A note on transfinite dimension

by

A. R. Pears (London)

1. Introduction. This paper is concerned with the following sum theorem for large transfinite inductive dimension: if $X = A \cup B$ where A and B are closed and $\operatorname{Ind} A$, $\operatorname{Ind} B$ exist then $\operatorname{Ind} X = \operatorname{Max}(\operatorname{Ind} A, \operatorname{Ind} B)$. An example due to B. T. Levšenko [3] shows that this is not generally true even if X is a compact metric space. Recently D. W. Henderson has proved this result in the case that X is a hereditarily paracompact Hausdorff space and $\operatorname{Ind} A \cap B$ is finite ([2], Proposition 1). Since the proof only requires the sum theorem for Ind for finite dimensional summands, Henderson's result holds in the wider class of totally normal spaces. In Theorem 1 of the present paper we prove that if X is completely normal, $\operatorname{Ind} X \leqslant \operatorname{Max}(\operatorname{Ind} A, \operatorname{Ind} B) \oplus (\operatorname{Ind}(A \cap B)+1)$ where $\oplus$ denotes the 'lower sum' of ordinals introduced by G. H. Toulmin [6]. Henderson's proposition follows immediately from this (Corollary 2). Theorem 2 is a 'Urysohn inequality'. This is merely a reformulation of a theorem due to Levšenko ([4], Theorem 1). However, it might be of interest because it shows up the theorem more clearly as an extension of Yu. M. Smirnov's result [5] in the finite dimensional case. In § 2 the definition of large transfinite inductive dimension is given and some, of course well-known, properties required in the subsequent work are established. Toulmin's definition of the lower sum $\oplus$ is recalled in § 3. § 4 contains the two theorems described above.

2. Transfinite dimension. Large transfinite inductive dimension is defined by transfinite induction as follows. $\operatorname{Ind} X = -1$ if X is empty. $\operatorname{Ind} X \leqslant \alpha$ if for each closed set E and each open set G such that $E \subset G$ there exists an open set $\mathfrak{U}$ such that $E \subset \mathfrak{U} \subset G$ and $\operatorname{Ind} b(\mathfrak{U}) \leqslant \beta < \alpha$ where $b(\mathfrak{U}) = \overline{\mathfrak{U}} - \mathfrak{U}$ is the boundary of $\mathfrak{U}$. $\operatorname{Ind} X = \alpha$ if $\operatorname{Ind} X \leqslant \alpha$ and it is not true that $\operatorname{Ind} X \leqslant \beta$ for any $\beta < \alpha$. X is said to have transfinite dimension if there exists an ordinal α such that $\operatorname{Ind} X = \alpha$. To establish that X has transfinite dimension it is enough to establish that $\operatorname{Ind} X \leqslant \alpha$ for some ordinal α, for the ordinals are well-ordered and hence there exists a first ordinal for which the inequality is satisfied.

We shall write $b(V)$ to denote the boundary of a subset V of a space X. If A is a subspace of X and $W \subset A$, the boundary of W in A will be denoted by $b_A(W)$.

LEMMA 1. *If X has transfinite dimension and A is a closed subset of X, then A has transfinite dimension and* $\operatorname{Ind} A \leqslant \operatorname{Ind} X$.

Proof. The proof is by transfinite induction. Suppose that the result is true for all spaces with transfinite dimension less than a and let $\operatorname{Ind} X = a$. Let $E \subset G \subset A$ where E is closed and G is open in A. Since A is closed, E is closed in X and $E \subset H$ where H is an open set of X such that $H \cap A = G$. Hence there exists an open set V such that

$$E \subset V \subset H \quad \text{and} \quad \operatorname{Ind} b(V) \leqslant \beta < a .$$

Let $U = V \cap A$. Then U is open in A, $E \subset U \subset G$ and

$$b_A(U) = \overline{U} - U \subset \overline{V} \cap A - V = b(V) \cap A .$$

Since $b_A(U)$ is a closed subset of $b(V)$ by the induction hypotheses $\operatorname{Ind} b_A(U) \leqslant \beta < a$. Hence $\operatorname{Ind} A \leqslant a$. It follows that A has transfinite dimension and $\operatorname{Ind} A \leqslant \operatorname{Ind} X$.

LEMMA 2. *If X is the topological sum of spaces A, B which have transfinite dimension, then X has transfinite dimension and*

$$\operatorname{Ind} X = \operatorname{Max}(\operatorname{Ind} A, \operatorname{Ind} B) .$$

Proof. The proof is by transfinite induction. Suppose that the result is true for the topological sum of spaces of transfinite dimension less than a and let $\operatorname{Ind} A \leqslant a$, $\operatorname{Ind} B \leqslant a$. If $E \subset G \subset X$ where E is closed and G is open then there exists V open in A such that $E \cap A \subset V \subset G \cap A$ and $\operatorname{Ind} b_A(V) < a$ and there exists W open in B such that $E \cap B \subset W \subset G \cap B$ and $\operatorname{Ind} b_B(W) < a$. If $U = V \cup W$ then U is open in X, $E \subset U \subset G$ and the boundary $b(U)$ of U is the topological sum of $b_A(V)$ and $b_B(W)$. Hence by the inductive hypothesis $\operatorname{Ind} b(U) < a$ and so $\operatorname{Ind} X \leqslant a$. Thus

$$\operatorname{Ind} X \leqslant \operatorname{Max}(\operatorname{Ind} A, \operatorname{Ind} B) .$$

Hence X has transfinite dimension and it follows from Lemma 1 that the reverse inequality holds. Thus we have

$$\operatorname{Ind} X = \operatorname{Max}(\operatorname{Ind} A, \operatorname{Ind} B)$$

as was to be shown.

LEMMA 3. *If X is a completely normal space and A is a subspace such that $\operatorname{Ind} A \leqslant a$ then if $E \subset G \subset X$ where E is closed and G is open there exists an open set U such that*

$$E \subset U \subset \overline{U} \subset G \quad \text{and} \quad \operatorname{Ind} b(U) \cap A < a .$$

Proof. Since X is normal there exist open sets L, M such that

$$E \subset L \subset \overline{L} \subset M \subset \overline{M} \subset G .$$

$\overline{L} \cap A$ is closed in A, $M \cap A$ is open in A and $\overline{L} \cap A \subset M \cap A$. Hence there exists H open in A such that

$$\overline{L} \cap A \subset H \subset M \cap A \quad \text{and} \quad \operatorname{Ind} b_A(H) < a .$$

Let $P = E \cup H$, $Q = A - \overline{H}$. Then $\overline{P} = E \cup \overline{H}$ and it is clear that $\overline{P} \cap Q = \emptyset$. $Q \subset A - H$ which is closed in A so that $\overline{Q} \cap A \subset A - H$ and it follows that $H \cap \overline{Q} = \emptyset$. And $A - H \subset X - L$ which is closed in X so that $\overline{Q} \subset X - L$ and it follows that $E \cap \overline{Q} = \emptyset$. Hence $P \cap \overline{Q} = \emptyset$. Since X is completely normal there exists W open in X such that

$$E \cup H \subset W \quad \text{and} \quad \overline{W} \cap (A - \overline{H}) = \emptyset .$$

Let $U = M \cap W$. Then U is open in X and $E \subset U \subset \overline{U} \subset G$. Since $U \subset W$, $\overline{U} \cap (A - \overline{H}) = \emptyset$ and so $\overline{U} \cap A \subset \overline{H}$. But $H \subset U$ and so we have $\overline{U} \cap A = \overline{H} \cap A$. Hence

$$b(U) \cap A = \overline{U} \cap A - U = \overline{H} \cap A - U \subset \overline{H} \cap A - H = b_A(H) .$$

$b(U) \cap A$ is closed in A and so is closed in $b_A(H)$. Thus by Lemma 1, $\operatorname{Ind}\big(b(U) \cap A\big) < a$.

3. Ordinal arithmetic. We recall some definitions due to Toulmin. A well-ordered set A is said to be a *shuffling* of well-ordered sets B and C if $A = B' \cup C'$ where B' and C' are disjoint sets order-isomorphic with B and C respectively. An ordinal number a is a *shuffling* of ordinals β, γ if there are sets A, B, C with ordinals a, β, γ respectively such that A is a shuffling of B and C. If β, γ are given ordinals then there is at least one ordinal shuffling β and γ, namely the sum $\beta + \gamma$, for it is clear that the well-ordered set $B + C$ (i.e. B followed by C) shuffles B and C. The *lower sum* $\beta \oplus \gamma$ is the least ordinal shuffling β and γ. Clearly $\oplus$ is commutative and associative.

Toulmin obtains a shuffling of well-ordered sets which gives rise to the lower sum of their ordinals. Let B and C be disjoint well-ordered sets with ordinals β, γ respectively. Then there is an order-isomorphism between the initial segment of the ordinals $\{\xi \mid \xi < \beta\}$ and B; let the image of ξ under this isomorphism be b_ξ. Similarly we can label the elements of C (in order) c_η for $0 \leqslant \eta < \gamma$. Then $B \cup C$ with the order $\precsim$ given by

$$b_\xi \precsim c_\eta \quad \text{if and only if} \quad \xi \leqslant \eta$$

is a well-ordered set whose ordinal is $\beta\oplus\gamma$. From this Toulmin obtains the following rule: if $\alpha=\alpha'+p$, $\beta=\beta'+q$ where α', β' are limit ordinals and p, q are non-negative integers then

$$\alpha\oplus\beta=\begin{cases}\alpha & \text{if} \quad \alpha'>\beta',\\ \alpha+q=\beta+p & \text{if} \quad \alpha'=\beta',\\ \beta & \text{if} \quad \alpha'<\beta'.\end{cases}$$

We make the following convention if just one of the summands is -1.

$$\alpha\oplus(-1)=(-1)\oplus\alpha=\begin{cases}\alpha-1 & \text{if} \quad 0\leqslant\alpha<\omega\\ \alpha & \text{if} \quad \alpha\geqslant\omega\end{cases}$$

where ω is the first infinite ordinal.

For each ordered pair (α,β) of ordinals such that $\alpha\geqslant\beta$ let

$$\theta(\alpha,\beta)=\alpha\oplus(\beta+1)\,.$$

And for each ordered pair (α,β) of ordinals let

$$\psi(\alpha,\beta)=\operatorname{Max}(\alpha,\beta)\oplus\big(\operatorname{Min}(\alpha,\beta)+1\big)\,.$$

It is clear that θ,ψ are non-decreasing in each argument.

In the next section we shall need the following strict inequalities.

LEMMA 4. (a) *If* $\gamma<\alpha$ *then* $\theta(\alpha,\gamma)<\theta(\alpha,\alpha)$.

(b) *If* $\beta\leqslant\gamma<\alpha$ *then* $\theta(\gamma,\beta)<\theta(\alpha,\beta)$.

(c) *If* $\alpha\geqslant\beta$ *and* $\gamma<\alpha$ *then* $\psi(\gamma,\beta)<\psi(\alpha,\beta)$.

Proof. (a) Let $\alpha=\alpha'+p$ where α' is a limit ordinal and $p\geqslant 0$ is an integer.

If $\gamma<\alpha'$ then

$$\theta(\alpha,\gamma)=\alpha<\alpha+p+1=\theta(\alpha,\alpha)\,.$$

If $\gamma=\alpha'+s$ where s is an integer and $0\leqslant s<p$ then

$$\theta(\alpha,\gamma)=\alpha+s+1<\alpha+p+1=\theta(\alpha,\alpha)\,.$$

(b) Let $\alpha=\alpha'+p$ where α' is a limit ordinal and $p\geqslant 0$ is an integer. Then if $\beta\leqslant\gamma<\alpha'$

$$\theta(\gamma,\beta)<\alpha'\leqslant\alpha=\theta(\alpha,\beta)\,.$$

If $\beta<\alpha'\leqslant\gamma$ then

$$\theta(\gamma,\beta)=\gamma<\alpha=\theta(\alpha,\beta)\,.$$

If $\beta=\alpha'+q$, $\gamma=\alpha'+s$ where q, s are integers and $0\leqslant q\leqslant s<p$ then

$$\theta(\gamma,\beta)=\alpha'+s+q+1<\alpha'+p+q+1=\theta(\alpha,\beta)\,.$$

(c)

$$\psi(\alpha,\beta)=\begin{cases}\theta(\alpha,\beta) & \text{if} \quad \alpha\geqslant\beta,\\ \theta(\beta,\alpha) & \text{if} \quad \alpha<\beta.\end{cases}$$

Thus if $\beta\leqslant\gamma<\alpha$

$$\psi(\gamma,\beta)=\theta(\gamma,\beta)<\theta(\alpha,\beta)=\psi(\alpha,\beta)\,.$$

If $\gamma<\beta<\alpha$ then

$$\psi(\gamma,\beta)=\theta(\beta,\gamma)<\theta(\alpha,\gamma)\leqslant\theta(\alpha,\beta)=\psi(\alpha,\beta)\,.$$

Finally if $\gamma<\alpha$, then

$$\psi(\gamma,\alpha)=\theta(\alpha,\gamma)<\theta(\alpha,\alpha)=\psi(\alpha,\alpha)\,.$$

4. The theorems.

THEOREM 1. *Let X be a completely normal space which is the union of two closed subsets A and B. If* $\operatorname{Ind}A$, $\operatorname{Ind}B$ *exist then*

$$\operatorname{Ind}X\leqslant\operatorname{Max}(\operatorname{Ind}A,\operatorname{Ind}B)\oplus\big(\operatorname{Ind}(A\cap B)+1\big)\,.$$

Proof. If $\operatorname{Max}(\operatorname{Ind}A,\operatorname{Ind}B)=\alpha$ and $\operatorname{Ind}(A\cap B)=\beta$ then, in the notation of the preceding paragraph, we must prove that $\operatorname{Ind}X\leqslant\theta(\alpha,\beta)$. The proof is by transfinite induction.

Let $[\alpha',\beta']$ denote the statement: if A' and B' are closed subsets of X and $\operatorname{Max}(\operatorname{Ind}A',\operatorname{Ind}B')=\alpha'$, $\operatorname{Ind}A'\cap B'=\beta'$, then $\operatorname{Ind}A'\cup B'\leqslant\theta(\alpha',\beta')$. By Lemma 2, $[\alpha',-1]$ is true. Let us suppose that $[\alpha',\beta']$ is true if $\beta'<\beta$ or if $\beta'=\beta$ and $\beta\leqslant\alpha'<\alpha$ and prove that $[\alpha,\beta]$ is true. Thus let A, B be closed subsets of X such that $\operatorname{Max}(\operatorname{Ind}A,\operatorname{Ind}B)=\alpha$, $\operatorname{Ind}A\cap B=\beta$ and let $Y=A\cup B$.

Suppose first that $\alpha=\beta$. If $E\subset G\subset Y$ where E is closed and G open in Y then by Lemma 3 there exists U open in Y such that

$$E\subset U\subset G \quad \text{and} \quad \operatorname{Ind}b(U)\cap A\cap B=\gamma<\alpha\,.$$

Now $b(U)=\big(b(U)\cap A\big)\cup\big(b(U)\cap B\big)$. Moreover $\operatorname{Ind}b(U)\cap A\leqslant\alpha$ and $\operatorname{Ind}b(U)\cap B\leqslant\alpha$. Thus since $[\alpha,\gamma]$ is true by hypothesis, we have

$$\operatorname{Ind}b(U)\leqslant\theta(\alpha,\gamma)\,.$$

But by Lemma 4 (a), $\theta(\alpha,\gamma)<\theta(\alpha,\alpha)$. Thus we have an open set U of Y such that

$$E\subset U\subset G \quad \text{and} \quad \operatorname{Ind}b(U)<\theta(\alpha,\alpha)\,.$$

Hence $\operatorname{Ind}Y<\theta(\alpha,\alpha)$.

Now suppose $\alpha>\beta$. If $E\subset G\subset Y$ where E is closed and G is open in Y, then by Lemma 3 there exists U open in Y such that

$$E\subset U\subset G \quad \text{and} \quad \operatorname{Ind}b(U)\cap A<\alpha\,.$$

And there exists V open in Y such that

$$E \subset V \subset \overline{V} \subset U \quad \text{and} \quad \operatorname{Ind} b(V) \cap B < \alpha.$$

Let W be the interior (in Y) of $(A \cap U) \cup (B \cap V)$. Then W is open in Y and since $V \subset W \subset U$ we have $E \subset W \subset G$. Clearly $W - A = V - A$. If $x \notin A$ and $x \notin \overline{V}$ then $X - A \cup \overline{V}$ is an open neighbourhood of x which does not meet V and so does not meet W. Thus $\overline{W} - A = \overline{V} - A$ and it follows that

$$b(W) - A = b(V) - A \subset b(V) \cap B.$$

Similarly $b(W) - B \subset b(U) \cap A$. Thus

$$b(W) \subset \big(b(U) \cap A\big) \cup \big(b(V) \cap B\big) \cup (A \cap B).$$

Since $\overline{V} \subset U$, $b(U)$ and $b(V)$ are disjoint. Thus if $C = \big(b(U) \cap A\big) \cup \big(b(V) \cap B\big)$, C is the topological sum of $b(U) \cap A$ and $b(V) \cap B$ and it follows that $\operatorname{Ind} C = \alpha' < \alpha$. Hence $\operatorname{Max}(\operatorname{Ind} C, \operatorname{Ind} A \cap B) = \operatorname{Max}(\alpha', \beta) = \gamma < \alpha$ and $\operatorname{Ind}(C \cap A \cap B) = \delta \leqslant \beta$. By hypothesis $[\gamma, \delta]$ is true and so

$$\operatorname{Ind} C \cup (A \cap B) \leqslant \theta(\gamma, \delta) \leqslant \theta(\gamma, \beta).$$

But by Lemma 4 (b), $\theta(\gamma, \beta) < \theta(\alpha, \beta)$. Thus, since $b(W)$ is a closed subset of $C \cup (A \cap B)$ we have $\operatorname{Ind} b(U) < \theta(\alpha, \beta)$. It follows that $\operatorname{Ind} Y \leqslant \theta(\alpha, \beta)$. This completes the proof.

COROLLARY 1. *Let X be a completely normal space which is the union of two closed subsets A and B. If $\operatorname{Ind} A \cap B$ is finite, $\operatorname{Ind} A$, $\operatorname{Ind} B$ exist, and at least one of $\operatorname{Ind} A$, $\operatorname{Ind} B$ is infinite then*

$$\operatorname{Ind} X = \operatorname{Max}(\operatorname{Ind} A, \operatorname{Ind} B).$$

Proof. If $\operatorname{Ind} A \cap B$ is finite, and $\operatorname{Max}(\operatorname{Ind} A, \operatorname{Ind} B)$ is infinite then the theorem gives

$$\operatorname{Ind} X \leqslant \operatorname{Max}(\operatorname{Ind} A, \operatorname{Ind} B).$$

Since X has transfinite dimension the reverse inequality holds (Lemma 1) and the proof is complete.

COROLLARY 2. *Let X be a totally normal space which is the union of two closed subsets A and B. If $\operatorname{Ind} A \cap B$ is finite and $\operatorname{Ind} A$, $\operatorname{Ind} B$ exist then*

$$\operatorname{Ind} X = \operatorname{Max}(\operatorname{Ind} A, \operatorname{Ind} B).$$

Proof. The case in which $\operatorname{Max}(\operatorname{Ind} A, \operatorname{Ind} B)$ is infinite is dealt with in Corollary 1. But if $\operatorname{Ind} A$ and $\operatorname{Ind} B$ are finite then the result is also known to be tıue ([1], Theorem 3).

THEOREM 2. *Let X be a completely normal space which is the union of subsets A and B. If $\operatorname{Ind} A$ and $\operatorname{Ind} B$ exist then*

$$\operatorname{Ind} X \leqslant \operatorname{Max}(\operatorname{Ind} A, \operatorname{Ind} B) \oplus \big(\operatorname{Min}(\operatorname{Ind} A, \operatorname{Ind} B) + 1\big).$$

Proof. We must prove that if $\operatorname{Ind} A = \alpha$ and $\operatorname{Ind} B = \beta$ then $\operatorname{Ind} X \leqslant \psi(\alpha, \beta)$. The proof is by transfinite induction.

Let $\{\alpha', \beta'\}$ be the statement: if $A', B' \subset X$ and $\operatorname{Ind} A' = \alpha'$, $\operatorname{Ind} B' = \beta'$ then $\operatorname{Ind} A' \cup B' \leqslant \psi(\alpha', \beta')$. Let us suppose that $\{\alpha', \beta'\}$ is true if $\operatorname{Min}(\alpha', \beta') < \beta$ or if $\operatorname{Min}(\alpha', \beta') = \beta$ and $\operatorname{Max}(\alpha', \beta') < \alpha$ and then prove $\{\alpha, \beta\}$ where $\alpha \geqslant \beta$.

Let $A, B \subset X$ and let $\operatorname{Ind} A = \alpha$, $\operatorname{Ind} B = \beta$ where $\alpha \geqslant \beta$. Let $Y = A \cup B$ and let $E \subset G \subset Y$ where E is closed and G is open. By Lemma 3 there exists U open in Y such that $E \subset U \subset G$ and $\operatorname{Ind} b(U) \cap A = \gamma < \alpha$. And by Lemma 1, $\operatorname{Ind} b(U) \cap B = \delta \leqslant \beta$. But

$$b(U) = \big(b(U) \cap A\big) \cup \big(b(U) \cap B\big)$$

and by the inductive hypothesis $\{\gamma, \delta\}$ is true. Thus

$$\operatorname{Ind} b(U) \leqslant \psi(\gamma, \delta) \leqslant \psi(\gamma, \beta) < \psi(\alpha, \beta)$$

where the last inequality is supplied by Lemma 4 (c). It follows that $\operatorname{Ind} A \cup B \leqslant \psi(\alpha, \beta)$ which completes the proof.

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QUEEN ELIZABETH COLLEGE
UNIVERSITY OF LONDON

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