) $A\delta B$ implies $A\neq\emptyset$, $B\neq\emptyset$,

(iv) $A\cap B\neq\emptyset$ implies $A\delta B$,

(v) $A\delta B$ and $b\delta C$ for each $b\in B$ implies $A\delta C$.

[We frequently write x for $\{x\}$.]

The pair (X,δ) is called a LO-*space*. If, in addition, δ satisfies,

(vi) $x\delta y$ implies $x=y$,

then δ is said to be *separated*. If δ satisfies (i) through (iv) and

(v′) $A\bar{\delta}B$ implies there exist E and F such that $A\bar{\delta}(X-E)$, $B\bar{\delta}(X-F)$ and $E\bar{\delta}F$,

then δ is called an EF-*proximity*. The pair (X,δ) is called an EF-*space*.

Every EF-space is a LO-space but the converse does not hold. Every LO-proximity δ on X induces a topology $\tau(\delta)$ on X as follows: $G\in\tau(\delta)$ iff for each $x\in G$, $x\bar{\delta}(X-G)$. A topology τ on X such that $\tau=\tau(\delta)$ is said to be *compatible with* δ. A topological space (X,τ) is said to be R_0 (*weakly regular* or *symmetric*) iff either of the following equivalent conditions is satisfied:

(a) $x\in G\in\tau$ implies $x^-\subset G$;

(b) $x\in y^-$ implies $y\in x^-$.

If δ is a (separated) LO-proximity, then $\tau(\delta)$ is R_0 (respectively T_1); conversely, every R_0-(T_1-) space (X,τ) has a compatible (respectively separated) LO-proximity δ_0 defined by;

(2.2) $A\delta_0B$ *iff* $A^-\cap B^-\neq\emptyset$.

As for EF-proximity, it is well known that a topological space is completely regular (Tychonoff) if and only if it has a compatible (respectively separated) EF-proximity. In particular, every completely regular space X has a compatible *functionally distinguishable* proximity δ_F defined by:

(2.3) $A\bar{\delta}_F B$ *iff there is a continuous function* $f\colon X\to[0,1]$ *such that* $f(A)=0$, $f(B)=1$.

It is worthwhile to note that δ_0 as defined by (2.2) is a compatible separated EF-proximity on X if and only if X is T_4. We also note that δ_0 is the unique compatible EF-proximity on a compact Hausdorff space.

The following result, which derives easily from the definitions, is frequently needed:

(2.4) *In a* LO-*space* (X,δ), $A\delta B$ *iff* $A^-\delta B^-$.

(2.5) DEFINITION. Let (X,δ_1) and (Y,δ_2) be LO-spaces. Then a function $f\colon X\to Y$ is *proximally continuous* iff $A\delta_1B$ implies $f(A)\delta_2f(B)$.

The following result is known:

(2.6) *Every proximally continuous function is continuous. The converse holds if either* $\delta_1=\delta_0$ (LO-*proximity*) *or* $\delta_1=\delta_F$ (EF-*proximity*).

Bunches and clusters correspond to ultrafilters and play rôles in proximity spaces analogous to the rôle of ultrafilters in topological spaces.

(2.7) DEFINITIONS. A non-empty family σ of subsets of a LO-space (X,δ) is called a *bunch* (Lodato [7]) iff:

(a) $A,B\in\sigma$ implies $A\delta B$,

(b) $(A\cup B)\in\sigma$ iff $A\in\sigma$ or $B\in\sigma$,

(c) $A\in\sigma$ iff $A^-\in\sigma$.

σ is a cluster (Leader [6]) if it satisfies (a), (b) and

(c′) $A\delta B$ for every $B\in\sigma$ implies $A\in\sigma$.

The following results in a LO-space X (unless otherwise stated) are either immediate consequences of the above definitions or are known results that are needed in the sequel.

(2.8) (i) *Every cluster is a bunch but not conversely.*

(ii) *Every bunch is contained in a maximal bunch.*

(iii) *If* $\mathfrak{L}$ *is an ultrafilter in* X, *then*

$$b(\mathfrak{L})=\{A\subset X\colon A^-\in\mathfrak{L}\},$$

is a bunch generated by $\mathfrak{L}$.

(iv) *For each* $x\in X$,

$$\sigma_x=\{A\subset X\colon A\delta x\},$$

is a cluster called the point cluster.

(v) *If σ is a bunch and $\{x\} \in \sigma$, then σ is the point cluster σ_x.*

(vi) *A family σ of subsets of an EF-space X is a cluster iff there exists an ultrafilter $\mathfrak{L}$ in X such that*

$$\sigma = \sigma(\mathfrak{L}) = \{A \subset X: A\delta L \text{ for every } L \in \mathfrak{L}\}.$$

It is called the cluster generated by $\mathfrak{L}$.

(vii) *If σ is a bunch, $A \in \sigma$ and $A \subset B$, then $B \in \sigma$.*

We say that a bunch σ in a LO-space X *converges to* $x \in X$ iff the neighbourhood filter $\mathcal{N}_x$ of x is a subclass of σ. Clearly a cluster σ in an EF-space converges to x iff $\sigma = \sigma_x$. Leader [6] has shown that an EF-space is compact iff every cluster in it is a point cluster. The following analogue holds for LO-spaces.

(2.9) Lemma. *A separated* LO-*space X is compact if and only if every bunch $b(\mathfrak{L})$ generated by a closed ultrafilter $\mathfrak{L}$ in X (see (2.8) (iii)) is a point cluster.*

Proof. Let $\mathfrak{L}$ be a closed ultrafilter in X. Then $b(\mathfrak{L}) = \sigma_{x_0}$ for some $x_0 \in X$ implies $\{x_0\} \in b(\mathfrak{L})$. This shows that $x_0 \in L$ for every $L \in \mathfrak{L}$ (since each L is closed) and so $\{x_0\} \in \mathfrak{L}$, since $\mathfrak{L}$ is maximal. Conversely, if X is compact and $\mathfrak{L}$ is a closed ultrafilter, then $\mathfrak{L}$ has a cluster point x. Since $\mathfrak{L}$ is maximal, $\{x\} \in \mathfrak{L} \subset b(\mathfrak{L})$ and so $b(\mathfrak{L}) = \sigma_x$ ((2.8) (v)).

The following result is a generalization of Lemma (5.7) of [10].

(2.10) Lemma. *Let $\mathcal{F}$ be a ring of subsets of X (e.g. closed subsets of X or z-sets of a Tychonoff space X). Suppose $\mathfrak{S} \subset \mathcal{F}$ is such that* (i) $\emptyset \notin \mathfrak{S}$, (ii) *For A, B in $\mathcal{F}$ $(A \cup B) \in \mathfrak{S}$ iff $A \in \mathfrak{S}$ or $B \in \mathfrak{S}$ and,* (iii) *$A \in \mathfrak{S}$, $A \subset B \in \mathcal{F}$ implies $B \in \mathfrak{S}$. Then given an $A_0 \in \mathfrak{S}$, there exists a prime $\mathcal{F}$-filter $\mathfrak{L}$ such that $A_0 \in \mathfrak{L} \subset \mathfrak{S}$. (Recall that $\mathfrak{L}$ is prime means that for elements A, B of $\mathcal{F}$, $(A \cup B) \in \mathfrak{L}$ implies $A \in \mathfrak{L}$ or $B \in \mathfrak{L}$.) If $\mathcal{F}$ is the power set of X, then $\mathfrak{L}$ is an ultrafilter.*

(2.11) Lemma. *In an* EF-*space a family of subsets is a cluster if and only if it is a maximal bunch.*

Proof. We need prove only that if σ is a maximal bunch then it is a cluster. By (2.10) there is an ultrafilter $\mathfrak{L} \subset \sigma$. Clearly $\sigma \subset \sigma(\mathfrak{L}) = \{A: A\delta L$ for every $L \in \mathfrak{L}\}$ which is a cluster in X (see (2.8) (vi)). Since σ is maximal, $\sigma = \sigma(\mathfrak{L})$.

(2.12) Lemma. *In an* EF-*space X, every bunch is contained in a unique cluster.*

Proof. If b is a bunch in X, then by (2.8) (ii) and (2.11), b is contained in a cluster. To show uniqueness, suppose on the contrary that b is contained in two different clusters σ_1, σ_2. Then there are sets $A_i \in \sigma_i$ $(i = 1, 2,)$ such that $A_1 \bar{\delta} A_2$. By (2.1) (v′) there exist $E_i \subset X$ such that $A_i \bar{\delta} (X - E_i)$ $(i = 1, 2)$ and $E_1 \bar{\delta} E_2$. Since $A_i \bar{\delta} (X - E_i)$, $(X - E_i) \notin \sigma_i$ $(i = 1, 2)$ and hence $(X - E_i) \notin b$. This implies that $E_i \in b$ and hence $E_1 \delta E_2$, a contradiction.

Let $\mathfrak{S}_X$, $\mathfrak{S}_Y$ be rings of subsets of topological spaces X, Y respectively, and let $f: X \to Y$ be a function such that for each $E \in \mathfrak{S}_Y$, $f^{-1}(E) \in \mathfrak{S}_X$. It is easy to prove that if $\mathcal{F}$ is a prime $\mathfrak{S}_X$-filter, then

$$f^{\#}(\mathcal{F}) = \{E \in \mathfrak{S}_Y: f^{-1}(E) \in \mathcal{F}\}, \tag{2.13}$$

is a prime $\mathfrak{S}_Y$-filter in Y (cf. Gillman and Jerison [5], p. 59).

Finally, we prove a slightly stronger version of (2.8) (vi).

(2.14) Lemma. *Let X be a Tychonoff space and let X have the functionally distinguishable proximity $\delta = \delta_F$ (2.3). If $\mathfrak{L}$ is a prime z-filter in X, then $\sigma(\mathfrak{L}) = \{A \subset X: A\delta L$ for every $L \in \mathfrak{L}\}$ is a cluster in X.*

Proof. Clearly if Z_1 and Z_2 are z-sets in X, then $Z_1 \bar{\delta} Z_2$ iff $Z_1 \cap Z_2 = \emptyset$ (see Gillman and Jerison [5], p. 17). The only non-trivial part is to show that if $A_1, A_2 \in \sigma(\mathfrak{L})$, then $A_1 \delta A_2$. If $A_1 \bar{\delta} A_2$, they are functionally distinguishable and hence $A_i \subset Z_i$ $(i = 1, 2)$ and $Z_1 \cap Z_2 = \emptyset$. Then there are Z'_i such that $Z_i \subset X - Z'_i$ $(i = 1, 2)$ and $Z'_1 \cup Z'_2 = X$. Since $Z_i \in \sigma(\mathfrak{L})$ and $Z_i \bar{\delta} Z'_i$, $Z'_i \notin \mathfrak{L}$ $(i = 1, 2)$. This implies that $X = Z'_1 \cup Z'_2 \notin \mathfrak{L}$, a contradiction.

3. Fundamental results.

In this section we prove the basic results concerning extensions of mappings.

(3.1) Definition. Let (X, δ) be a LO-space and Σ_X the family of all bunches in X. Let $\Sigma \subset \Sigma_X$. A set $E \subset X$ is said to *absorb* $\mathcal{A} \subset \Sigma$ iff $E \in \sigma$ for every $\sigma \in \mathcal{A}$.

The proof of the following two lemmas is similar to that in Lodato [7] and is omitted.

(3.2) Lemma. *For $\mathcal{A} \subset \Sigma \subset \Sigma_X$,*

$$\mathrm{Cl}(\mathcal{A}) = \{\sigma \in \Sigma: E \text{ absorbs } \mathcal{A} \text{ implies } E \in \sigma\},$$

defines a Kuratowski closure operator on X.

(The resulting topology on Σ is called the *absorption* or *A-topology.*)

(3.3) Lemma. *The A-topology on Σ is*

(i) *T_1 if and only if $\sigma_1, \sigma_2 \in \Sigma$, $\sigma_1 \neq \sigma_2$ implies $\sigma_1 \not\subset \sigma_2$ and $\sigma_2 \not\subset \sigma_1$;*

(ii) *T_2 if either $A \in \sigma_1$ or $B \in \sigma_2$ $(\sigma_1, \sigma_2 \in \Sigma)$ for all subsets A, B of X such that $A \cup B = X$, then $\sigma_1 = \sigma_2$.*

(3.4) Theorem. *Let (X, δ) be a* LO-*space and let $\varphi = \varphi_X: X \to \Sigma_X$ be a function defined by $\varphi(x) = \sigma_x$, the point cluster* (2.8 (iv)). *Then φ is* *continuous and closed and $\varphi(X)$ is dense in Σ_X. If δ is separated, then X is homeomorphic to $\varphi(X)$.*

Proof. That φ is continuous and closed follows from the fact $x\delta A$ iff $A \in \sigma_x$ i.e. $x \in A^-$ iff $\sigma_x \in \mathrm{Cl}(\varphi(A))$. If δ is separated then $x \neq y$ implies $\sigma_x \neq \sigma_y$ and φ is one-to-one. Thus φ is a homeomorphism from X to $\varphi(X)$. Finally $\mathrm{Cl}\varphi(X) = \{\sigma \in \Sigma_X\colon X \in \sigma\} = \Sigma_X$.

(3.5) COROLLARY. *If* (i) $\varphi(X) \subset \Sigma \subset \Sigma_X$, (ii) $A\delta B$ *implies there exists a* $\sigma \in \Sigma$ *such that* $A, B \in \sigma$ *and* (iii) *the* A-*topology on* Σ *is* T_1, *then* φ *is a proximal isomorphism between* X *and* $\varphi(X)$, *the latter having the subspace* LO-*proximity of* δ_0 *on* Σ.

Proof. The proximal isomorphism follows from the fact that under the conditions stated, $A\delta B$ iff $\mathrm{Cl}\varphi(A) \cap \mathrm{Cl}\varphi(B) \neq \emptyset$.

We now prove our main result.

(3.6) THEOREM. *Let* (X, δ_1), (Y, δ_2) *be* LO-*spaces and* $f\colon X \to Y$ *be proximally continuous. Then there exists an associated function*

$$f_\Sigma\colon \Sigma_X \to \Sigma_Y,$$

defined by $f_\Sigma(\sigma) = \{E \subset Y\colon f^{-1}(E^-) \in \sigma\}$. *The map* f_Σ *is continuous with respect to the* A-*topologies on* Σ_X, Σ_Y *and* $f_\Sigma(\sigma_x) = \sigma_{f(x)}$ *for each* $x \in X$.

Proof. We first show that if $\sigma \in \Sigma_X$, then $f_\Sigma(\sigma) \in \Sigma_Y$, by verifying (2.7) (a), (b), (c).

(a) If $A, B \in f_\Sigma(\sigma)$, then $f^{-1}(A^-)$, $f^{-1}(B^-) \in \sigma$. This implies that $f^{-1}(A^-)\,\delta_1 f^{-1}(B^-)$ and since f is proximally continuous, $A^-\delta_2 B^-$. By (2.4) we have $A\delta_2 B$.

(b) $(A \cup B) \in f_\Sigma(\sigma)$ iff $f^{-1}[(A \cup B)^-] \in \sigma$ iff $f^{-1}(A^-) \cup f^{-1}(B^-) \in \sigma$ iff $A \in f_\Sigma(\sigma)$ or $B \in f_\Sigma(\sigma)$.

(c) Obviously $A \in f_\Sigma(\sigma)$ iff $A^- \in f_\Sigma(\sigma)$.

To show that f_Σ is continuous we must show that if $\sigma \in \mathrm{Cl}\,(\mathcal{A})$ for $\mathcal{A} \subset \Sigma_X$, then $f_\Sigma(\sigma) \in \mathrm{Cl}(f_\Sigma(\mathcal{A}))$. If this is not the case, then there is a set $E \subset Y$ which absorbs $f_\Sigma(\mathcal{A})$ but does not belong to $f_\Sigma(\sigma)$. This implies that $f^{-1}(E^-)$ absorbs $\mathcal{A}$ but is not in σ i.e. $\sigma \notin \mathrm{Cl}(\mathcal{A})$, a contradiction. Finally, if $x \in X$,

$$\begin{aligned} f_\Sigma(\sigma_x) &= \{A \subset Y\colon f^{-1}(A^-) \in \sigma_x\} \\ &= \{A \subset Y\colon x\delta_1 f^{-1}(A^-)\} \\ &= \{A \subset Y\colon x \in f^{-1}(A^-)\} \\ &= \{A \subset Y\colon f(x) \in A^-\} \\ &= \sigma_{f(x)}. \end{aligned}$$

If in the above theorem δ_1, δ_2 are separated, then φ_X, φ_Y (as defined in Theorem (3.4)) are homeomorphisms. By Theorem (3.6), we have the following diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi_X"'] & Y \arrow[d, "\varphi_Y"] \\ \Sigma_X \arrow[r, "f_\Sigma"] & \Sigma_Y \end{tikzcd}$$

$$f_\Sigma\varphi_X = \varphi_Y f.$$

Identifying X with $\varphi(X)$ and Y with $\varphi(Y)$ we have:

(3.7) FUNDAMENTAL EXTENSION THEOREM. *Let* (X, δ_1) *and* (Y, δ_2) *be separated* LO-*spaces. Then every proximally continuous function* $f\colon X \to Y$ *has a continuous extension* $f_\Sigma\colon \Sigma_X \to \Sigma_Y$.

(3.8) THEOREM. *Let* X *be a dense separated* LO-*subspace of a* LO-*space* $(\alpha X, \delta_1)$ [δ_1 *need not be separated*]. *Then the function* $\psi = \psi_{\alpha X}\colon \alpha X \to \Sigma_X$ *defined by*

$$\psi(x) = \sigma^x = \{E \subset X\colon x\delta_1 E\},$$

for each $x \in \alpha X$ *is continuous. If* $x \in X$, *then* $\psi(x) = \sigma_x$ *the point cluster, i.e.* $\psi|X = \varphi_X$.

Proof. We first verify that σ^x is a bunch in X. Since X is dense in αX, $X \in \sigma^x$ and $\sigma^x \neq \emptyset$. (a) If $A, B \in \sigma^x$, then $x\delta_1 A$ and $x\delta_1 B$ and consequently (by (2.1) (i), (v)), $A\delta_1 B$. (b) $(A \cup B) \in \sigma^x$ iff $(A \cup B)\delta_1 x$ iff $A\delta_1 x$ or $B\delta_1 x$ ((2.1) (ii)) iff $A \in \sigma^x$ or $B \in \sigma^x$. (c) $A \in \sigma^x$ iff $x\delta_1 A$ iff $x \in A^-$ iff $A^- \in \sigma^x$.

In order to show that ψ is continuous, we must prove that if $x \in E^-$ where $E \subset \alpha X$, then $\psi(x) \in \mathrm{Cl}(\psi(E))$. If $\psi(x) \notin \mathrm{Cl}(\psi(E))$, then there is a set $A \subset X$ which absorbs $\psi(E)$ and $A \notin \psi(x) = \sigma^x$. This implies that $E^- \subset A^-$ and $x \notin A^-$, a contradiction. Hence ψ is continuous. The last part is obvious.

(3.9) COROLLARY. *If in Theorem* (3.8), αX *is* T_3, *then* ψ *is a homeomorphism of* αX *into* Σ_X.

Proof. If x_1, x_2 are two distinct points of αX, then there are disjoint neighbourhoods N_1, N_2 of x_1, x_2 respectively. Clearly $X \cap N_1 \in \sigma^{x_1} - \sigma^{x_2}$ and $X \cap N_2 \in \sigma^{x_2} - \sigma^{x_1}$, i.e. $\sigma^{x_1} \neq \sigma^{x_2}$. This shows that ψ is 1-1. To prove that ψ is a homeomorphism it is sufficient to show that ψ is closed. Suppose $G \subset \alpha X$ and $x \notin G^-$. Since αX is T_3, there are disjoint neighbourhoods N_G, N_x of G and x respectively. Since X is dense in αX, $(N_G \cap X)$ absorbs $\psi(G)$ but does not belong to σ^x. Hence $\psi(x) = \sigma^x \notin \mathrm{Cl}(\psi(G))$, i.e., ψ is closed.

Leader [6] has shown that the Smirnov compactification $\mathfrak{X}$ of a separated EF-space (X, δ) is the family of all clusters in X with the A-topology. Also to each $\sigma \in \Sigma_X$ there corresponds a unique cluster σ_θ in X (see (2.12)) containing σ.

(3.10). THEOREM. *If (X, δ) is a separated* EF-*space, then the map $\theta = \theta_X$: $\Sigma_X \to \mathfrak{X}$ given by $\theta(\sigma) = \sigma_\theta$ is continuous. Moreover, $\theta(\sigma_x) = \sigma_x$.*

Proof. To show that θ is continuous, we must prove that if $\sigma \in \mathrm{Cl}(\mathcal{A})$, $\mathcal{A} \subset \Sigma_X$, then $\theta(\sigma) \in \mathrm{Cl}(\theta(\mathcal{A}))$. If, on the contrary, $\theta(\sigma) \notin \mathrm{Cl}(\theta(\mathcal{A}))$, then since $\mathfrak{X}$ is compact T_2, there are non-near neighbourhoods U_1, U_2 of $\theta(\sigma)$, $\theta(\mathcal{A})$ respectively. Clearly $U_2 \cap X$ absorbs $\mathcal{A}$ but does not belong to σ, i.e., $\sigma \notin \mathrm{Cl}(\mathcal{A})$, a contradiction. That $\theta(\sigma_x) = \sigma_x$ is obvious.

If X is compact T_2, then $X = \mathfrak{X}$ (identifying x with σ_x) and the map θ in the above theorem is onto X. Hence we have,

(3.11) COROLLARY. *If X is compact T_2 with the* EF-*proximity δ_0, the map $\theta = \theta_X$: $\Sigma_X \to X$ given by $\theta(\sigma) = x_\sigma$ (where σ converges to x_σ) is continuous.*

The method of proof in Theorem (3.10) actually shows that the following stronger result is true.

(3.12) THEOREM. *Let (X, δ_0) be a T_3* LO-*space and let Σ be a subset of Σ_X such that each $\sigma \in \Sigma$ converges to a (unique) $x_\sigma \in X$. Then the map $\theta = \theta_X$: $\Sigma \to X$ given by $\theta(\sigma) = x_\sigma$ is continuous.*

Leader's result quoted before Theorem (3.10) together with Lemma (2.11) provides a motivation for the following generalization.

(3.13) THEOREM. *Let (X, δ) be a separated* LO-*space and X^* be the am ily of all maximal bunches in X with the A-topology. Then X^* is a compact T_1-space containing a dense homeomorphic copy of X.*

Proof. From (3.3) (i) and (3.4), X is homeomorphic to $\varphi(X)$, which is dense in the T_1-space X^*. We need to prove that X^* is compact and it is sufficient to show that if $\{A_\alpha^*: \alpha \in \Lambda\}$ [where $A_\alpha^* = \{\sigma \in X^*: A_\alpha \in \sigma$, A_α closed in $X\}$] has the finite intersection property (f.i.p.), then $\bigcap_{\alpha \in \Lambda} A_\alpha^* \neq \emptyset$. (This is due to the fact that the family $\{A_\alpha^*: A_\alpha$ closed in $X\}$ is a base for closed sets in Σ_X in the A-topology.) Since $\{A_\alpha^*: \alpha \in \Lambda\}$ has the f.i.p., the corresponding family $\mathcal{F} = \{A_\alpha: \alpha \in \Lambda\}$ of closed subsets of X has the property:

(3.14) *every finite subfamily of $\mathcal{F}$ is a subclass of some $\sigma \in X^*$.*

Let $\mathfrak{G}$ be the family of all collections $\mathcal{G}$ of closed subsets of X such that

(i) $\mathcal{F} \subset \mathcal{G}$,

and

(ii) $G_1, G_2, \ldots, G_n \in \mathcal{G}$ implies there exists a $\sigma \in X^*$ such that $G_i \in \sigma$, $= 1, \ldots, n$.

By Zorn's Lemma $\mathfrak{G}$ has a maximal element $\mathcal{M}$. It can be verified that $b(\mathcal{M}) = \{E \subset X: E^- \in \mathcal{M}\}$ is a bunch in X. By (2.8) (ii), $b(\mathcal{M})$ is contained in $\sigma_0 \in X^*$. Clearly $\sigma_0 \in \bigcap_{\alpha \in \Lambda} A_\alpha^*$ and X^* is compact.

4. Applications to LO-spaces. In [7] Lodato's motivation for introducing a LO-space (X, δ) was a part of the well-known theorem of Smirnov: Every separated EF-space (X, δ) is proximally isomorphic to a dense subspace of a compact T_2 space $\mathfrak{X}$ with the EF-proximity δ_0. Lodato enquired whether there exists a set of axioms for the binary relation δ on the power set of X such that δ satisfies these conditions iff there exists a topological space Y in which X can be embedded so that

(4.1) *$A \delta B$ in X iff $A^- \cap B^- \neq \emptyset$ in Y.*

Granting that such an embedding exists, it is easy to verify that δ must be a LO-proximity (2.1). Lodato's generalization [7] is as follows:

(4.2) LODATO'S THEOREM. *Given a set X and a binary relation δ on the power set of X, the following are equivalent:*

(a) *There exists a T_2-space Y in which X is embedded so that* (4.1) *holds;*

(b) *(X, δ) is a separated* LO-*space possessing a family $\mathcal{B}$ of bunches such that* (i) *$A \delta B$ implies there is a $\sigma \in \mathcal{B}$ such that $A, B \in \sigma$ and* (ii) *$\sigma_1, \sigma_2 \in \mathcal{B}$ and either $A \in \sigma_1$ or $B \in \sigma_2$ for all A, B such that $A \cup B = X$, then $\sigma_1 = \sigma_2$.*

The above theorem follows from (3.3) (ii), (3.5), and the fact that $A \delta B$ iff $\mathrm{Cl}(\varphi(A)) \cap \mathrm{Cl}(\varphi(B)) \neq \emptyset$. This result of Lodato is a partial generalization of the complete theorem of Smirnov, which is as follows:

(4.3) SMIRNOV'S THEOREM. *Let (X, δ) be a separated* EF-*space.*

Then (i) *there exists a compact T_2-space $\mathfrak{X}$ containing a dense homeomorphic copy of X,* (ii) *$A \delta B$ iff $A^- \cap B^- \neq \emptyset$ in $\mathfrak{X}$, and* (iii) *if f: $(X, \delta_1) \to (Y, \delta_2)$ is proximally continuous, then f has a continuous extension $\bar{f}$: $\mathfrak{X} \to \mathfrak{Y}$. Further,* (iv) *any $\mathfrak{X}$ satisfying* (i) *and* (ii) *is unique up to a proximal isomorphism and can be described as the space of all clusters in X with the A-topology.*

Clearly (4.2) generalizes only (4.3) (i). Our results in the previous section contain generalizations of (4.3) (i), (ii), (iii) as follows: (3.13) of (4.3) (i), (3.5) of (4.3) (ii) and (3.7) of (4.3) (iii). No doubt the extension spaces are not the same for all these. However, if we require that (X, δ) satisfies (4.2) (b) (i) (which is already contained in the hypothesis of (3.5)), we get the following analogue of Theorem (4.3) due to Smirnov:

(4.4) THEOREM. *Let (X, δ) be a separated* LO-*space such that if $A \delta B$, then there is a bunch in X containing both A and B. Then* (i) *there exists a compact T_1-space $\mathfrak{X}$ (the space of all maximal bunches in X with the A-topology) containing a dense homeomorphic copy of X;* (ii) *$A \delta B$ iff*

$\mathrm{Cl}\varphi(A) \cap \mathrm{Cl}\varphi(B) \neq \emptyset$ *in* $\mathfrak{X}$; (iii) *if* $f\colon (X, \delta) \to (Y, \delta_2)$ *is proximally continuous (where* Y *is a separated* LO-*space), then* f *has a continuous extension* $\bar{f}\colon \mathfrak{X} \to \Sigma_Y$.

If Y is separated EF, then a method of proof used in (3.10) shows that the map from Σ_Y to $\mathfrak{Y}$ which assigns to each $\sigma \in \Sigma_Y$, a maximal bunch $\sigma' \in Y$ (see (2.8) (ii)) is continuous. And so in (iii) above one may consider $\bar{f}$ as a continuous function from $\mathfrak{X} \to \mathfrak{Y}$. Since in an EF-space X (i) every maximal bunch is a cluster (2.11) and (ii) $A \delta B$ implies there is a cluster in X which contains both A and B, we see that Theorem (4.4) generalizes (4.3) (i), (ii), (iii). No doubt we have lost the uniqueness, which we cannot expect without requiring that $\mathfrak{X}$ should be T_2, but in that case $\mathfrak{X}$ becomes an EF-space.

5. An extension of Taimanov's Theorem. In Taimanov's Theorem (1.1) the range space Y is compact T_2. In this section we take Y to be a separated EF-space and obtain a generalization which will be used in the sequel to get other extension results as well. First we note that if X is a T_1-dense subspace of a LO-space $(\alpha X, \delta_0)$ and if Y is dense in the separated LO-space (Y^*, δ_0), a necessary condition for a continuous function $f\colon X \to Y$ to have a continuous extension $\bar{f}\colon \alpha X \to Y^*$ is that f be proximally continuous (2.6). The important part of Taimanov's Theorem is that this condition is also sufficient. It follows that in all extension theorems, we need only prove the sufficiency of the condition of proximal continuity.

(5.1) Generalization of Taimanov's Theorem. *Let* X *be a* T_1*-dense subspace of an* R_0*-space* αX *and let* X *be assigned the* LO-*subspace-proximity induced by* δ_0 *on* αX. *Let* (Y, δ) *be a separated* EF-*space and* $\mathfrak{Y}$ *be its Smirnov compactification. Then a continuous function* $f\colon X \to Y$ *has a continuous extension* $\bar{f}\colon \alpha X \to \mathfrak{Y}$ *if and only if* f *is proximally continuous.*

Proof. We need only to prove sufficiency and this results from the following diagram, where the numbers refer to results proved earlier.

$$\begin{tikzcd}
\alpha X \arrow[r, "\psi_{\alpha X}", "(3.8)"'] & \Sigma_X \arrow[r, "f_\Sigma", "(3.7)"'] & \Sigma_Y \arrow[r, "\theta_y", "(3.10)"'] & \mathfrak{Y} \\
X \arrow[u, "\mathrm{id}_X"] \arrow[ur, "\varphi_X"', "(3.4)"] \arrow[rr, "f"] & & Y \arrow[u, "\varphi_Y", "(3.4)"'] &
\end{tikzcd}$$

Clearly $\bar{f} = \theta_Y f_\Sigma \psi_{\alpha X}\colon \alpha X \to \mathfrak{Y}$ is a continuous extension of f.

There are two ways in which we may specialize Theorem (5.1). First, we may ask for the conditions under which $\bar{f}$ is a function from αX to Y. An easy answer is that when Y is compact T_2, Y is homeomorphic to $\mathfrak{Y}$ and we get Taimanov's Theorem (1.1).

Secondly, we may ask for the largest subspace of αX to which f has a continuous extension if the range of the extension is Y. McDowell [8] solved this problem as follows: Suppose, in addition to the conditions in Theorem (5.1), functionally separated sets in X have disjoint closures in αX (recall that subsets A, B of X are functionally separated iff there is a continuous function $f\colon X \to [0, 1]$ such that $f(A) = 0$, $f(B) = 1$). If $f\colon X \to Y$ is continuous, then it is easily checked that f is proximally continuous: if $C \bar{\delta} D$ in Y, then C and D are functionally separated in Y. Since f is continuous, $f^{-1}(C)$, $f^{-1}(D)$ are functionally separated in X and hence $f^{-1}(C)\ \bar{\delta}_0\, f^{-1}(D)$. By Theorem (5.1), f has a continuous extension $\bar{f}\colon \alpha X \to \mathfrak{Y}$. We now find $\bar{f}^{-1}(Y)$. Clearly $\bar{f}(x) = y \in Y$ iff $f_\Sigma(\sigma^x)$ converges to y iff the neighbourhood filter $\mathcal{N}_y \subset f_\Sigma(\sigma^x)$ iff $x \in \bigcap_{N_y \in \mathcal{N}_y} \mathrm{Cl}_{\alpha X} f^{-1}(Ny)$ $= \alpha X_y$(say). Thus $\bar{f}^{-1}(Y) = \bigcup_{y \in Y} \alpha X_y$ and we get a slightly generalized version of one of McDowell's results in [8].

(5.2) Theorem. *Let* X *be a* T_1*-dense subspace of an* R_0*-space* αX; *let functionally separated sets in* X *have disjoint closures in* αX *and let* Y *be a Tychonoff space. Then every continuous function* $f\colon X \to Y$ *can be extended to a continuous function* $\bar{f}\colon \alpha X_Y \to Y$ *defined by* $\bar{f}(x) = y$ *iff* $x \in \alpha X_y$. *Moreover,* $\alpha X_Y = \bigcup_{y \in Y} \alpha X_y$ *is the largest subspace of* αX *to which* f *has a continuous extension.*

6. Applications to real-compact spaces. This section deals with extension theorems involving either real-compact spaces or real-compactifications of Tychonoff spaces. We also prove that the Hewitt real-compactification of X is the space of all clusters $\sigma(\mathfrak{L})$ $\mathfrak{L}$ a real z-ultrafilter in X (see (2.14)) with the A-topology. (Cf. this with the Smirnov compactification (4.3) and Alo–Shapiro [1]). This space is homeomorphic to the space of all real z-ultrafilters in X with the Wallman topology.

We now show that a result proved by Blefko [2] and Engelking [4] follows from Theorem (5.1). We recall that a Tychonoff space X is real-compact iff every real prime z-filter in X converges. (Gillman and Jerison [5], p. 120).

(6.1) Theorem. *Let* X *be a* T_1 *dense subspace of an* R_0*-space* αX *and let* Y *be a Tychonoff real-compact space. A continuous function* $f\colon X \to Y$ *has a continuous extension* $\bar{f}\colon \alpha X \to Y$ *if and only if, for every sequence* (F_n) *of closed subsets of* Y, $\bigcap_{n=1}^{\infty} F_n = \emptyset$ *implies* $\bigcap_{n=1}^{\infty} \mathrm{Cl}_{\alpha X} f^{-1}(F_n) = \emptyset$.

Proof. We prove the necessity in the same way as Blefko or Engelking. To prove the sufficiency, let αX and Y be assigned respectively the LO-proximity δ_0 and any compatible EF-proximity δ. By (2.6) f is proximally continuous and by (5.1) f has a continuous extension

$\bar{f}\colon aX \to \mathcal{Y}$. In view of Theorem (3.12) it is sufficient to show that for each $x \in aX$, $f_\Sigma(\sigma^x)$ converges to a certain $y_x \in Y$. We first note that the zero sets in $f_\Sigma(\sigma^x)$ have the countable intersection property; for if they did not and $\bigcap_{n=1}^{\infty} Z_n = \emptyset$ for z-sets $Z_n \in f_\Sigma(\sigma^x)$, then we would have $\bigcap_{n=1}^{\infty} \mathrm{Cl}_{aX} f^{-1}(Z_n) = \emptyset$, which contradicts the fact that $x \in \bigcap_{n=1}^{\infty} \mathrm{Cl}_{aX} f^{-1}(Z_n)$. By (2.10) there is a prime z-filter $\mathfrak{L}$ contained in $f_\Sigma(\sigma^x)$. Obviously $\mathfrak{L}$ has the c.i.p. and so converges to a certain $y_x \in Y$.

We next give a concrete realization of the Hewitt real-compactification of a Tychonoff space X. Let X be assigned the EF-proximity $\delta = \delta_F$ (2.3). Let vX be the family of all clusters generated by real z-ultrafilters in X (see (2.14)) and let it be assigned the A-topology. Let Y be any real-compact space and let it be assigned any compatible EF-proximity. By (3.7), every continuous function $f\colon X \to Y$ has a continuous extension $f_\Sigma\colon vX \to \Sigma_Y$. In view of Theorem (3.12), in order to prove that vX is the Hewitt real-compactification of X (Gillman and Jerison [5], p. 118), it is sufficient to show that for each $\sigma \in vX$, $f_\Sigma(\sigma)$ converges to $y_\sigma \in Y$. If $\sigma \in vX$, then $\sigma = \sigma(\mathfrak{L})$ for some real z-ultrafilter $\mathfrak{L}$ in X. Clearly $f^{\#}(\mathfrak{L}) \subset f_\Sigma(\sigma)$ and $f^{\#}(\mathfrak{L})$ is a prime z-filter in Y (2.13). Since Y is real-compact, $f^{\#}(\mathfrak{L})$ converges to $y_\sigma \in Y$ and consequently $f_\Sigma(\sigma)$ also converges to y_σ. Thus we have

(6.2) THEOREM (See Alo and Shapiro [1]). *Let X be a Tychonoff space and let it be assigned the proximity δ_F. The Hewitt real-compactification of X is the space of all clusters generated by the real-z-ultrafilters in X, the space being assigned the A-topology.*

The above result throws some light on a result due to Gillman–Jerison quoted in McDowell [8] and provides an easy proof of it. First we note that from (i) 6F (3), p. 94 and (ii) 8F, p. 126 of Gillman and Jerison [5] it follows that:

(6.3) LEMMA. *If X is dense in a real-compact space aX, then every real prime z-filter in X converges in aX.*

Let X and Y be Tychonoff spaces with the EF-proximity δ_F. If $f\colon X \to Y$ is continuous, then, as in the argument preceding Theorem (6.2), $f_\Sigma(\sigma)$ (for $\sigma \in vX$) is a bunch containing a real prime z-filter in Y. By Lemma (6.3), $f_\Sigma(\sigma)$ converges to a point $y_\sigma \in vY$. By (3.12) the map $f_v\colon vX \to vY$ which assigns to each $\sigma \in yX$ the unique limit of $f_\Sigma(\sigma)$ in vY is continuous. Thus we have

(6.4) THEOREM. *Every continuous function f from a Tychonoff space X to a Tychonoff space Y can be extended to a continuous function $f_v\colon vX \to vY$.*

7. Application to Wallman extensions. Let X and Y be T_1-spaces and let each be assigned the LO-proximity δ_0. The Wallman extension (compactification) wX of X is the space of all closed ultrafilters with the Wallman or w-topology, whose closed subsets have the family $\{A^*\colon A$ closed in $X\}$, where $A^* = \{\mathcal{F} \in wX\colon A \in \mathcal{F}\}$, as a base. It is easy to show that the Wallman extension is homeomorphic to the space of bunches generated by all closed ultrafilters in X with the A-topology, and we assume this. (Note that if $\mathfrak{L}$ is a closed ultrafilter, then $b(\mathfrak{L}) = \{E \subset X\colon E^- \in \mathfrak{L}\}$ is a bunch.)

In general, if $\mathfrak{L}$ is a closed ultrafilter in X and $f\colon X \to Y$ is continuous, then $f^{\#}(\mathfrak{L}) = \{E\colon E$ closed in Y and $f^{-1}(E) \in \mathfrak{L}\}$ is a prime closed filter. However, we have

(7.1) LEMMA. *Let $f\colon X \to Y$ be continuous and closed. Then if $\mathfrak{L}$ is a closed ultrafilter in X, $f^{\#}(\mathfrak{L})$ is a closed ultrafilter in Y.*

Proof. We first note that $f(\mathfrak{L}) = \{f(L)\colon L \in \mathfrak{L}\} \subset f^{\#}(\mathfrak{L})$. This follows from the fact that $f(L)$ is closed in Y (for L closed in X) and $f^{-1}(f(L)) \supset L \in \mathfrak{L}$. To show that $f^{\#}(\mathfrak{L})$ is maximal we must show that if E is closed in Y and $E \cap M \neq \emptyset$ for every $M \in f^{\#}(\mathfrak{L})$, then $E \in f^{\#}(\mathfrak{L})$. In particular, $E \cap f(L) \neq \emptyset$ for each $L \in \mathfrak{L}$ (since $f(\mathfrak{L}) \subset f^{\#}(\mathfrak{L})$), i.e., $f^{-1}(E) \cap L \neq \emptyset$ for each $L \in \mathfrak{L}$. But $\mathfrak{L}$ is maximal and so $f^{-1}(E) \in \mathfrak{L}$, i.e., $E \in f^{\#}(\mathfrak{L})$.

We now prove a result due to Ponomarev [11].

(7.2) THEOREM. *Let X and Y be T_1-spaces and let $f\colon X \to Y$ be continuous and closed. Then f has a continuous extension $wf\colon wX \to wY$; further, if f is onto, then so is wf.*

Proof. Let X and Y have the LO-proximity δ_0. Then f is proximally continuous (2.6), and so f has a continuous extension $f_\Sigma\colon wX \to \Sigma_Y$. (We are assuming that wX is the set of bunches generated by the closed ultrafilters). But in view of Lemma (7.1) and the easily verified fact that $f_\Sigma\big(b(\mathfrak{L})\big) = b\big(f^{\#}(\mathfrak{L})\big) \in wY$, f_Σ maps wX to wY. If f is onto, and if $b(\mathfrak{L}')$ is a bunch generated by a closed ultrafilter $\mathfrak{L}'$ in Y, then $\{f^{-1}(L')\colon L' \in \mathfrak{L}'\}$ s a closed filter base in X and is contained in a closed ultrafilter $\mathfrak{L}$ in X. Clearly $\{f\big(f^{-1}(L')\big)\colon L' \in \mathfrak{L}'\} \subset \{f(L)\colon L \in \mathfrak{L}\}$ and since f is closed and $\mathfrak{L}'$ is maximal, $\mathfrak{L}' = f(\mathfrak{L})$. Thus $f_\Sigma\big(b(\mathfrak{L})\big) = b(\mathfrak{L}')$ and wf is onto.

Note added in proof. For further rnvestigations on Wallman extensions see authors' paper *Wallman compactifications and Wallman realcompactifications*, J. Austr. Math. Soc. (in press).

References

[1] R. A. Alo and H. L. Shapiro, *$\mathfrak{L}$-realcompactifications and normal bases*, J. Austr. Math. Soc. 9 (1969), pp. 489–495.

[2] R. L. Blefko, *On E-compact spaces*, Ph. D. Thesis, The Pennsylvania State University, 1965.

[3] V. A. Efremovič, *The geometry of proximity, I*, Mat. Sb. 31 (1952), pp. 189–200.
[4] R. Engelking, *Remarks on real-compact spaces*, Fund. Math. 55 (1964), pp. 303–308.
[5] L. Gillman and M. Jerison, *Rings of continuous functions*, Princeton, N. J., 1960.
[6] S. Leader, *On clusters in proximity spaces*, Fund. Math. 47 (1959), pp. 205–213.
[7] M. W. Lodato, *On topologically induced generalized proximity relations*, Proc. Amer. Math. Soc. 15 (1964), pp. 417–422; II, Pacific J. Math. 17 (1966), pp. 131–135.
[8] R. H. McDowell, *Extensions of functions from dense subspaces*, Duke Math. J. 25 (1958), pp. 297–304.
[9] C. J. Mozzochi, *Symmetric generalized topological structures*, Trinity College, Hartford, Conn., 1968.
[10] S. A. Naimpally and B. D. Warrack, *Proximity spaces*, Cambridge, Mathematical Tract 59 (1970).
[11] V. I. Ponomarev, *On closed mappings*, Uspehi Mat. Nauk 14, No 4, (1959), pp. 203–206.
12] A. D. Taimanov, *On extension of continuou mappings of topological spaces*, Mat. Sb. 31 (1952), pp. 459–463.

MATHEMATICS DEPARTMENT,
INDIAN INSTITUTE OF TECHNOLOGY,
Kanpur, India

Reçu par la Rédaction le 22. 12. 1969

Cardinal algebras of functions and integration

by

Rolando Chuaqui (Santiago)

Introduction. The purpose of this work is to apply the methods of [1] to cardinal algebras of functions instead of algebras of sets. I believe that the results become more elegant in this way and somewhat stronger, because it is possible to obtain the integral directly.

In the first part it is proved that the non-negative Baire functions are a cardinal algebra, which is an interesting result in its own right. Finally, a complete characterization of the Lebesgue integral (and Lebesgue measure) is obtained in terms of translations.

As in the previous paper, [1], I shall quote the theorems in Tarski's book [2] by their number and a T.

I. Cardinal algebras of functions. I am first going to prove that the class of non-negative Baire functions is a cardinal algebra. I will adopt the following notation: $\mathcal{R}^+$ is the set of non-negative real numbers; $\wedge, \vee$ the lattice operations on the class of functions; ${}^{B}A$ the class of functions from B into A.

THEOREM 1.1. *Let $F \subseteq {}^{E}\mathcal{R}^+$ such that*

(i) *If $f, g \in F$ then $f+g$, $(f-g)\vee 0$, $f\wedge g \in F$,*

(ii) *If for every $n<\infty$, $f_n \in F$, $f_n \leqslant f_{n+1}$ and $\lim f_n = f < \infty$, then $f \in F$,*

(iii) *If for every $n<\infty$, $f_n \in F$ and $f_{n+1} \leqslant f_n$, then $\lim f_n \in F$.*

Then $\langle F, +, \sum\rangle$ is a finitely closed generalized cardinal algebra where $\sum_{i<\infty} f_i$ is defined only if $\sum_{i<\infty} f_i < \infty$.

Proof. We note that

(1) If $\sum_{i<\infty} f_i < \infty$ with $f_i \in F$, $\sum_{i<\infty} f_i \in F$.

So

(2) If $\sum_{i<\infty} f_i \in F$, $\sum_{i<\infty} f_{i+n} \in F$ for all $n<\infty$.

(3) If $\sum_{i<\infty} f_i \in F$, $g_i \leqslant f_i$ and $g_i \in F$ for all $i<\infty$, then $\sum_{i<\infty} g_i \in F$.