On composants of Hausdorff continua *

by

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Introduction. Sorgenfrey [4] proved that every compact nondegenerate unicoherent metric continuum which is not a triod is irreducible (has more than one composant). In this paper it is shown that every compact nondegenerate unicoherent Hausdorff continuum which is not a triod and which is not irreducible contains a nondegenerate indecomposable continuum which is not irreducible. Certain results of Miller's on E-subcontinua of metric continua [2] are proven for Hausdorff continua. A definition is given which is used to characterize the number of composants in compact metric continua and compact decomposable Hausdorff continua.

Definitions and notation. Throughout this paper M denotes a nondegenerate compact Hausdorff continuum. Only in the last two theorems and corollary is M assumed to be metric. M is decomposable into H and K means H and K are proper subcontinua of M whose sum is M. A point p is said *to have property E with respect to* M if and only if p belongs to M and M is not decomposable into two continua each containing p. E_M denotes the set to which p belongs if and only if p has property E with respect to M. A composant of M is a point set K such that, for some point p of M, x belongs to K if and only if there is a proper subcontinuum of M containing both p and x. A continuum is irreducible if and only if it is irreducible between two of its points, i.e. has more than one composant. The continuum M is a triod means there exists a subcontinuum K of M such that $M-K$ has at least three components. M is unicoherent means that if M is decomposable into H and K then $H \cap K$ is a continuum.

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1. On Hausdorff continua.

THEOREM 1. *If E_M is a connected proper subset of M then M is decomposable into two continua, H and K, such that H is indecomposable and does not intersect every composant of K, and K intersects every composant of H.*

Proof. M is decomposable into two continua one containing no point of E_M. So there exists a collection G of all subcontinua of M such that if H is any one of them then H contains no point of E_M and there exists another subcontinuum H' of M such that M is decomposable into H and H'. Suppose G' is a nested subcollection of G such that I, the common part of all the elements of G', does not belong to G. I is a continuum. Let p denote some point of I. So p is not in E_M. Therefore, there exists a proper subcontinuum H_p of M containing p and E_M. If $H_p \cup I$ is M, I belongs to G, a contradiction. So $H_p \cup I$ is not M. Let q denote a point of M not in $H_p \cup I$. Since q is not in I, some element J of G' does not contain q. So there exists a proper subcontinuum J' of M such that M is decomposable into J and J' and J' contains E_M. Since q is not in $H_p \cup J$, $H_p \cup J$ is a proper subcontinuum of M. So M is decomposable into $H_p \cup J$ and J'. But H_p and J' contain E_M, which is impossible. So I belongs to G. By the minimal principle, there exists an element H of G such that it contains no other element of G. Supposing H is decomposable lends easily to a contradiction. So H is indecomposable.

With the aid of Theorem 2–16 and Theorem 3–47 of [1] the following can be established: If J is a compact continuum intersecting H but not intersecting the composant Z of H, every subcontinuum of $H \cup J$ intersecting Z and J contains H.

Suppose J is a continuum such that M is decomposable into H and J, Z is a composant of H not intersected by J, and p is a point of Z. Since p is in H, p does not have property E with respect to M, so M is decomposable into two continua, U and V both containing p such that V contains a point q of E_M. V is a proper subcontinuum of $H \cup J$ containing the point p of Z and the point q of J. So by the results of the previous paragraph, V contains H. Since V contains q and H, and M is $J \cup H$, M is $J \cup V$. V contains q and q is in E_M, so q is in J. Since M is decomposable into J and V, q does not have property E with respect to M, a contradiction. So if J is a continuum such that M is decomposable into H and J, then J intersects every composant of H.

From the definition of G, M is decomposable into H and H'. Suppose $H'-(H \cap H')$ is not connected. Since H' contains E_M and H contains no point of E_M, some component T of $H'-(H \cap H')$ contains a point q of E_M. Suppose W is another component of $H'-(H \cap H')$. Since H' is a continuum and $H \cap H'$ is closed, the closure of each component of $H'-(H \cap H')$ intersects $H \cap H'$ and therefore intersects H. So $\overline{T} \cup H$ is a continuum. Since T and W are components of $H'-(H \cap H')$, $\overline{T}$ contains no point of W. Since H does not intersect W, the continuum $\overline{T} \cup H$ is a proper subcontinuum of M. Since M is the sum of H and H', M is decomposable into $H \cup \overline{T}$ and H'. Since H' contains the point q of E_M and $H \cup \overline{T}$ contains q, q does not have the property E with respect to M. This involves a contradiction. So $H'-(H \cap H')$ is connected. It follows that $\overline{H'-(H \cap H')}$ is a continuum and M is decomposable into H and $\overline{H'-(H \cap H')}$.

Let K denote the continuum $\overline{H'-(H \cap H')}$. Suppose H intersects every composant of K, o is a point of E_M and Z is the composant of K such that the point x belongs to Z if and only if there exists a proper subcontinuum of K containing o and x. Since H intersects every composant of K, there exists a point p of Z in H and a proper subcontinuum po of K containing p and o. Since po contains the point p of H, $H \cup po$ is a continuum. If $H \cup po$ is M, then po contains $H'-(H \cap H')$ and since po is closed it contains $\overline{H'-(H \cap H')}$, which is K. So po is not a proper subcontinuum of K, a contradiction. So $H \cup po$ is not M, and is therefore a proper subcontinuum of M.

Since M is decomposable into H and H', and $H \cup po$ is a proper subcontinuum of M, M is decomposable into $H \cup po$ and H'. Since o is a point of E_M, H' contains o. But since $H \cup po$ also contains o, o does not have property E with respect to M, a contradiction.

THEOREM 2. *If M is a compact continuum such that E_M has two components then E_M has only two components and no proper subcontinuum of M intersects both of them.*

Proof. Suppose E_M has two components one containing the point a and another containing the point b, and for each two continua whose sum is M one contains both a and b. G denotes the collection to which H belongs if and only if H is a proper subcontinuum of M containing a and b such that for some proper subcontinuum H' of M, M is $H \cup H'$. Suppose G' is a nested subcollection of G and I denotes the common part of all the elements of G'. I is a proper subcontinuum of M containing $\{a\} \cup \{b\}$. Where q denotes a point of I not in E_M, M is decomposable into two continua each containing q. Let U and V denote two such continua where U contains a and b. If the continuum $V \cup I$ is not M then M is decomposable into $V \cup I$ and U, and a does not have property E with respect to M. So $V \cup I$ is M and therefore I belongs to G. By the minimal principle there exists an element H of G which contains no other element of G and a subcontinuum T of M such that M is decomposable into H and T.

Supposing H is indecomposable and K denotes the composant of H containing a leads to these contradictory statements: 1) K is a subset of E_M and 2) b belongs to a component of E_M different from the one containing a, and b is a limit point of K.

Let X and Y denote two proper subcontinua of H whose sum is H. Suppose X contains a and b, and q is a point of $T \cap H$. M is not $X \cup T$ and M is not $Y \cup T$, since U and V are proper subcontinua of H. If q is in X, M is decomposable into H and $X \cup T$, and a is in their common part, a contradiction. If q is in Y, M is decomposable into X and $Y \cup T$, and X is a proper subset of H in the collection G. Since q is in H, this contradicts X containing a and b. Similarly Y does not contain a and b.

Suppose T intersects X. M is not $T \cup X$ since $T \cup X$ does not contain both a and b. So M is decomposable into $T \cup X$ and Y. So by the first supposition, either $T \cup X$ or Y contains a and b, which is impossible. It follows that M is decomposable into two continua U and V, one containing a and the other containing b.

Suppose T is a proper subcontinuum of M containing a and b. $U \cup T$ is a continuum and since U contains a or b, and T contains a and b, $U \cup T$ is not M. So M is decomposable into $U \cup T$ and V, leading to either a or b not lying in E_M. So no proper subcontinuum of M contains a and b. Now assuming that E_M has three components leads easily to a contradiction.

Corollary. *If E_M has two components, no point of E_M belongs to every composant of M.*

Theorem 3. *If T is a component of E_M then either T is closed or $\overline{T}$ is indecomposable.*

Proof. Suppose T is not closed, $\overline{T}$ is decomposable into H and K, and p is a point of $\overline{T}-T$. p is not in E_M, so M is decomposable into two continua X and Y each containing p. T lies in X or Y and p is in H or K. Suppose T is a subset of X, and p is in H. $H \cup Y$ is a proper subcontinuum of M containing a point t of T. M is decomposable into $H \cup Y$ and X and each contains t, a contradiction.

Theorem 4. *Suppose M is not indecomposable. Then either M has only one composant or M has only three composants. Furthermore, M has only one composant if and only if either no point has property E with respect to M or E_M is a connected proper subset of M, and M has only three composants if and only if E_M is not connected.*

Proof. By Theorem 2, either no point has property E with respect to M or E_M is a connected proper subset of M, or E_M is not connected and has only two components. If E_M has two components, U and V, it follows from Theorem 2 and its corollary that M has only three composants, namely M, $M-U$, and $M-V$. If E_M is a connected proper subset of M it follows from Theorem 1 that M is not irreducible between any two of its points. If no point has property E with respect to M it follows from the definition of property E that M is not irreducible between any two of its points. So in each of these two cases M has only one composant.

Theorem 5. *If M is irreducible between two of its points and decomposable then each of the two components of E_M is a complement in M of a composant of M. Furthermore, each is a continuum unless its closure is indecomposable.*

Proof. This follows immediately through the use of Theorem 3 and the argument used in Theorem 4.

Theorem 6. *If M is unicoherent, not irreducible, and not a triod, then M contains a nondegenerate indecomposable continuum which is not irreducible.*

Proof. Suppose M contains no such indecomposable continuum. Since M is not indecomposable it follows from Theorem 4 that either no point has property E with respect to M or E_M is a connected proper subset of M. From Theorem 1 and M being unicoherent it follows that no point has property E with respect to M.

M is decomposable into two continua, H and K, and since M is not a triod, $M-(H \cap K)$ has only two components, U and V, where $U = H-(H \cap K) = M-K$ and $V = K-(H \cap K) = M-H$.

Suppose $\overline{U}$ is indecomposable and q is a point of U not in the composant of $\overline{U}$ intersected by the continuum $\overline{U} \cap K$. Since no point of M has property E with respect to M, M is decomposable into two continua, X and Y, each containing q. It follows that both X and Y contain $\overline{U}$. Since $\overline{U}-U = \overline{U} \cap K$ and is a continuum, $(X \cap Y)-U$ is a continuum. Furthermore, $M-[(X \cap Y)-U] = U \cup [M-(X \cap Y)]$ and $M-(X \cap Y) = A \cup B$ where A and B are mutually separated. So $M-[(X \cap Y)-U] = U \cup A \cup B$ and since U and $M-(X \cap Y)$ are mutually separated, $(X \cap Y)-U$ is a subcontinuum of M such that its complement in M has three components. So M is a triod, contradicting $\overline{U}$ being indecomposable. Similarly $\overline{V}$ is not indecomposable. Supposing some point of $\overline{U} \cap K$ does not belong to $E_{\overline{U}}$ will also contradict M not being a triod. Similarly, every point of $\overline{V} \cap H$ belongs to $E_{\overline{V}}$.

Suppose $E_{\overline{U}}$ is a connected proper subset of $\overline{U}$. It follows from Theorem 1 that $\overline{U}$ is decomposable into two continua, H' and K', where H' is indecomposable, K' intersects every composant of H', and K' contains $E_{\overline{U}}$. $K \cup K'$ is a proper subcontinuum of M intersecting every composant of H' and $M = H' \cup (K \cup K')$. So since M is unicoherent, H' has only one composant, a contradiction. So $E_{\overline{U}}$ is not a connected proper subset of $\overline{U}$. A similar statement is true about $\overline{V}$.

Since neither $\overline{U}$ nor $\overline{V}$ is indecomposable, it follows from Theorem 2 that $E_{\overline{U}}$ has only two components, E and E', and $E_{\overline{V}}$ has only two components, F and F', where E' contains $\bar{U} \cap K$ and F' contains $\bar{V} \cap H$. There exists a proper subcontinuum W of M which contains a point of E and a point of F. So W intersects U and V, therefore intersecting $H \cap K$. Since every subcontinuum of M intersecting E and $\bar{U} \cap K$ must contain $\bar{U}$, W contains $\bar{U}$. Similarly W contains $\bar{V}$. $M-W$ has at most two components. Let X denote a component of $M-W$ and if $M-W$ has two components, Y denotes the other.

Since $M = W \cup X \cup Y$, $(W \cup Y) \cap \bar{X}$ is a continuum containing some point q. $W \cap K$ and $W \cap H$ are continua containing q, and $W = (W \cap K) \cup (W \cap H)$. Since $W \cup Y$ is a continuum, $W \cup Y$ is decomposable into two continua, I and J where $I = (W \cap K) \cup Y$ and $J = (W \cap H) \cup Y$, and each contains q. So $M = \bar{X} \cup (I \cup J)$. If L denotes the continuum $[\bar{X} \cap (I \cup J)] \cup [I \cap (\bar{X} \cup J)] \cup [J \cap (\bar{X} \cup I)]$ then $M-L$ has three components, one a subset of $\bar{X}$, one a subset of I, and another a subset of J. This contradicts M not being a triod.

2. On metric continua. For the remainder of this paper M denotes a nondegenerate compact metric continuum.

THEOREM 7. *If M is decomposable and some point has property E with respect to M, then E_M is not connected.*

Proof. By Theorem 2, either E_M is a connected proper subset of M or E_M has only two components. Suppose E_M is a connected proper subset of M. By Theorem 1, M is decomposable into two continua, H and K, such that H is indecomposable and K intersects every composant of H. Let p denote a point of M. If p is in $H \cap K$, p does not have property E with respect to M. Suppose p is in $H-(H \cap K)$. There exists a proper subcontinuum L of H containing p and intersecting K. So M is decomposable into H and $K \cup L$ and each contains p. So p does not have property E with respect to M.

Suppose p is in $K-(H \cap K)$, q is a point of $H-(H \cap K)$ and $R_1, R_2, R_3, \ldots$ is a sequence of open sets closing down on q, $\bar{R}_1$ does not contain a point of K. By Theorem 136 from Chapter 1 of [3], the composant of H containing q is the sum of countably many proper subcontinua of $M, K_1, K_2, K_3, \ldots$ For each positive integer n, H_n, denotes the set to which h belongs if and only if h is a component of $H-(R_n \cap H)$ which contains a point of K. Suppose x is a limit point of the sum of the elements of H_n. Since each element of H_n is a continuum, there exists an infinite sequence of elements of H_n such that x is in the limiting set of that sequence. Let W denote the limiting set of such a sequence. Since each element of that sequence intersects K, W must intersect K. It follows that the continuum W is a subset of some element of H_n. So the sum of the elements of H_n is a closed point set.

Suppose h is a point of H not in the composant of H containing q. There exists a proper subcontinuum L of M containing h and intersecting K, and a positive integer n such that R_n contains no point of L. So L is a subset of some element of H_n. Since, for each positive integer n, the sum of K_n and all the elements of H_n is a closed point set, it follows that H is the sum of countably many closed point sets. With the use of Theorem 53 from Chapter 1 of [3], it follows that there exists a positive integer n such that H_n^*, (the sum of the elements of H_n) contains a set D which is open with respect to H. So $H-H_n^*$ is the sum of two mutually separated point sets U and V. Since $H_n^* \cup K$ is a continuum, $H_n^* \cup K \cup U$ and $H_n^* \cup K \cup V$ are continua, and since $H_n^* \cup K$ does not intersect U or V, M is decomposable into those two continua, and each contains p. So p does not have property E with respect to M. This contradicts the hypothesis that some point has property E with respect to M.

COROLLARY. *If M is not irreducible then if p is a point of M, M is decomposable into two continua each containing p.*

Proof. This follows easily with the use of Theorems 4 and 7.

THEOREM 8. *M has only one composant if and only if no point has property E with respect to M, M has only three composants if and only if E_M is not connected, and M has uncountably many composants if and only if E_M is M.*

Proof. This follows with the use of Theorems 4 and 7 and Theorem 139 from Chapter 1 of [3].

References

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