Completely regular compactifications

by

Anthony J. D'Aristotle (New York City)

1. Introduction. In [3] Frink studied Wallman-type Hausdorff compactifications of Tychonoff spaces. In this paper we generalize Frink's results to obtain completely regular (and normal) compactifications of arbitrary completely regular spaces which are not Hausdorff or T_1. We also derive necessary and sufficient conditions for extending continuous functions defined on the original space to continuous functions on the compactification.

In [3] Frink provided an internal characterization of Tychonoff or completely regular T_1 spaces in terms of the notion of a *normal base* for the closed sets of a space X. A normal base Z for the closed sets of a topological space X is a base for the closed sets which is a *disjunctive ring* of sets, disjoint members of which may be separated by disjoint complements of members of Z. Frink showed that if Z is a normal base for a T_1 space X, then the Wallman space $\omega(Z)$ consisting of the Z-*ultrafilters*, is a Hausdorff compactification of X. It is then clear that X is a Tychonoff space. On the other hand, if X is Tychonoff then the family of all zero sets of real continuous functions over X is a normal base for the closed sets. Therefore, a T_1 space is Tychonoff if and only if it has a normal base. Frink pointed out that by choosing different normal bases Z for X, we may obtain different Hausdorff compactifications of X in the form of Wallman spaces $\omega(Z)$.

In this paper we modify Frink's technique to produce a Wallman space $\chi(Z)$ which is a completely regular but not necessarily Hausdorff compactification of a given topological space X with normal base Z. Instead of taking all the Z-ultrafilters on X, we form $\chi(Z)$ by adding the free Z-ultrafilters $\mathcal{U}$ of X to the original points of the space. It may then be seen that the family of all sets A^* of the form $A \cup \{\mathcal{A} \in \mathcal{U} | A \in \mathcal{A}\}$ for $A \in Z$ is a base (of closed sets) for a topology on $\chi(Z)$. We show that many of the basic results about the spaces $\omega(Z)$ can be carried over to this more general setting.

2. Definitions. A family $\mathcal{F}$ of closed sets of a topological space X is said to be *disjunctive* if, given any closed set A and any point x not in A there exists a closed set $F \in \mathcal{F}$ which contains x and is disjoint from A.

A family of sets is called a *ring of sets* if it contains all finite unions and finite intersections of its members. Every ring of sets is a lattice.

A base Z for the closed sets of a topological space X is called a *normal base* if it is a disjunctive ring of closed sets such that any two disjoint members A and B of Z are subsets respectively of disjoint complements C' and D' of members C and D of Z; that is, $A \subseteq C'$, $B \subseteq D'$, and $C' \cap D' = \emptyset$. More generally, a family $\mathcal{F}$ of sets of a space X is said to be a normal family if any two disjoint members A and B of $\mathcal{F}$ are contained in disjoint complements C' and D' of members C and D of $\mathcal{F}$.

A family $\mathcal{F}$ of closed sets of a topological space X is said to be *separating* if it separates points from closed sets; that is, given any closed set S and any point x not in S, there exists sets A and B in $\mathcal{F}$ such that $x \in A$, $S \subseteq B$, and $A \cap B = \emptyset$. It is clear that a family $\mathcal{F}$ of closed sets is separating if and only if it is a disjunctive family which is a base for the closed sets.

Members of a normal base will be called *Z-sets* and their complements *Z-complements*. The Z-complements form a base for the open sets of the space.

A proper subset of a normal base Z is called a *Z-filter* if it is closed under finite intersection and contains every superset in Z of each of its members. No Z-filter contains the empty set $\emptyset$.

A *Z-ultrafilter* is a maximal Z-filter. It follows from Zorn's lemma that every Z-filter is contained in at least one Z-ultrafilter.

A Z-filter $\mathcal{A}$ on a topological space X is said to be *fixed* if there is an element p of X with $p \in \bigcap \{A\colon A \in \mathcal{A}\}$. A Z-filter which is not fixed is said to be *free*.

3. Generalization of Frink's result. We have already noted that in [3], Frink provided an internal description of Tychonoff spaces (completely regular and T_1). A T_1 space X is Tychonoff if and only if it has a normal base. In [8], Steiner extended this result to cover all completely regular spaces. A topological space is completely regular if and only if it possesses a normal separating family of closed sets. We shall refer to this result in proving the following theorem.

THEOREM 1. *Let X be a completely regular topological space. Then to each normal base Z for X there corresponds a completely regular compactification $\chi(Z)$ of X.*

Proof. If Z is a normal base for X, let $\mathfrak{U}$ be the collection of all free Z-ultrafilters on X and let $\chi(Z) = X \cup \mathfrak{U}$. We define a topology for $\chi(Z)$ by taking as a base for the closed sets the family $\mathcal{F}$ of all sets A^* of the form $A \cup \{\mathcal{A} \in \mathfrak{U}\colon A \in \mathcal{A}\}$ where $A \in Z$. The fact that $A_1^* \cup A_2^* = (A_1 \cup A_2)^*$ implies that the sets A^* do indeed form a base.

We note that the topological space X with base Z is homeomorphic to X considered as a subspace of $\chi(Z)$. This is easy to see because the basic closed sets for $X \subseteq \chi(Z)$ are of the form $A^* \cap X = A$. That X is dense in $\chi(Z)$ may be seen as follows. If $X \subseteq A^* = A \cup \{\mathcal{A} \in \mathfrak{U}\colon A \in \mathcal{A}\}$ we must have $A = X$. But $X \in \mathcal{A}$ for all $\mathcal{A} \in \mathfrak{U}$ and so $X \subseteq A^*$ implies that $A^* = X^* = \chi(Z)$.

We next show that $\chi(Z)$ is a compact topological space. For if $\{A_\lambda^*\}_{\lambda \in \Lambda}$ is a family of basic closed sets with the finite intersection property, then the corresponding family $\{A_\lambda\}_{\lambda \in \Lambda}$ of Z-sets has the finite intersection property. To see this note that $\bigcap_{i=1}^{i=n} A_{\lambda_i} = \emptyset$ for $\lambda_i \in \Lambda$ together with $\bigcap_{i=1}^{i=n} A_{\lambda_i}^* \neq \emptyset$ would imply there exists $\mathcal{B} \in \mathfrak{U}$ with $A_{\lambda_i} \in \mathcal{B}$ for $1 \leqslant i \leqslant n$. Therefore we must have $\bigcap_{i=1}^{i=n} A_{\lambda_i} \in \mathcal{B}$ and so $\bigcap_{i=1}^{i=n} A_{\lambda_i} \neq \emptyset$ which is a contradiction. Now either there exists a $p \in \bigcap_{\lambda \in \Lambda} A_\lambda$ in which case $p \in \bigcap_{\lambda \in \Lambda} A_\lambda^*$ or $\bigcap_{\lambda \in \Lambda} A_\lambda = \emptyset$. In the latter situation the familly $\{A_\lambda\}_{\lambda \in \Lambda}$ generates a free Z-ultrafilter $\mathcal{C}$ and clearly $\mathcal{C} \in \bigcap_{\lambda \in \Lambda} A_\lambda^*$. Hence this family has a non-empty intersection.

Finally it may be seen that $\chi(Z)$ is a completely regular space. In light of the remarks made prior to the statement of this theorem, it suffices to exhibit a normal separating family of closed sets in $\chi(Z)$. It is easily verified that the family $\mathcal{F}$ of sets A^* for A in Z is such a family. This completes the proof of the theorem.

Since a compact regular space is normal, it follows that the spaces $\chi(Z)$ are normal compactifications of X.

In [3], as we noted in the Introduction, Frink showed that if Z is a normal base for a T_1 space X, then the Wallman spaces $\omega(Z)$ consisting of the Z-ultrafilters, is a Hausdorff compactification of X. If we require that X be a T_1 space, then it is quite easy to verify that the space $\chi(Z)$ is homeomorphic to $\omega(Z)$.

4. Continuous extensions. Frink also showed that the real functions over a Tychonoff space X which may be extended to continuous real functions over the compactification $\omega(Z)$ are those which are *Z-uniformly* continuous. In light of the above remarks, it is natural to try to generalize this result. In making this extension, we found a proof which is simpler and more direct than the original proof.

Definition. A real function $f(x)$ defined over a completely regular space X with normal base Z is said to be *Z-uniformly continuous* if for every positive epsilon there exists an open cover of X by Z-complements, on each of which the oscillation of $f(x)$ is less than epsilon.

Theorem 2. *A real function $f(x)$ defined over a completely regular space X with normal base Z can be extended to a real continuous function over the compactification $\chi(Z)$ if and only if $f(x)$ is Z-uniformly continuous.*

Proof. We first prove that the condition is necessary. For suppose $f(y)$ is a continuous real function defined over the compact space $\chi(Z)$. Given a positive epsilon, it is clear that $\chi(Z) \subseteq \bigcup_{y\in\chi(Z)} \{f^{-1}(S(f(y), \varepsilon/2))\}$ where $S(f(y), \varepsilon/2)$ is the spherical neighborhood of $f(y)$ and $\varepsilon/2$. But each set $f^{-1}(S(f(y), \varepsilon/2))$ is a union of basic open sets of the form $\chi(Z)-A^* = (X-A) \cup \{\mathcal{A} \in \mathfrak{U} \mid \exists P \in \mathcal{A}$ with $P \subseteq X-A\}$. Since $\chi(Z)$ is compact we may extract a finite cover of $\chi(Z)$ consisting of basic open sets, on each of which the oscillation of $f(y)$ is less than epsilon. We have that $\chi(Z) \subseteq (\chi(Z)-A_1^*) \cup (\chi(Z)-A_2^*) \cup \ldots \cup (\chi(Z)-A_n^*)$. It then becomes obvious that the Z-complements $X-A_1, X-A_2, \ldots, X-A_n$ cover X, and on each of them the oscillation of the restriction $f(x)$ of $f(y)$ to X is less than epsilon. Hence $f(x)$ is Z-uniformly continuous.

Conversely, suppose the real function $f(x)$ is Z-uniformly continuous on a completely regular space X with normal base Z. We define a function g which extends f from X to $\chi(Z)$ as follows. Now $\chi(Z) = X \cup \mathfrak{U}$ and if $x \in X$ we let $g(x) = f(x)$. If $\mathcal{A} \in \mathfrak{U}$ then the family $S_{\mathcal{A}} = \{f(A): A \in \mathcal{A}\}$ has the finite intersection property and is therefore a subbase for the filter $\mathcal{F}_{\mathcal{A}}$ consisting of all supersets of finite intersections of members of $S_{\mathcal{A}}$. The filter $\mathcal{F}_{\mathcal{A}}$ is a Cauchy filter and therefore converges uniquely to a real number which we call $g(\mathcal{A})$.

That g is continuous at each point of X is readily verified. It remains to show that g is continuous at each point $\mathcal{B} \in \mathfrak{U}$. Let the family $\{X-C_j\}_{j=1}^{j=m}$ be a finite cover of X by Z-complements, on each member of which the oscillation of $f(x)$ is less than $\varepsilon/3$. We may suppose that $C_1 \notin \mathcal{B}$ so that there is an element $Q \in \mathcal{B}$ with $Q \subseteq X-C_1$. We show that

$$g[(X-C_1) \cup \{\mathcal{A} \in \mathfrak{U}: \exists P \in \mathcal{A} \text{ with } P \subseteq X-C_1\}] \subseteq S(g(\mathcal{B}), \varepsilon).$$

Now $g(\mathcal{B}) \in \mathrm{cl}_R f(Q)$ and we choose $q \in Q$ so that $|g(\mathcal{B})-f(q)| < \varepsilon/3$. If $y \in X-C_1$ we then have

$$|g(\mathcal{B})-g(y)| \leqslant |g(\mathcal{B})-f(q)|+|f(q)-g(y)| < \frac{\varepsilon}{3}+\frac{\varepsilon}{3} < \varepsilon.$$

It therefore follows that $g(X-C_1) \subseteq S(g(\mathcal{B}), \varepsilon)$. If $\mathcal{A} \in \mathfrak{U}$ and there is

an $S \in \mathcal{A}$ with $S \subseteq X-C_1$, we choose a point $s \in S$ satisfying $|g(\mathcal{A})-f(s)| < \varepsilon/3$. The points q and s are members of $X-C_1$ and so

$$|g(\mathcal{A})-g(\mathcal{B})| \leqslant |g(\mathcal{A})-f(s)|+|f(s)-f(q)|+|f(q)-g(\mathcal{B})| < \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} = \varepsilon.$$

Thus we have established that g is a continuous, real-valued function on $\chi(Z)$.

If X is a Hausdorff space and Z is the normal base consisting of the zero sets of X, then $\chi(Z)$ is the Stone–Čech compactification of X and it follows that X is C^*-embedded in $\chi(Z)$. The above theorem therefore implies that every bounded, continuous real-valued function on X is Z (zero set)-uniformly continuous. We get immediately the following result.

Corollary 1. *Let X be a completely regular topological space and let Z be the normal base consisting of the zero sets of X. Then every bounded, continuous real-valued function on X has an extension to a bounded, continuous real-valued function on $\chi(Z)$.*

Proof. We note that every bounded, continuous real-valued function on X is zero-set uniformly continuous and appeal to Theorem 2.

5. The one-point compactification. In [2], Brooks has shown that the one-point Hausdorff compactification of a *locally compact* Hausdorff space may always be obtained as a Wallman space $\omega(Z)$, where the normal base Z consists of the zero sets of those continuous real functions on X which are constant on the complement of a compact set. (A topological space X is said to be *locally compact* if each point of the space is contained in a compact neighborhood.) We now generalize this result and note that the work of Alo and Shapiro in [1] was most helpful in doing so.

The following lemma is easily verified.

Lemma 1. *Let X be a topological space with normal base Z, and let $\chi(Z) = X \cup \mathfrak{U}$ be the completely regular compactification of X corresponding to Z. Then each point $\mathcal{C} \in \mathfrak{U}$ is a closed subset of $\chi(Z)$.*

It is known that if X is a locally compact, Hausdorff space, then the zero sets of continuous real-valued functions which are constant on the complement of a compact subset of X form a normal base for X. It isn't very difficult to obtain the following result.

Lemma 2. *Let X be a completely regular, locally compact topological space. Then Z equal to the collection of zero sets of continuous, real-valued functions which are constant on the complement of a compact subset of X is a normal base for X.*

Lemma 3. *Let X be a completely regular, locally compact topological space. If we take Z to be the normal base for X consisting of the zero sets of continuous, real-valued functions on X which are constant on the comple-*

ment of a compact subset of X, then to each $p \in X$ there exists an $A_p \in Z$ with A_p compact and $p \in A_p^0 \subseteq A_p$. ($A_p^0$ denotes the interior of A_p.)

Proof. If $p \in X$ there is an open set U in X with $p \in U \subseteq \mathrm{cl}_X U$ where $\mathrm{cl}_X U$ is a compact subset of X. There is a continuous mapping h from X to $[0,1]$ with $h(p) = 0$ and $h(X-U) = 1$. If we let $A_p = \{x \in X: h(x) \leqslant \frac{1}{2}\}$, then A_p is compact since it is a closed subset of the compact set $\mathrm{cl}_X U$. Moreover, $p \in A_p^0$ since $\{x \in X: h(x) < \frac{1}{2}\}$ is an open set containing p and contained in A_p. Also, it is clear that $A_p \in Z$ since it is the zero set of the continuous function $h(x) - \frac{1}{2} + |h(x) - \frac{1}{2}|$ which is constant on the complement of the compact set $\mathrm{cl}_X U$.

THEOREM 3. *Let X be a completely regular, locally compact topological space. Then there is a normal base Z for X such that the one-point compactification $Y = X \cup \{\infty\}$ of X is homeomorphic to $\chi(Z)$.*

Proof. Let Z be the collection of zero sets of continuous, real-valued functions on X which are constant on the complement of a compact subset of X. Z is a normal base for X by Lemma 2. If we let $\mathcal{B} = \{A \in Z |\ A$ is closed but not compact$\}$, it is easily verified that $\mathcal{B}$ is the only free Z-ultrafilter on X.

We now verify that $Y = X \cup \{\infty\}$ is homeomorphic to $\chi(Z) = X \cup \mathcal{U}$. We define a function f from $X \cup \{\infty\}$ to $X \cup \mathcal{U}$ by $f(x) = x$ for $x \in X$ and $f(\infty) = \mathcal{B}$. Then f is obviously 1-1, and it is an onto map since we have shown above that $\mathcal{U} = \{\mathcal{B}\}$.

That f is continuous may be seen as follows. Each basic closed set in $\chi(Z)$ is of the form $A \cup \{\mathcal{A} \in \mathcal{U}: A \in \mathcal{A}\} = A^*$ where $A \in Z$. Since $\mathcal{U} = \{\mathcal{B}\}$ the basic closed sets of $\chi(Z)$ are the closed, compact member of Z together with sets of the form $A \cup \{\mathcal{B}\}$ where $A \in \mathcal{B}$. If A is a closed, compact member of Z, then $f^{-1}(A) = A$ is a closed, compact subset of X and therefore closed in $Y = X \cup \{\infty\}$. If $A \in \mathcal{B}$ then $f^{-1}(A \cup \{\mathcal{B}\}) = A \cup \{\infty\}$ which is closed in Y. The inverse image of each basic closed in $\chi(Z)$ is closed in $Y = X \cup \{\infty\}$ and so f is continuous.

It remains to show that f is a closed map. The closed sets in $Y = X \cup \{\infty\}$ consist of the closed, compact subsets of X, and subsets of Y which are of the form $F \cup \{\infty\}$ where F is a closed subset of X. If Q is a closed, compact subset of X, then from Lemma 3 we see that to each $q \in Q$ there is a compact member A_q of Z with $q \in A_q^0$. Therefore the sets A_q^0 form an open covering of Q, and since Q is compact there is a finite number of elements $q_1, q_2, \ldots, q_n \in Q$ with $Q \subseteq A_{q_1}^0 \cup A_{q_2}^0 \cup \ldots \cup A_{q_n}^0$. Letting $B = A_{q_1} \cup \ldots \cup A_{q_n}$ we see that B is a compact member of Z with $Q \subseteq B$. Now Z is a base for the closed subsets of X and so Q is an intersection of members of Z. Suppose $Q = \bigcap_{\lambda \in \Lambda} \{A_\lambda: A_\lambda \in Z\}$. Clearly we may assume $A_\lambda = B$ for some $\lambda \in \Lambda$. Let Λ_1 be the set of all $\sigma \in \Lambda$ such that A_σ is a closed but not compact member of Z, and let Λ_2 be the set of all $\varrho \in \Lambda$ with A_ϱ a closed, compact member of Z. We note that $\Lambda_2 \neq \emptyset$. Thus, $f(Q) = Q = [\bigcap_{\sigma \in \Lambda_1} (A_\sigma \cup \{\mathcal{B}\})] \cap [\bigcap_{\varrho \in \Lambda_2} A_\varrho]$, and it is now clear that $f(Q)$ is closed in $\chi(Z)$. We now consider the case where the closed set K of Y is of the form $F \cup \{\infty\}$ where F is closed in X. If F is, in addition, compact then $f(F \cup \{\infty\}) = f(F) \cup \{\mathcal{B}\}$. We have just verified that $f(F)$ is closed in $\chi(Z)$, and from Lemma 1 we have that $\{\mathcal{B}\}$ is closed there. It follows that $f(K)$ is closed in $\chi(Z)$. If F is closed but not compact, suppose $F = \bigcap_{\delta \in \Delta} A_\delta$ where each A_δ is a member of Z. We note that each A_δ is not compact for otherwise F would be compact. Then $f(F \cup \{\infty\}) = f(F) \cup \{\mathcal{B}\} = F \cup \{\mathcal{B}\} = [\bigcap_{\delta \in \Delta} A_\delta] \cup \{\mathcal{B}\} = \bigcap_{\delta \in \Delta} (A_\delta \cup \{\mathcal{B}\})$. But each $A_\delta \cup \{\mathcal{B}\}$ is a basic closed set in $\chi(Z)$ and so $f(F \cup \{\infty\})$ is closed in $\chi(Z)$. We conclude that f is a closed map and hence a homeomorphism from $Y = X \cup \{\infty\}$ to $\chi(Z) = X \cup \{\mathcal{B}\}$.

References

[1] R. A. Alo and H. L. Shapiro, *A note on compactifications and semi-normal spaces*, J. Austral. Math. Soc. 8 (1968), pp. 102–108.

[2] R. M. Brooks, *On Wallman compactifications*, Fund. Math. 60 (1967), pp. 157–173.

[3] Orrin Frink, *Compactifications and semi-normal spaces*, Amer. J. Math. 86 (1964), pp. 602–607.

[4] L. Gillman and M. Jerison, *Rings of continuous functions*, New York 1960.

[5] J. R. Kelley, *General topology*, New York 1955.

[6] W. J. Pervin, *Foundations of general topology*, New York 1964.

[7] W. Rudin, *Principles of mathematical analysis*, New York 1964.

[8] E. J. Steiner, *Normal families and completely regular spaces*, Duke Math. J. 33 (1966), pp. 743–745.

[9] H. Wallman, *Lattices and topological spaces*, Ann. of Math. 42 (2) (1938), pp. 112–126.

THE CITY COLLEGE OF NEW YORK

Reçu par la Rédaction le 23. 12. 1969