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Reçu par la Rédaction le 21. 6. 1969

Metric spaces in which a strengthened form of Blumberg's theorem holds

by

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Introduction. Let the following statement be referred to as

Proposition A. *If f is a function from X into Y, then there is a dense subset D of X such that $f|D$ is continuous.*

Henry Blumberg first proved [2] that Proposition A holds if X is the plane and Y is the space R of real numbers, and he later proved [3] that Proposition A holds if X and Y are Euclidean spaces. Blumberg observes in [3] that according to his construction, the set D is countable, and that it cannot be made to be otherwise, for Sierpinski and Zygmund have shown ([9], p. 118; [10]) that if the continuum hypothesis is true, there exists a real function f with domain R such that if D is any uncountable subset of R, then $f|D$ has a point of discontinuity. Block and Cargal [1] show that Proposition A holds for fairly general topological spaces X and Y, with suitable restrictions on the categoric nature of the open sets in X. In [6] Goffman gave an example to show that D cannot be constructed so as to make $f|D$ a homeomorphism from D onto $f(D)$. He also gave an example [6] which shows that Proposition 4 of [5], which states that Proposition A holds for every pair of metric spaces X and Y, is false. Bradford and Goffman [4] proved that if X is a metric space, then Proposition A holds (where Y is R) if and only if every open subset of X is of second category.

The main purpose of this paper is to establish theorems analogous to those of Bradford and Goffman concerning the following two propositions, each of which is stronger than Proposition A (where $Y = R$).

Proposition B. *If f is a function from X into R, then there exists an uncountably dense subset W of X and a dense subset D of W such that $f|W$ is continuous at each element of D.*

Proposition C. *If f is a function from X into R, then there exists a c-dense subset W of X and a dense subset D of W such that $f|W$ is continuous at each element of D.*

The statement that W is uncountably dense (c-dense) in a metric space X means that every open subset of X contains uncountably many (c-many) elements of W.

The paper is divided into two parts. In Section I the notion of a metric space being "typically dense" in itself is defined, and it is shown that Proposition B holds in every metric space which is typically dense in itself. It is then shown that if X is a metric space which is not typically dense in itself, then Proposition C fails to hold in X. In Section II the notion of a metric space being "c-typically dense" in itself is defined, and it is shown that Proposition C holds in every metric space which is c-typically dense in itself and that this property characterizes the separable metric spaces in which Proposition C holds. It follows as a corollary that Proposition C holds for real valued functions with a complete metric domain which is dense in itself.

I. Typical density and Propositions B and C. Suppose that M is a subset of a metric space X. The statement that M is *nowhere dense* means that if S is an open subset of X, then there is an open subset T of S which does not intersect M. The statement that M is of *first category* means that M is the union of countably many nowhere dense sets. The statement that M is of *second category* means that M is not of first category. The statement that M is a *Lusin* set means that M has no uncountable nowhere dense subset. The statement that M is of *first type* means that M is the union of a first category set and a Lusin set. The statement that M is of *second type* means that M is not of first type. If S is an open subset of X, then the statement that M is *dense* in S (*categorically dense* in S) (*typically dense* in S) means that if T is an open subset of S, then $T \cap M$ is non-empty (second category) (second type). The statement that M is *nowhere categorically dense* (*nowhere typically dense*) means that if T is an open set, there is an open subset V of T such that $M \cap V$ is first category (first type).

If every set of second category had an uncountable nowhere dense subset, it would not be necessary to introduce the notions of "first type" and "typically dense". However, N. Lusin has shown ([8], [9], p. 36) that if the continuum hypothesis is true, there is an uncountable number set which has no uncountable nowhere dense subset, and this set is of second category in R.

Notice that every subset of a nowhere dense (first category) (Lusin) (first type) (nowhere categorically dense) (nowhere typically dense) set is nowhere dense (first category) (Lusin) (first type) (nowhere categorically dense) (nowhere typically dense). Every countable set is a Lusin set. However, there can be finite sets (of isolated points) which are not of first category. First category sets and Lusin sets are of first type.

LEMMA 1. *Suppose Q is a property and every open subset of a metric space X has an open subset with property Q. Then there exists a collection G of mutually exclusive open subsets of X such that G^* (the union of the sets in G) is dense in X and every set in G has property Q.*

LEMMA 2. *The union of countably many first type subsets of a metric space X is of first type.*

LEMMA 3. *If M is a nowhere categorically dense subset of a metric space X, then M is first category.*

Proof. If S is an open subset of X, there is an open subset T of S such that $M \cap T$ is first category. Thus, from Lemma 1 it follows that there is a collection G of mutually exclusive open subsets of X such that G^* is dense in X and if T is in G, $M \cap T$ is first category. For each T in G, let $M(T,1), M(T,2), \ldots$ be a sequence of nowhere dense sets such that $M(T,1) \cup M(T,2) \cup \ldots$ is $M \cap T$, and let $A = M - G^*$. For each positive integer j, let $M_j = A \cup \{M(T,j) |\ T \text{ is in } G\}^*$. Then $M_1, M_2, \ldots$ is a sequence of nowhere dense sets with union M.

Remark 1. A nowhere typically dense subset of a separable metric space X has to be of first type, but this may not be true in non-separable spaces X. Suppose the continuum hypothesis is true and that M is a Lusin subset of $I = [0,1]$ such that M is second category. Let X be $\{(x,y) |\ x \text{ and } y \text{ are in } I\}$ with the metric d: $d[(x,y),(u,v)] = 1$ if $x \neq u$, and $d[(x,y),(u,v)] = |y-v|$ if $x = u$. Let $N = \{(x,y) |\ x \text{ is in } I \text{ and } y \text{ is in } M\}$. N is nowhere typically dense in X, but is not of first type. Nevertheless, the following lemma does hold.

LEMMA 4. *The union of countably many nowhere typically dense subsets of a metric space X is nowhere typically dense.*

Proof. Suppose M is typically dense in an open subset S of a metric space X, and $M_1, M_2, \ldots$ is a sequence of nowhere typically dense subsets of X with union M. Assume that M is a subset of S. Suppose that for each positive integer j and each open subset T of S for which $M_j \cap T$ is first type, $M_j \cap T$ is actually of first category. Then it follows (Lemma 3) that each M_j would be of first category, and M is of first category, which is a contradiction. Therefore, there is an open subset T of S and a positive integer n such that $M_n \cap T$ is of first type but not of first category. $M_n \cap T$ is the union of a first category set A and a Lusin set B, and there must be an open subset V of T in which B is dense. Let $N = M \cap V$, and for each positive integer j, let $N_j = M_j \cap V$. N is typically dense in V, each N_j is nowhere typically dense, and N_n is the union of a first category set $A' = A \cap V$ and a Lusin set $B' = B \cap V$ which is dense in V. Notice that since N is typically dense in V, then no open subset of V is degenerate. Now, suppose j is a positive integer different from n.

N_j is nowhere typically dense, so there must be a collection G of mutually exclusive open subsets of V such that G^* is dense in V and if W is in G, $N_j \cap W$ is of first type. Suppose G is uncountable. B' is dense in V, so let B'' be a subset of B' which consists of just one element of B' in each set in G. Since no open subset of V is degenerate, it follows that B'' is an uncountable nowhere dense subset of B, and this is a contradiction. Thus, G is countable, and N_j is the union of countably many sets of first type and must be of first type. Then $N = N_1 \cup N_2 \cup \ldots$ is also of first type. This is a contradiction.

LEMMA 5. *If M is a typically dense subset of an open set S in a metric space X and N is a subset of M which is nowhere typically dense, then $M-N$ is typically dense in S.*

Proof. If $M-N$ is not typically dense in S, then there is an open subset T of S such that $T \cap (M-N)$ is of first type. But there is an open subset V of T such that $N \cap V$ is of first type, so $M \cap V = [(M-N) \cap V] \cup \cup (N \cap V)$ is of first type, and M is not typically dense in S.

LEMMA 6. *If G is a collection of open subsets of a metric space X such that G^* is dense in an open subset S of X and M is a subset of X which is typically dense in each set of G, then M is typically dense in S.*

Proof. If M is not typically dense in S, then there is an open subset T of S such that $T \cap M$ is of first type. T must intersect an open set V of G, and $T \cap V$ is an open subset W of V such that $M \cap W$ is of first type, which means that M is not typically dense in V. This is a contradiction.

LEMMA 7. *If M is a typically dense subset of an open subset S of a metric space X, and $H_1, H_2, \ldots, H_n$ is a finite sequence of mutually exclusive subsets of M with union M, then there exists a collection G of mutually exclusive open subsets of S such that* (1) *G^* is dense in S, and* (2) *if T is in G, there is a positive integer $i \leqslant n$ such that H_i is typically dense in T.*

Proof. The proof is by induction on n. The lemma holds for $n=1$. Suppose the lemma holds for $n=k$ and $H_1, H_2, \ldots, H_{k+1}$ is a $k+1$ term sequence of mutually exclusive subsets of M with union M. Let g be the union of all the open subsets of S in which H_{k+1} is typically dense. It follows from Lemma 6 that H_{k+1} is typically dense in g. Let $T = S-\mathrm{Cl}(g)$. $T \cap H_{k+1}$ must be nowhere typically dense, so $M-H_{k+1} = H_1 \cup H_2 \cup \ldots \cup H_k$ is typically dense in T, and there is a collection G_T of mutually exclusive open subsets of T such that (1) G_T^* is dense in T, and (2) if V is in G_T, there is a positive integer $i \leqslant k$ such that H_i is typically dense in V. The collection $G = (g) \cup G_T$ is the desired collection.

NOTATION. If f is a function from a subset of a metric space into R and $a<b$, then the symbol $[a<f\leqslant b]$ shall denote the set $\{x|\ a<f(x)\leqslant b\}$.

LEMMA 8. *Suppose $a<b$ and f is a real valued function with domain a typically dense subset M of an open subset S of a metric space X, and $a<f(x)\leqslant b$ for each x in M. Then there is a subset N of M such that N is typically dense in S and $f|N$ is continuous at some element x of N.*

Proof. Assume without loss of generality that $b=a+1$. Since M is typically dense in S, it follows that if T is an open subset of S, then $T \cap M$ is not a Lusin set and is therefore uncountable. For each positive integer n, let P_n denote the set of all n-term sequences of zero's and one's, and if $z = \{i_1, i_2, \ldots, i_n\}$ belongs to P_n, let $t(z)$, $G(z)$, and $M(z)$ be defined as follows: (1) $t(z) = a+i_1/2+i_2/2^2+\ldots+i_n/2^n$, (2) $G(z)$ is the union of all open subsets T of S such that $[t(z)<f\leqslant t(z)+1/2^n]$ is typically dense in T, and (3) $M(z) = [t(z)<f\leqslant t(z)+1/2^n] \cap G(z)$. For each positive integer n, let $V_n = \{G(z)|\ z \text{ is in } P_n\}^*$, let $H_n = [S-V_n] \cap M$, and let $K_n = M-\{M(z)|\ z \text{ is in } P_n\}^*$. It follows from Lemma 7 that if n is a positive integer, V_n is dense in S, so that H_n is nowhere dense. Furthermore, if n is a positive integer, K_n must be nowhere typically dense, otherwise there would be a z in P_n such that part of K_n would be in $M(z)$ (Lemma 7). Thus, $K_1 \cup K_2 \cup \ldots$ is nowhere typically dense (Lemma 4), as is $J = (H_1 \cup H_2 \cup \ldots) \cup (K_1 \cup K_2 \cup \ldots)$. Therefore $M-J$ must be typically dense in S (Lemma 5). Let x be in $M-J$. Notice that if n is a positive integer, and z' is in P_n, and z is in P_{n+1}, and z' and z agree in the first n terms, then $G(z)$ is a subset of $G(z')$ and $M(z)$ is a subset of $M(z')$. Now, for each positive integer n, there is an element z of P_n such that x is in $M(z)$, so there must be one infinite sequence $Z = \{i_1, i_2, \ldots\}$ of zero's and one's such that if n is a positive integer and $z = \{i_1, i_2, \ldots, i_n\}$, then x is in $M(z)$. It follows that $f(x) = a+i_1/2+i_2/2^2+\ldots$ Now, let $R_1, R_2, \ldots$ be a sequence of spherical open neighborhoods of x such that for each positive integer n, (1) $\mathrm{Cl}(R_n)$ is a proper subset of $G(i_1, i_2, \ldots, i_n)$, (2) $\mathrm{Cl}(R_{n+1})$ is a proper subset of R_n, and (3) R_n has radius less than $1/n$. (The above set inclusions can be made proper since no open subset of S is degenerate.) Now, let $A_1, A_2, \ldots$ be such that $A_1 = M-\mathrm{Cl}(R_1)$, and if n is an integer greater than 0, $A_{n+1} = M(i_1, i_2, \ldots, i_n) - \mathrm{Cl}(R_{n+1})$. A_1 is typically dense in $S-\mathrm{Cl}(R_1)$ and for each positive integer n, A_{n+1} is typically dense in $R_n-\mathrm{Cl}(R_{n+1})$, so $N = (x) \cup A_1 \cup \cup A_2 \cup \ldots$ is typically dense in S (Lemma 6).

In order to show that $f|N$ is continuous at x, suppose $\varepsilon>0$. Let n be a positive integer such that $1/2^n<\varepsilon$. $N \cap R_n$ is a subset of $M(i_1, i_2, \ldots, i_n)$, so if y is in $N \cap R_n$,

$$a+i_1/2+\ldots+i_n/2^n < f(y) \leqslant a+i_1/2+\ldots+i_n/2^n+1/2^n,$$

and $|f(x)-f(y)| \leqslant 1/2^n < \varepsilon$.

THEOREM 1. *If X is a metric space which is typically dense in itself, then Proposition* B *holds for X.*

Proof. Since X is typically dense in itself, X has no degenerate open subsets. If for each integer t, the set $[t < f \leqslant t+1]$ is nowhere typically dense, then it follows from Lemma 4 that X is nowhere typically dense. Therefore, there must be an integer t such that $[t < f \leqslant t+1]$ is typically dense in some open subset of X. In fact, it follows from Lemma 1 that there must exist a collection G_1 of mutually exclusive open subsets of X such that G_1^* is dense in X and for each set S in G_1, there is an integer t_S such that $M_S = [t_S < f \leqslant t_S+1] \cap S$ is typically dense in S. Now, an infinite sequence of steps, each step involving four stages, is defined inductively as follows:

Step A1. Let G_1 be the collection described above, and for each S in G_1, let t_S and M_S be as described above.

Step B1. For each S in G_1, let N_S and x_S be such that N_S is a subset of M_S, N_S is typically dense in S, and x_S is an element of N_S at which $f|N_S$ is continuous.

Step C1. For each S in G_1, let H_S be an uncountable nowhere dense subset of N_S such that H_S contains x_S (H_S can be made to contain x_S because X has no degenerate open subsets), and let K_S be a collection of mutually exclusive spherical open subsets of S of radius less than 1/1 such that K_S^* is dense in S but does not intersect H_S.

Step D1. For each S in G_1, and for each V in K_S, let $t_V = t_S$ and let $M_V = N_S \cap V$ (which is typically dense in V).

Now, for each integer $n > 1$, Steps An, Bn, Cn, and Dn can be defined inductively as follows:

Step An. Let $G_n = \{K_S|\ S \text{ is in } G_{n-1}\}^*$.

Step Bn. (Same as Step B1, except "G_n" replaces "G_1").

Step Cn. (Same as Step C1, except "G_n" replaces "G_1" and "$1/n$" replaces "1/1").

Step Dn. (Same as Step D1, except "G_n" replaces "G_1").

Notice that if n is an integer greater than 1 and S is in G_n, then there is an S' in G_{n-1} such that S is an element V of $K_{S'}$, $t_S = t_{S'}$, and $M_S = N_{S'} \cap S$, so that M_S, N_S, and H_S are subsets of $N_{S'}$.

Now, let $W = \{x|$ there is a positive integer n and a set S in G_n such that x is in $H_S\}$, and let $D = \{x|$ there is a positive integer n and a set S in G_n such that $x = x_S\}$. D is a subset of W. In order to show that W is uncountably dense in X and D is dense in X, suppose T is a spherical open subset of X of radius ε. Let n be a positive integer such that $1/n < \varepsilon/3$. The open sets of G_{n+1} are spherical with radius less than $1/n$, so

there must be an open set S in G_{n+1} which lies inside T. H_S is an uncountable subset of $W \cap T$ and x_S is an element of $D \cap T$.

Now suppose x is an element of D. Let n be a positive integer and S be an element of G_n such that $x = x_S$. $W \cap S$ is a subset of $N_S \cap S$ and $f|N_S$ is continuous at x_S, so $f|W$ is continuous at x. This completes the proof of Theorem 1.

THEOREM 2. *If X is a metric space which is not typically dense in itself, then Proposition* C *fails to hold in X.*

Proof. If X is not c-dense in itself, then the theorem holds vacuously, so assume every open subset of X has cardinality at least c. Suppose X is not typically dense in itself. There is an open subset S of X which is of first type. Then $S = A \cup M_1 \cup M_2 \cup \ldots$, where A is a Lusin set, M_n is nowhere dense for each positive integer n, and A, M_i, and M_j are mutually exclusive if $i \neq j$. First, suppose S is actually first category (A is empty) or that A has cardinality less than c. Then, as in [4], let $f(x) = 0$ if x is not in $M_1 \cup M_2 \cup \ldots$, and let $f(x) = j$ if x is in M_j. Suppose there is a c-dense subset W of X and a dense subset D of W such that $f|W$ is continuous at each element of D. Let x be an element of $D \cap S$, and let V be an open subset of S such that x is in V and $|f(x)-f(y)| < 1/2$ for each y in $W \cap V$. $W \cap V$ has cardinality at least c, so it is not a subset of A. Therefore, there is a positive integer j such that $W \cap V$ is a subset of M_j. Since M_j is nowhere dense, there is an open subset V' of V which contains no element of M_j. Then V' contains no element of W, and this is a contradiction.

Assume S is not of first category. There must be an open subset T of S in which A is dense. Let $B = T \cap A$, and for each positive integer j, let $N_j = T \cap M_j$. Assume B has cardinality at least c. Since B has no uncountable nowhere dense subset, and B is dense in T, and no open subset of T is degenerate, then T cannot have uncountably many mutually exclusive open subsets. Thus B, considered as a subspace of X, must be separable and of cardinality c. Then it follows from the theorem of Sierpiński and Zygmund ([7], p. 422; [10]) that there is a function g from B into the segment $(0, 1)$ such that if M is a subset of B of cardinality c, then $g|M$ has a point of discontinuity. Now, let f be defined so that $f(x) = 0$ if x is in $X-T$, $f(x) = g(x)$ if x is in B, and $f(x) = j+1$ if x is in N_j for some positive integer j. Now, suppose there is a c-dense subset W of X such that $f|W$ is continuous at each element of a dense subset D of W. Let x be an element of $D \cap T$, and let V be an open subset of T such that x is in V and $|f(x)-f(y)| < 1/2$ if y is in $W \cap V$. $W \cap V$ is either a subset of B or a subset of one of the sets N_j. If $W \cap V$ is a subset of one of the sets N_j, then the contradiction reached earlier occurs again. Suppose $W' = W \cap V$ is a subset of B. Since $f|W$ is continuous at a dense subset of W' and W' is dense in V, then the set E of points of W' at which $f|W$ is discontinuous is first category. But since W' is a Lusin set, it follows that E is countable. Since W' has cardinality c and $f|(W'-E) = g|(W'-E)$, then $W'-E$ is a subset M of B of cardinality c such that $f|M = g|M$ is continuous. This is a contradiction and completes the proof of Theorem 2.

Remark 2. The set W constructed in the proof of Theorem 1, while uncountably dense in X, is nevertheless of first category. It cannot be made to be otherwise, for suppose the continuum hypothesis is true, and consider the function f of Sierpiński and Zygmund mentioned in the introduction. If there is an uncountably dense subset W of R such that W is of second category and $f|W$ is continuous at each element of a dense subset of W, then $f|W$ is continuous on an uncountable subset of W, which is a contradiction.

Remark 3. There are clearly metric spaces in which Proposition A holds and Proposition B fails, for a metric space X can satisfy the condition of [4] and have degenerate open subsets, whereas in order for Proposition B to hold in X, every open set in X must be uncountable. Furthermore, if the continuum hypothesis is true, and M is an uncountable Lusin subset of the line, then M would have to be categorically dense in some segment S. Then $N = M \cap S$, considered as a metric space, would satisfy the condition of [4] and would have no countable open subsets, but would not be typically dense in itself.

II. c-typical density and Proposition C. Suppose that M is a subset of a metric space X. The statement that M is a *c-Lusin* set means that if N is a nowhere dense subset of M, then $N = N_1 \cup N_2 \cup \ldots$, where each N_i is of local cardinality less than c (i.e. such that if x belongs to N_i for some positive integer i, then there is an open set U containing x such that $U \cap N_i$ has cardinality less than c). The resulting definitions of *first c-type*, *second c-type*, *c-typically dense*, and *nowhere c-typically dense* are analogous to definitions in Section I.

A Lusin set is a c-Lusin set. A first type set is a first c-type set. If a metric space is c-typically dense in itself, it is typically dense in itself.

The lemmas and theorems of this section are analogous to lemmas and theorems of Section I and are numbered so as to indicate that analogy.

Lemma 2′. *The union of countably many first c-type subsets of a metric space X is of first c-type.*

Proof. Suppose $M = M_1 \cup M_2 \cup \ldots$, where each set $M_i = A_i \cup B_i$ such that A_i is first category and B_i is a c-Lusin set. Let $A = A_1 \cup A_2 \cup \ldots$ and $B = B_1 \cup B_2 \cup \ldots$ A is first category. Suppose N is a nowhere dense subset of B. For each positive integer i, let G_i be a countable collection of subsets of B_i, each of local cardinality less than c, such that the union of the sets in G_i is $B_i \cap N$. Then $G = G_1 \cup G_2 \cup \ldots$ is a countable collection of subsets of B, each of local cardinality less than c, such that the union of the sets in G is N. Therefore B is a c-Lusin set and M is of first c-type.

Lemma 3′. *If M is a nowhere c-typically dense subset of a metric space X then M is of first c-type.*

Proof. There exists a collection G of mutually exclusive open subsets of X such that G^* is dense in X and if T is in G, then $M \cap T$ is the union of a first category set A_T and a c-Lusin set B_T. Let B be the union of all the sets B_T such that T is in G, and let $A = M-B$. As in the proof of Lemma 3, A is first category. Now, suppose N is a nowhere dense subset of B. For each set T of G, let $N(T,1), N(T,2), \ldots$ be a sequence of sets with union $B_T \cap N$ such that each set $N(T,i)$ is of local cardinality less than c. Now, for each positive integer i, let N_i be the union of all the sets $N(T,i)$ such that T is in G. Then $N_1, N_2, \ldots$ is a sequence of sets with union N, and each set N_i is of local cardinality less than c. Thus, B is a c-Lusin set and M is of first c-type.

Lemmas 4′, 5′, 6′, 7′, and 8′, will not be stated, but their statements are analogous to the statements of Lemmas 4, 5, 6, 7, and 8, respectively, with the notion of c-typical density replacing the notion of typical density. The proofs are either obvious, or else analogous to the proofs of the earlier lemmas.

Theorem 1′. *If X is a metric space which is c-typically dense in itself, then Proposition* C *holds for X.*

Proof. Analogous to the proof of Theorem 1, except on "Step Cn" H_S is made to have cardinality c.

Corollary. *If X is a complete metric space which is dense in itself, then Proposition* C *holds for X.*

Proof. Suppose T is an open subset of X such that $T = B \cup M_1 \cup M_2 \cup \ldots$, where B is a c-Lusin set and each M_i is nowhere dense. Let $G_1, G_2, \ldots$ be a sequence of collections of neighborhoods such that (1) G_1 contains just one neighborhood and that neighborhood is a subset of T and (2) if n is an integer greater than 1 and g belongs to G_{n-1}, then G_n contains just two neighborhoods h and k which intersect g, and the closures of h and k lie in g, are mutually exclusive, have radii less than $1/n$, and fail to intersect M_n. $N = G_1^* \cap G_2^* \cap \ldots$ is a separable Cantor subset of B of cardinality c. Since B is a c-Lusin set, $N = N_1 \cup N_2 \cup \ldots$, where each N_i is of local cardinality less than c. Since N is separable, it follows that each N_i is the union of countably many sets of cardinality less than c. Therefore, each N_i is itself of cardinality less than c ([9], p. 7), so N is of cardinality less than c. This is a contradiction. Thus X is c-typically dense in itself, and Proposition C holds for X.

THEOREM 2′. *If X is a separable metric space which is not c-typically dense in itself, then Proposition* C *fails to hold in* X.

Proof. Analogous to the proof of Theorem 2, except the set B, considered as a metric subspace of X, is now separable simply because X is separable.

Remark 4. If the continuum hypothesis is true, then the notions of typically dense and c-typically dense are the same, as are Propositions B and C, and Theorems 1 and 2 yield a characterization of all metric spaces in which Proposition C holds. However, except in the case of separable spaces, this paper does not yield an outright characterization of the metric spaces in which Proposition C (or Proposition B) holds. Hopefully, there might exist a generalization of the theorem of Sierpiński and Zygmund which would lead to a proof of Theorem 2′ without the hypothesis of separability of X.

Remark 5. When this paper was originally submitted, a c-Lusin set was defined to be a set which had no nowhere dense subset of cardinality c. This author found it necessary to have the additional hypothesis of separability of X in Lemmas 3′ through 8′ and Theorem 1′ at that time. The referee observed that changing the definition of a c-Lusin set to its present form would so strengthen the property of a space being c-typically dense in itself that the hypothesis of separability of X would become superflous in Lemmas 3′ through 8′ and Theorem 1′. The author is indebted to the referee for this suggestion.

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Reçu par la Rédaction le 7. 7. 1969