On convex metric spaces II

by

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§ 1. Introduction. R. H. Bing has raised [1] the following problem, which we quote in the formulation of K. Borsuk (cf. [4]):

Is it true that every n-dimensional continuum which is SC (strongly convex) and WR (without ramifications) must be topologically an n-cell?

Lelek and Nitka [4] solved the problem positively for $n \leqslant 2$, Rolfsen [6] for $n = 3$. For $n > 3$ only partial solutions are known. E.g., Rolfsen has shown [6] that every SC-WR compact n-manifold is a cell for $n \neq 4, 5$, and Toranzos [7] claims that it is so for every n including 4 and 5.

The main result of the present paper is the following (cf. Theorem 2 in [5]):

Main Theorem. *If $\langle X, \varrho \rangle$ is a compact* SC-WR-CT-*space and* $\dim X = n$, *then X is an n-cell.*

It follows from 8.3 and 8.1 below that an n-dimensional SC-WR compact space is a cell if it contains a convex n-cell. And the property CT defined in § 10 implies the existence of a convex n-cell in X (see 15.3).

Paragraphs 9, 11, 12, and 13 are unnecessary for the proof of the main theorem. However, they have been inserted here for the sake of completeness. In particular, in § 9 it is shown that in an SC-WR-cell each metric ball with a centre in the interior is also a cell. And on the other hand, the existence of such a metric ball in an SC-WR compact space implies that the space is a cell.

In § 11 equivalent forms of CT-condition are studied (cf. Theorem 1 in [5]).

In § 12 we prove that 2-dimensional compact SC-WR-spaces are CT-spaces (cf. Theorem 3 in [5]). This gives another proof of the main theorem of [4].

As is shown in § 13, the CT-property is independent of SC-WR for higher dimensions.

Notions and symbols not defined in the paper are derived from [4].

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§ 2. Preliminary lemmas. The lemmas below are concerned with notions of betweenness, convexity, SC, and WR (cf. [4], § 2 and 3).

Let $\langle X, \varrho\rangle$ be a metric space and let $p, q, r, s \in X$.

2.1. *If pqr and prs, then pqs and qrs, and conversely.*

In a convex space this is equivalent to

2.2. *If pqr and $p \neq r$, then $\overline{pq} \cup \overline{qr}$ is a segment.*

Frequently the following propositions will be used:

2.3. *A complete metric space is convex iff for every $p, q \in X$ and for every $0 \leqslant t \leqslant 1$ there exists at least one point $z \in X$ such that*

$$\varrho(p, z) = (1-t)\cdot\varrho(p, q) \quad \text{and} \quad \varrho(z, q) = t\cdot\varrho(p, q)\,. \tag{1}$$

From 2.3 and the definition of an SC-space we infer

2.4. *A metric space is an SC-space iff for every pair of points $p, q \in X$ and for every $0 \leqslant t \leqslant 1$ there exists exactly one point z such that (1) holds.*

2.5. *If $\langle X, \varrho\rangle$ is a compact SC-space, $p_i, z_i, q_i \in X$, $p_i z_i q_i$ for every $i = 0, 1, 2, \ldots$, $\lim p_i = p_0$, $\lim q_i = q_0$, and $\lim \varrho(p_0, z_i) = \varrho(p_0, z_0)$, then the sequence $\{z_i\}$ is convergent to z_0.*

Note also the following obvious propositions on WR-spaces.

2.6. *If $\langle X, \eta\rangle$ is a WR-space, then pqr, pqs, $p \neq q$ and $\varrho(p, r) \leqslant \varrho(p, s)$ imply prs.*

2.7. *A metric space $\langle X, \varrho\rangle$ is a WR-space iff pqr, pqs, $p \neq q$, and $\varrho(p, r) = \varrho(p, s)$ imply $r = s$.*

§ 3. *B*-cones and cells. Let I denote the closed segment $[0, 1]$ with the natural topology. The space obtained from the Cartesian product $M \times I$ by the identification of the set $M \times 0$ to a point w will be called a *bounded cone over M* and denoted by $\mathrm{B\,cone}\, M$. A point w will be called a *cone vertex*, the image of the set $M \times 1$ in $\mathrm{B\,cone}\, M$ will be denoted shortly by M.

By a geometrical n-cell K^n we mean the unit solid sphere of the Euclidean n-space E^n with the topology induced by the Euclidean norm, i.e. $K^n = \{p \in E^n\colon |p| \leqslant 1\}$. If $|p| < 1$, we call p an *interior point of K^n* and write $p \in \mathrm{Int} K^n$. If $|p| = 1$, p is a *boundary point* and we write $p \in \mathrm{Bd} K^n$. The set $\mathrm{Bd} K^n$ is a geometrical $(n-1)$-sphere.

A subset Q^n of a topological space X is called an *n-cell* if there exists a homeomorphism h of K^n onto Q^n. The image $h(\mathrm{Int} K^n)$ is called the *interior of Q^n* and denoted by $\mathrm{Int} Q^n$. In an analogous way we define the *boundary* $\mathrm{Bd} Q^n$ of Q^n.

It is known that the interior and the boundary of an n-cell do not depend on the choice of the homeomorphism h.

A 2-cell will be called a *disk*.

Recall two useful facts:

3.1. *If $M = Q^n$ is an n-cell, then $\mathrm{B\,cone}\, M$ is an $(n+1)$-cell and $\mathrm{Int\,B\,cone}\, M = (\mathrm{B\,cone\,Int}\, M) \setminus (M \cup \{w\})$, $\mathrm{Bd\,B\,cone}\, M = (\mathrm{B\,cone\,Bd}\, M) \cup M$.*

3.2. *If $M = \mathrm{Bd} Q^n$, then $\mathrm{B\,cone}\, M$ is an n-cell with the boundary equal to M.*

§ 4. Maximal prolongation, ϱ-cones. Throughout this paragraph we suppose $\langle X, \varrho\rangle$ to be a compact SC-WR-space. Take a closed non-void subset $A \subset X$ and a segment $\overline{pq} \subset X$. We say that $\overline{pr}$ is a *prolongation in A* (through the point q) of the segment $\overline{pq}$ if $\overline{pr} \supset \overline{pq}$ and $r \in A$. In particular, if $A = X$, then such a segment is called shortly a prolongation (through the point q) of $\overline{pq}$.

A prolongation $\overline{pr}$ of $\overline{pq}$ in A is called a *maximal prolongation* in A and segment $\overline{pr}$ is called a *maximal segment in A* if for every $r' \in A$ the betweenness prr' implies $r = r'$.

4.1. *If $p \in X$, $q \in A$, and $p \neq q$, then there exists a unique maximal prolongation of $\overline{pq}$ in A.*

Proof. Set $Z = \{z \in A\colon pqz\}$ contains point q and by 2.5 is closed in the compact set A. Take $r \in Z$ such that $\varrho(p, r) = \sup_{z \in Z} \varrho(p, z)$. It is easy to check that $\overline{pr}$ is a unique maximal prolongation of $\overline{pq}$ in A.

The union of all segments joining a set A with a point $v \in X$ is called, following (1) § 5 in [4], a *ϱ-cone* over A with the vertex v and denoted by $C_\varrho(A, v)$. The set of all points x of $C_\varrho(A, v)$ for which vxx' and $x' \in A$ implies $x = x'$ is called a *base of ϱ-cone* $C_\varrho(A, v)$ and denoted by $B_\varrho(A, v)$.

4.2. *$A \subset B$ implies $C_\varrho(A, v) \subset C_\varrho(B, v)$.*

4.3. *$C_\varrho(A, v)$ is closed for A closed.*

The proof depends on 2.5.

4.4. *$x \in B_\varrho(A, v)$ iff $\overline{vx}$ is maximal in A.*

4.5. *For every $x \in C_\varrho(A, v)$, $x \neq v$ there exists a unique point $b \in B_\varrho(A, v)$ such that vxb.*

Proof. The existence is obvious, the uniqueness follows from 4.4 and 4.1.

§ 5. ϱ-homotopies. Let $\langle X, \varrho\rangle$ be a compact SC-space, $v \in X$, $0 < k \leqslant 1$ and $0 \leqslant t \leqslant 1$. According to 2.4 there exists for every $x \in X$ exactly one point $z \in X$ such that

$$\varrho(v, z) = (1-kt)\cdot\varrho(v, x) \quad \text{and} \quad \varrho(z, x) = kt\cdot\varrho(v, x)\,. \tag{2}$$

Regarding v and k as fixed and putting $z = H_{v,k}(x, t)$, we obtain a mapping $H_{v,k}\colon X \times I \to X$ called *ϱ-homotopy*.

Evidently,

5.1. *If* $A \subset X$ *and* $C = C_\varrho(A, v)$, *then the partial function* $H_{v,k}/C \times I$ *maps* $C \times I$ *into* C.

5.2. $H_{v,k}$ *is continuous.*

The proof follows from 2.5.

5.3. *If* $\langle X, \varrho\rangle$ *is a compact* SC-WR-space *and* $kt_0 - 1 \neq 0$, *then* $H_{v,k}(x, t_0)$ *is a homeomorphism of* X *into* X.

Proof. By the compactness of X and 5.2 it suffices to show that $H_{v,k}(x, t_0)$ is a 1-1 mapping. Indeed, the equality $H_{v,k}(x_1, t_0) = H_{v,k}(x_2, t_0) = z$ implies vzx_1 and vzx_2. From $kt_0 - 1 \neq 0$ and from formula (2) we have $\varrho(v, x_1) = \varrho(v, x_2)$. If now $v = z$, we have by (2) $x_1 = x_2 = v$, for otherwise z would be a ramification point, contrary to 2.7.

A compact SC-WR-space possesses a kind of homogeneity (comp. [3], p. 49):

5.4. *If* $\langle X, \varrho\rangle$ *is a compact* SC-WR-*space*, $v, p, q \in X$, $v \neq p$, *and* vpq, *then there exists a homeomorphism* h: $X \to X$ *such that* $h(q) = p$.

Proof. Take $k = 1$, $t_0 = \varrho(p, q) : \varrho(v, q)$. By hypothesis, $1 - kt_0 \neq 0$ and so, by 5.3, $h(x) = H_{v,k}(x, t_0)$ is a homeomorphism from X into X. By the definition of h we have $vh(x)q$ and $\varrho(h(q), q) = \varrho(p, q)$. Hence, by 2.4, $h(q) = p$.

There is also another kind of homogeneity:

5.5. *If* $\langle X, \varrho\rangle$ *is a compact* SC-WR-*space*, $v \in X$, *and* $\varepsilon > 0$, *then there exists a homeomorphism* h: $X \to X$ *such that* $h(X)$ *is contained in a metric ball* $B(v, \varepsilon)$ *with the centre* v *and radius* ε.

Proof. Let m be a diameter of X, $t_0 = 1$, and let $k = 1 - \varepsilon : (m + \varepsilon)$; then $1 - kt_0 \neq 0$. By 5.3, $h(x) = H_{v,k}(x, t_0)$ is a homeomorphism, h: $X \to X$, and for every $x \in X$ we have $\varrho(v, h(x)) = (1 - kt_0)\cdot\varrho(v, x) < \varepsilon$.

With the proper choice of k we have a homotopy which moves every point not more than for a given $\varepsilon > 0$:

5.6. *If* $v \in X$, $A \subset X$, $C = C_\varrho(A, v)$ *and* $\varepsilon > 0$, *then there exists a continuous function* g: $C \times I \to C$ *such that*

1° $g(x, 0) = x$ *for every* $x \in C$,

2° $\varrho(g(x, t), x) < \varepsilon$ *for every* $(x, t) \in C \times I$,

3° $vg(x, t)x$ *for every* $(x, t) \in C \times I$,

4° $g(x, 1) \neq x$ *for every* $x \neq v$.

Proof. Take $k = \varepsilon : (m + \varepsilon)$, m being the diameter of a space X, and consider $g = H_{v,k}/C \times I$. The proof follows from 5.1, 5.2 and formula (2).

§ 6. Natural bounded homeomorphism. Suppose that $\langle X, \varrho\rangle$ is a compact SC-WR-space, $v \in X$, and $A \neq \{v\}$ is a closed subset of X. If $x \in C_\varrho(A, v)$, $x \neq v$; then, according to 4.5, there exists a unique point $y \in B_\varrho(A, v)$ with vxy. Putting $p_c(x) = y$, we define the mapping p_c: $C_\varrho(A, v)\setminus\{v\} \to B_\varrho(A, v)$ (comp. § 6 in [4]).

Putting

$$(3) \qquad H_c(x) = \begin{cases} (p_c(x), r(x)) & \text{for } x \in C_\varrho(A, v)\setminus\{v\}, \\ w & \text{for } x = v, \end{cases}$$

where $r(x) = \varrho(v, x) : \varrho(v, p_c(x))$ and $w = B_\varrho(A, v) \times \{0\}$, we obtain the mapping H_c: $C_\varrho(A, v) \to \mathrm{Bcone}\, B_\varrho(A, v)$.

6.1. *If* $B_\varrho(A, v)$ *is closed, then the mapping* H_c *is a homeomorphism of* $C_\varrho(A, v)$ *onto* $\mathrm{Bcone}\, B_\varrho(A, v)$.

The proof that H_c is 1-1 and continuous is on similar lines to that of 6.2 in [4]. It remains to show that H_c is a mapping onto. Take $(y, t) \in \mathrm{Bcone}\, B_\varrho(A, v)\setminus\{w\}$; this means that $y \in B_\varrho(A, v)$, $0 < t \leqslant 1$. By hypothesis $y \neq v$, and for every $z \in \overline{vy}$ we have $p_c(z) = y$. The function r is continuous in $\overline{vy}$, and $r(v) = 0$, $r(y) = 1$; hence there exists a point $x \in \overline{vy}$ such that $r(x) = t$, $x \neq v$. Therefore, $H_c(x) = (p_c(x), r(x)) = (y, t)$.

The equality $C_\varrho(A, v) = C_\varrho(B_\varrho(A, v), v)$ together with 3.1 and 6.1 implies that

6.2. *If* $B_\varrho(A, v) = Q^k$ *is a* k-*cell, then* $C_\varrho(A, v) = C_\varrho(Q^k, v) = Q$ *is a* $(k+1)$-*cell and* $\mathrm{Bd}\,Q = C_\varrho(\mathrm{Bd}\,Q^k, v) \cup Q^k$, $\mathrm{Int}\,Q = C_\varrho(\mathrm{Int}\,Q^k, v)\setminus(Q^k \cup \{v\})$.

Similarly we infer from 3.2 and 6.1 that

6.3. *If* $B_\varrho(A, v) = \mathrm{Bd}\,Q^k = S^{k-1}$, *then* $C_\varrho(A, v) = Q$ *is a* k-*cell and* $\mathrm{Bd}\,Q = S^{k-1}$.

Finally, from 5.6 in [4] and 6.1 we infer that

6.4. *If* $\{v, a, b\}$ *is not linear, then* $D = C_\varrho(\overline{ab}, v)$ *is a disk and* $\mathrm{Bd}\,D = \overline{va} \cup \overline{ab} \cup \overline{bv}$.

§ 7. Labile points and n-cells in X. A point p of a metric space $\langle X, \varrho\rangle$ is called *homotopically labile in* X whenever for every $\varepsilon > 0$ there exists a continuous mapping g: $X \times I \to X$ fulfilling the following properties:

1° $g(x, 0) = x$ for every $x \in X$,

2° $\varrho(x, g(x, t)) < \varepsilon$ for every $(x, t) \in X \times I$,

3° $g(x, 1) \neq p$ for every $x \in X$.

The notion of a labile point was introduced by Borsuk and Jaworowski in [2], p. 160.

7.1. *If* $\langle X, \varrho\rangle$ *is a compact* SC-*space*, $A \subset X$ *is closed*, $v, p \in X$, $v \neq p$, *and* $\overline{vp}$ *is maximal in the* ϱ-*cone* $C = C_\varrho(A, v)$, *then* p *is homotopically labile in* C.

Proof. Applying 5.6 to the ϱ-cone C, we find a continuous function g which satisfies 1° and 2°. Moreover, we have $vg(x, 1)x$ for every $x \in C$.

If for some $x \in C$ we had $g(x,1) = p$, we would have vpx, whence by the maximality of vp in C it would follow that $p = x$, which contradicts 4^o in 5.6.

Borsuk–Jaworowski's Corollary (see [2], p. 168): *If X is an n-dimensional space and $Q \subset X$ is an n-cell, then any point of the interior of Q is not homotopically labile in X.*

7.2. *If $\langle X, \varrho \rangle$ is a compact* SC-WR-*space, $Q \subset X$ is a cell,* $\dim X = \dim Q = n$, $v \in X$, $p \in \operatorname{Int} Q$, *and $v \neq p$, then there exists a point $q \in X$ such that $p \neq q$ and vpq.*

Proof. Evidently, $X = C_\varrho(X, v)$. Take a maximal prolongation $\overline{vq}$ of the segment $\overline{vp}$. Such a prolongation does exist and is unique, see 4.1. By 7.1 the point q is homotopically labile, and so we must have $q \notin \operatorname{Int} Q$; otherwise we get a contradiction of Borsuk–Jaworowski's Corollary. This shows the inequality $p \neq q$.

7.3. *If $\langle X, \varrho \rangle$ is a compact* SC-WR-*space, $Q \subset X$ is a cell,* $\dim X = \dim Q = n$, $p \in \operatorname{Int} Q$, *and* $\overline{pq} \cap \operatorname{Bd} Q = 0$, *then* $pq \subset \operatorname{Int} Q$.

Proof. Suppose that $p \neq q$. Applying 7.2 to the segment $\overline{qp}$, we have a point $v \in X$ such that qpv and $p \neq v$. Now suppose, on the contrary, that $\overline{pq} \setminus Q \neq 0$. Since Q is compact, the set $\overline{pq} \setminus Q$ is open in $\overline{pq}$. Thus there exists a segment $\overline{p_1 q_1} \subset \overline{pq}$ such that $p_1 \neq q_1$, pp_1q_1, and $\overline{p_1 q_1} \cap Q = \{p_1\}$. Evidently, $p_1 \in \operatorname{Int} Q$. Take a maximal prolongation $\overline{vm}$ of the segment $\overline{vq}$; hence $p_1 \notin \overline{qm}$. Since $p_1 \in \operatorname{Int} Q$, $p_1 \notin \overline{qm}$, and $\overline{q_1 m} \cap Q$ is closed in Q, we can choose an n-cell $Q_1 \subset Q$ such that $p_1 \in \operatorname{Int} Q_1$ and $Q_1 \cap \overline{q_1 m} = 0$. Consider the ϱ-cone $C_1 = C_\varrho(Q_1, v)$. From $v \neq p_1$, $p_1 \in \operatorname{Int} Q_1$, vp_1m, the maximality of $\overline{vm}$ in X, and $\overline{p_1 m} \cap Q_1 = (\overline{p_1 q_1} \cup \overline{q_1 m}) \cap Q_1 = \{p_1\}$ we infer that the segment $\overline{vp_1}$ is maximal in C_1; hence p_1 is homotopically labile in C_1 contrary to $p_1 \in \operatorname{Int} Q_1$, $\dim Q_1 = \dim C_1 = n$ and to the Borsuk–Jaworowski's Corollary.

As has just been shown, in an n-dimensional space every segment passing through an interior point of an n-cell $Q \subset X$ is prolongable; moreover, such a prolongation is possible up to the boundary of Q. Namely,

7.4. *If $\langle X, \varrho \rangle$ is a compact* SC-WR-*space, $Q \subset X$ is a cell,* $\dim X = \dim Q$, $v \in X$, $p \in \operatorname{Int} Q$, *then there exists a point q such that vpq,* $\overline{pq} \subset Q$, *and* $\overline{pq} \cap \operatorname{Bd} Q = q$.

Proof. Denote by $\overline{vm}$ the maximal prolongation in Q of the segment $\overline{vp}$ and observe that $\overline{pm} \cap \operatorname{Bd} Q \neq 0$. Indeed, according to 7.3 and the maximality of $\overline{vm}$ in Q, we infer that $m \in \operatorname{Bd} Q$. It is easy to see that the point $q \in \overline{pm} \cap \operatorname{Bd} Q$ nearest to p satisfies our assertion.

Applying 7.3 we can show that the interior of an n-cell Q is an open set in an n-dimensional SC-WR-space X containing Q. More exactly,

7.5. *If $\langle X, \varrho \rangle$ is a compact* SC-WR-*space, $Q \subset X$ is a cell,* $\dim X = \dim Q$, *and* $p \in \operatorname{Int} Q$, *then for every* $\varepsilon > 0$

$$\varepsilon < \varrho(p, \operatorname{Bd} Q) \quad \textit{implies} \quad B(p, \varepsilon) \subset \operatorname{Int} Q\,. \tag{4}$$

By 4.4. and 7.4 we have

7.6. *If $\langle X, \varrho \rangle$ is a compact* SC-WR-*space, $Q \subset X$ is a cell,* $\dim X = \dim Q$, *and* $v \in \operatorname{Int} Q$, *then* $B_\varrho(X, v) \subset X \setminus \operatorname{Int} Q$. *Consequently,* $\varrho(v, B_\varrho(X, v)) \geqslant \varrho(v, \operatorname{Bd} Q) > 0$.

We are going to prove that

7.7. *Under the assumptions of* 7.6, *the base* $B_\varrho(X, v)$ *is closed.*

Proof. Supposing the contrary, we would have a sequence of points $\{p_i\}$ such that $p_i \in B_\varrho(X, v)$ for $i = 1, 2, \ldots$, $\lim p_i = p_0$, and $p_0 \notin B_\varrho(X, v)$. By 7.6, $\varrho(v, p_0) = \lim \varrho(v, p_i) > 0$ and hence $v \neq p_0$. Denote by p a point of the base $B_\varrho(X, v)$ such that vp_0p holds; then $v \neq p_0 \neq p$. Applying 5.4, we would find a homeomorphism $h\colon X \to X$ such that $h(v) = p_0$. Thus p_0 would become an interior point of the cell $Q_1 = h(Q)$. By $\lim p_i = p_0$ and by 7.5 we would have $p_i \in \operatorname{Int} Q_1$ for i sufficiently large. This contradicts the maximality of the segment $\overline{vp_i}$ and 7.2.

§ 8. Star-like cells, reduction of the main theorem. Let Q be a cell contained in a compact SC-WR-space, and let $v \in \operatorname{Int} Q$. We say that Q is *star-like* with respect to v if every segment passing through v meets a boundary of Q in at most one point, i.e. if vpq and $p, q \in \operatorname{Bd} Q$ imply $p = q$.

A cell which is star-like with respect to each of its interior points will be shortly called star-like.

8.1. *If $\langle X, \varrho \rangle$ is an* SC-WR-*space, and $Q \subset X$ is a convex cell, then Q is star-like.*

Proof. Take an arbitrary point $v \in \operatorname{Int} Q$ and let vpq, $p, q \in \operatorname{Bd} Q$. We must show that $p = q$. A space $\langle Q, \varrho \rangle$ is a compact SC-WR-space. Suppose that $p \neq q$. Applying 5.4, we find a homeomorphism $h\colon Q \to Q$ such that $h(v) = p$, and so p becomes an interior point of the cell $Q_1 = h(Q)$, $\dim Q_1 = \dim Q$, whence $p \in \operatorname{Int} Q$.

A very important property of a star-like cell is contained in the following proposition:

8.2. *If $\langle X, \varrho \rangle$ is a compact* SC-WR-*space, $Q \subset X$ is a cell,* $\dim X = \dim Q$, *and Q is star-like with respect to a point* $v \in \operatorname{Int} Q$, *then* $B_\varrho(X, v)$ *is homeomorphic to* $\operatorname{Bd} Q$.

Proof. Take an arbitrary point $x \in B_\varrho(X, v)$. According to 7.6 and 7.3, $\overline{vx} \cap \operatorname{Bd} Q \neq 0$. If p and q were two points of $\overline{vx} \cap \operatorname{Bd} Q$, we would have vpq or vqp. In both cases $p = q$. Putting $h(x) = y = \overline{vx} \cap \operatorname{Bd} Q$, we define a mapping $h\colon B_\varrho(X, v) \to \operatorname{Bd} Q$. Evidently, h is a 1-1 mapping and from 4.5 we deduce that h transforms $B_\varrho(X, v)$ onto $\operatorname{Bd} Q$. The continuity of h follows from 2.5 and from the fact that $\mathrm{Bd}Q$ is compact. Finally, by 7.7, $B_\varrho(X, v)$ is compact, which guarantees that h is a homeomorphism.

Joining 8.2 and 6.3, we get

8.3. *If $\langle X, \varrho\rangle$ is a compact* SC-WR-*space*, $\dim X = n$, *and X contains an n-cell Q which is star-like with respect to a point $v \in \mathrm{Int}Q$, then X is an n-cell.*

This result reduces a proof of the main theorem to a construction of an n-cell which is star-like with respect to some point.

§ 9. Geometry of a convex cell. Let $\langle X, \varrho\rangle$ be a compact SC-WR-space of finite dimension and let $Q \subset X$ be a cell such that $\dim Q = \dim X$. We shall show that:

9.1. *The following four properties of Q are equivalent*:

1° *Q is convex*,

2° *Q is star-like*,

3° *Q is strictly convex, i.e., if a segment $\overline{pq}$ has at least three points in common with* $\mathrm{Bd}Q$, *then* $\overline{pq} \cap \mathrm{Int}Q = 0$,

4° $\mathrm{Int}Q$ *is convex.*

The equivalence is a consequence of four implications, some of which will be proved without the assumption that X is a WR-space.

The first implication, $1° \to 2°$, is contained in 8.1.

The second implication is given in the following proposition:

9.2. *If $\langle X, \varrho\rangle$ is a metric space, then* $2° \to 3°$.

The proof follows from the observation that under assumption 2°, if $x \in \mathrm{Int}Q \cap \overline{pq}$, then x follows or precedes two points of $\mathrm{Bd}Q$.

9.3. *If $\langle X, \varrho\rangle$ is a compact* SC-WR-*space*, $\dim X = \dim Q$, *then* $3° \to 4°$.

In fact, applying 7.3, we find that every segment through an interior point which does not meet the boundary lies in $\mathrm{Int}Q$. On the other hand, by 7.4, every segment joining two interior points is prolongable up to the boundary of Q.

The last implication, $4° \to 1°$, follows from the more general observation:

9.4. *If $\langle X, \varrho\rangle$ is a compact* SC-*space and $A \subset X$ is convex, then the closure of A in X is also convex.*

The proof follows immediately from 3.4 in [4] or from 2.5.

In the Euclidean n-cube I^n every metric closed ball $\bar{B}(p, r)$, where p $\mathrm{Int}I^n$ and $r < \varrho(p, \mathrm{Bd}I^n)$, is an n-cell and every metric sphere $S(p, r)$ is an $(n-1)$-sphere.

Now we show that an SC-WR-cell has the same property.

9.5. *If $\langle Q, \varrho\rangle$ is an n-cell with an* SC-WR-*metric*, $p \in \mathrm{Int}Q$ *and* $r < \varrho(p, \mathrm{Bd}Q)$, *then* $\bar{B}(p, r) = \{x\colon \varrho(p, x) \leqslant r\}$ *is an n-cell and* $S(p, r) = \{x\colon \varrho(p, x) = r\}$ *is an $(n-1)$-sphere.*

Proof. Evidently, $Q = C_\varrho(Q, p)$. By 7.4 every maximal prolongation of a segment $\overline{pq}$ ends in $\mathrm{Bd}Q$. Conversely, if $q \in \mathrm{Bd}Q$, then by property 2° from 9.1 every segment $\overline{pq}$ is maximal. Consequently, by 4.4, $B_\varrho(Q, p) = \mathrm{Bd}Q$. Now if we assign to every $x \in \mathrm{Bd}Q$ a point $h(x) = \overline{px} \cap S(p, r)$, we get a homeomorphism of $\mathrm{Bd}Q$ to $S(p, r)$ (compare 8.2). On the other hand, $\bar{B}(p, r) = C_\varrho(S(p, r), p)$ and, evidently, the $(n-1)$-sphere $S(p, r)$ is a base of such a ϱ-cone. Applying 6.3, we find that $\bar{B}(p, r)$ is an n-cell.

In a similar way it can be proved that every metric closed ball $\bar{B}(p, r)$ is an n-cell if $p \in \mathrm{Int}Q$. However, we do not know whether this is also true if $p \in \mathrm{Bd}Q$.

All metric spheres are star-like with respect to the centre but, in general, we cannot expect them to be convex. Joining 8.3 with 9.5 we find that

9.6. *An* SC-WR *compact metric space $\langle X, \varrho\rangle$ of finite dimension is a cell iff there exists a metric closed ball $\bar{B}(p, r)$ which is a cell and for which p is an interior point.*

§ 10. CT-condition. An SC-space $\langle X, \varrho)$ has a convex-triangle property (and then ϱ is called an SC-CT-metric and X is called an SC-CT-space) if the following condition holds:

(CT) *for every triple $v, p, q \in X$, $C_\varrho(\overline{pq}, v)$ is convex.*

In an SC-CT-space the operation of taking a ϱ-cone over a segment is “associative”. We have

10.1. *If $\langle X, \varrho\rangle$ is a compact* SC-*space*, $p, q, v \in X$, *and one of the ϱ-cones $C_\varrho(\overline{pq}, v)$, $C_\varrho(\overline{vp}, q)$, $C_\varrho(\overline{qv}, p)$ is convex, then all three ϱ-cones are equal.*

Evidently, the CT-condition becomes trivial for a linear triple $\{p, q, v\}$. Otherwise, by 6.4, we infer that

10.2. *If $\langle X, \varrho\rangle$ is a compact* SC-WR-CT-*space and a triple $p, q, v \in X$ is not linear, then $C_\varrho(\overline{pq}, v)$ is an* SC-WR-*disk.*

Condition (CT) states the convexity of a ϱ-cone over a segment only. However, it implies also the convexity of a ϱ-cone over an arbitrary convex set:

10.3. *If $\langle X, \varrho\rangle$ is an* SC-CT-*space*, $v \in X$, *and $A \subset X$ is convex, then $C_\varrho(A, v)$ is convex.*

Proof. Take $p, q \in C_\varrho(A, v)$ and let p', q' be points of A such that vpp' and vqq'. By the convexity of A we have $\overline{p'q'} \subset A$. By (CT), ϱ-cone

$C_\varrho(\overline{p'q'}, v)$ is convex, and so it contains the segment $\overline{pq}$. According to 4.2, $C_\varrho(\overline{p'q'}, v) \subset C_\varrho(A, v)$.

§ 11. Equivalent forms of the CT-condition. Throughout § 11 we assume that $\langle X, \varrho\rangle$ is a compact SC-WR-space.

A triple $\{p, q, v\}$ determines three ϱ-cones and (CT) ensures the convexity of each. By 10.1 the convexity of one of those ϱ-cones implies the identity of all three.

Consider now a kind of the Pasch axiom

(P) *for every six points v, p, q, v', p', q', if $pv'q$, $qp'v$, $vq'p$, then there exists a point z such that vzv' and $p'zq'$.*

In an SC-space, let us call a triangle with the vertices p, q, v the union of three sides $\overline{pq}$, $\overline{qv}$ and $\overline{vq}$. Then

11.1. *In an* SC-*space* (P) *is equivalent to the condition that each segment joining two sides of the triangle meets each segment joining the vertex common to the two sides with a point of the opposite side.*

In [8] there is the following axiom:

(Wh) *for every five points v, p, q, p', q', if $vp'q$ and $vq'p$, then there exists a point z such that pzp' and qzq'.*

11.2. *In an* SC-*space axiom* (Wh) *is equivalent to the condition that two segments joining two vertices of a triangle with points on the opposite sides meet.*

Let us introduce the following condition, needed only in this and the next paragraph:

(5) *for every triple $\{p, q, v\}$ and every $x \in C_\varrho(\overline{pq}, v)$ we have $\overline{px} \subset C_\varrho(\overline{pq}, v)$.*

Condition (5) appears to be antisymmetric, but it suffices to observe that for every $x \in C_\varrho(\overline{pq}, v)$ we have $\overline{vx} \subset C_\varrho(\overline{pq}, v)$. On the other hand, from $C_\varrho(\overline{pq}, v) = C_\varrho(\overline{qp}, v)$ and (5) we infer that $\overline{qx} \subset C_\varrho(\overline{pq}, v)$.

Now we will show that in an SC-WR-space the four conditions just introduced are equivalent.

Note first that

11.3. *If a triple $\{p, q, v\}$ is linear in an* SC-*space, then* (CT), (P), (Wh) *and* (5) *are equivalent.*

Suppose now that a triple $\{p, q, v\}$ is not linear. According to 6.4, three ϱ-cones $C_v = C_\varrho(\overline{pq}, v)$, $C_p = C_\varrho(\overline{qv}, p)$ and $C_q = C_\varrho(\overline{vp}, q)$ are disks with the common boundary equal to $B = \overline{vp} \cup \overline{pq} \cup \overline{qv}$. Under this assumption

11.4. (CT) *implies* (P).

Proof. A convex disk C_v is strictly convex in the sense of 3° in § 9. Take v', p', q' as in (P) and consider the set $\overline{vv'} \cap B$. If this set has more than two points, then $\overline{vv'} \subset B$ and the segments $\overline{vv'}$ and $\overline{p'q'}$ have p' or q' in common. If $\overline{vv'} \cap B = \{v, v'\}$, then the segment $\overline{vv'}$ separates the disk C_v between every point $q' \in \overline{vp}$ $(q' \neq v)$ and every point $p' \in \overline{vq}$ $(p' \neq v)$; hence the arc $\overline{p'q'} \subset C_v$ meets $\overline{vv'}$. Since $C_v = C_p = C_q$, the proof runs analogously for every permutation of the triple $\{p, q, v\}$ and the corresponding permutation of $\{p', q', v'\}$.

11.5. (P) *implies* (Wh).

In fact, axiom (Wh) in the triangle C_v is a special case of condition (P) in the ϱ-cones C_p and C_q.

11.6. (Wh) *implies* (5).

Proof. Take a point $x \in C_v$. By the definition of a ϱ-cone there exists a point x' such that $x' \in \overline{pq}$ and vxx', and so we have $C'_v = C_\varrho(\overline{px'}, v) \subset C_v$. Now we shall show that the segment $\overline{px}$ is contained in C'_v. If the triple $\{v, p, x'\}$ is linear, the implication is trivial. Otherwise C'_v is a disk with the boundary $B' = \overline{vp} \cup \overline{px'} \cup \overline{x'v}$. Take an arbitrary point $t \in \overline{px'}$. Segments $\overline{px}$ and $\overline{vt}$ join the vertices p and v with the points of the opposite sides in the triangle with the vertices v, p, x' and therefore, by (Wh), there exists a point $z(t)$ such that $z(t) \in \overline{px} \cap \overline{vt}$. Evidently, $z(t) \in C_v$, $z(p) = p$ and $z(x') = x$. Moreover, by applying 2.5, we check the continuity of the function z. The continuous image of $\overline{px'}$ in $\overline{px}$ contains the points p and x. Hence $\overline{px} = z(\overline{px'})$, which proves $\overline{px} \subset C'_v \subset C_v$. The proof for a segment $\overline{qx}$ follows symmetrically.

11.7. (5) *implies* (CT).

Proof. Take two points $a, b \in C_v$, find a', b' in $\overline{pq}$ such that vaa', vbb' and put $C_1 = C_\varrho(\overline{a'b'}, v)$. Evidently, C_1 C_v and $a', b \subset C_1$; hence by (5) we have $\overline{a'b} \subset C_1$. Put $C_2 = C_\varrho(\overline{ba'}, v)$. We have $C_2 \subset C_1$ and applying (5) to C_2 we have $\overline{ba} \subset C_2$.

By implications 11.3–11.7

11.8. *In a compact* SC-WR-*space $\langle X, \varrho\rangle$ the properties* (CT), (P), (Wh) *and* (5) *are equivalent.*

§ 12. Two-dimensional spaces. The following proposition shows that in two-dimensional spaces the Pasch axiom is a consequence of SC-WR, compare [3] p. 52.

12.1. *If $\langle X, \varrho\rangle$ is a compact* SC-WR-*space and* $\dim X = 2$, *then X is a* CT-*space.*

Proof. In view of 11.8 it suffices to show that X possesses property (5). The implication is trivial if the triple $\{p, q, v\}$ is linear, so we may suppose that $C = C_\varrho(\overline{pq}, v)$ is a disk with the boundary $B = \overline{vp} \cup \overline{pq} \cup \overline{qv}$. Take an arbitrary point $x \in \operatorname{Int} C$. Applying 7.4 to the segment $\overline{px}$ and to the disk C, we get a point r such that pxr and

$\overline{xr} \cap B = \{r\}$. Evidently, the point r must belong to $\overline{vq}$ and $v \neq r \neq q$. We shall prove that $\overline{px} \cap B = \{p\}$. For if this were not true, we should have a point $z \in B$ such that pzx and $p \neq z$. By the transitivity of betweenness we should have also pzr and zxr, and so the point z could not belong to $\overline{vq}$. If $z \in \overline{pv}$, then from pzr, pzv and $p \neq z$ we would have prv or pvr. In both cases $\overline{pr} \subset B$, which contradicts $x \in \overline{pr}$ and $x \in \operatorname{Int} C$. In a similar way, if $z \in \overline{pq}$, then pzq, pzr and $p \neq z$ should imply pqr or prq and we should get the same contradiction. We have thus proved that, for every point $z \in \overline{px}$ distinct from p, $\overline{xz} \cap B = 0$, whence, by 7.3, $\overline{xz} \subset \operatorname{Int} C$. Taking a sequence $\{z_n\}$ such that $z_n \in \overline{px}$, $z_n \neq p$ and $\lim z_n = p$, we have $\overline{xz_n} \subset \operatorname{Int} C$; therefore $\overline{px} \subset C$. To complete the proof of the proposition, suppose that $x \in B$ and take a sequence $\{x_n\}$ of interior points converging to x. For every n we have $\overline{px_n} \subset C$, and so, by 2.5, $\overline{px} \subset C$.

§ 13. Product of convex spaces. Let $\langle X, \varrho_1 \rangle$, $\langle Y, \varrho_2 \rangle$ be metric spaces and $a > 0$. Let p_1 and p_2 be arbitrary points of the Cartesian product $X \times Y$, i.e. $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$, $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. We put

$$\varrho_a(p_1, p_2) = [\varrho_1^a(x_1, x_2) + \varrho_2^a(y_1, y_2)]^{1/2}. \tag{6}$$

It is known that $\langle X \times Y, \varrho_a \rangle$ is a metric space and the following implication holds (see [3], p. 42):

13.1. *If $p_1 = (x_1, y_1)$, $p = (x, y)$, $p_2 = (x_2, y_2)$ are points of $\langle X \times Y, \varrho_a \rangle$, then $p_1 p p_2$ holds iff $x_1 x x_2$, $y_1 y y_2$ and*

(7) *either* $\varrho_2(y_1, y) = \varrho_2(y_1, y_2) = 0$ *or* $\varrho_1(x_1, x) : \varrho_2(y_1, y) = \varrho_1(x, x_2) : \varrho_2(y, y_2) = k$.

From the definition of a centre and 13.1 we infer by a direct calculation that

13.2. *A point $p = (x, y)$ is a centre of a pair $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$ in $\langle X \times Y, \varrho_a \rangle$ iff x is a centre of the pair x_1, x_2 in $\langle X, \varrho_1 \rangle$ and y is a centre of the pair y_1, y_2 in $\langle Y, \varrho_2 \rangle$.*

From 2.3, 2.4, 2.7, and 13.2 we easily infer (comp. [3], p. 43) that:

13.3. *If complete metric spaces $\langle X, \varrho_1 \rangle$, $\langle Y, \varrho_2 \rangle$ are respectively convex,* SC *or* WR, *then the product $\langle X \times Y, \varrho_a \rangle$ is respectively convex,* SC *or* WR.

It is easy to find a disk D with an SC-WR-metric ϱ_0 such that $D = C_{\varrho_0}(\overline{ab}, c)$ and such that if a', b', c' are the centres of $\overline{bc}$, $\overline{ac}$, $\overline{ab}$, respectively, then a point z common to the segments $\overline{a'b'}$ and $\overline{cc'}$ is not the centre of $\overline{a'b'}$. Evidently, the disk $\langle D, \varrho_0 \rangle$ possesses the CT-property. Also the k-cube I^k with an ordinary Euclidean metric possesses the CT-property. Now we will show that CT-property is not productive even in the restricted case of the class of cells. More exactly,

13.4. *For every $k \geqslant 1$ the a-product of $\langle D, \varrho_0 \rangle$ by I^k is a $(k+2)$-cell with an* SC-WR*-metric and non-convex triangles.*

Proof. By 13.3, $\langle Q, \varrho_a \rangle = \langle D \times I^k, \varrho_a \rangle$ is an SC-WR-space. Put

$$A = (a, 0, \ldots, 0), \quad B = (b, 1, 0, \ldots, 0), \quad C = (c, \tfrac{1}{2}, 0, \ldots, 0),$$
$$A' = (a', \tfrac{3}{4}, 0, \ldots, 0), \quad B' = (b', \tfrac{1}{4}, 0, \ldots, 0), \quad C' = (c', \tfrac{1}{2}, 0, \ldots, 0).$$

By 13.2 A', B', C' are the centres of the segments $\overline{BC}$, $\overline{AC}$, $\overline{AB}$ respectively. Suppose now that there exists a point Z such that $Z \in Q$, i.e. $Z = (z, t_1, \ldots, t_k)$, where $z \in D$, $t_i \in I$, CZC' and $B'ZA'$. Applying 13.2 once again, we infer from CZC' that czc' and $t_1 = \frac{1}{2}$, $t_2 = \ldots = t_k = 0$. By $A'ZB'$ and $t_1 = \frac{1}{2}$ we see that the point z must be the centre of the segment $\overline{a'b'}$; therefore we get a contradiction of the property of the disk D that no point z common to the segments $\overline{a'b'}$ and $\overline{cc'}$ is the centre of $\overline{a'b'}$.

§ 14. ϱ-cones over convex cells. Some preliminary steps are needed in the inductive construction of a convex n-cell.

14.1. *If $\langle X, \varrho \rangle$ is a compact* SC-WR*-space, $Q \subset X$ is a convex cell, $v \notin Q$, $p \in \operatorname{Int} Q$, and $\overline{vp} \cap \operatorname{Bd} Q = 0$, then $p \in B\varrho(Q, v)$ and $\overline{vp} \cap Q = p$.*

Proof. In order to prove $p \in B_\varrho(Q, v)$ it suffices, by 4.4, to show that $\overline{vp}$ is maximal in Q. If it were not, a point s would exist such that $s \neq p$, $s \in Q$ and vps. Applying 7.4 to the compact SC-WR-space $\langle Q, \varrho \rangle$ and to the segment $\overline{sp}$, we could prolonge $\overline{sp}$ up to the boundary of Q; hence, we would find a point $q \in \operatorname{Bd} Q$ such that spq. From spq, spv, $s \neq p$, we would have pqv or pvq. In the first case we get a contradiction of $\overline{vp} \cap \operatorname{Bd} Q = 0$, in the second of $v \notin Q$. Now, if there were another point $p' \in \overline{vp} \cap Q$, then from $p' \in \operatorname{Int} Q$, $\overline{vp'} \cap \operatorname{Bd} Q = 0$ and from the preceding part of the proof applied to p' it would follow that $p' \in B_\varrho(Q, v)$, which contradicts the maximality of $\overline{vp'}$ in Q and $vp'p$.

The above result can be strenghtened in a CT-space. Namely,

14.2. *If $\langle X, \varrho \rangle$ is a compact* SC-WR-CT*-space, $Q \subset X$ is a convex k-cell, $k \geqslant 2$, $v \notin Q$, $p \in \operatorname{Int} Q$, and $\overline{vp} \cap \operatorname{Bd} Q = 0$, then $B_\varrho(Q, v) = Q$*

Proof. Since the boundary of the k-cell Q is closed in X and $p \in \operatorname{Int} Q$, we can find a cell $Q_1 \subset \operatorname{Int} Q$ such that $p \in Q_1$, $\dim Q_1 = k$, and for every $q \in Q_1$ we have $\overline{vq} \cap \operatorname{Bd} Q = 0$. The existence of Q_1 is ensured by 2.5 and by the elementary properties of cells. By 14.1 for every $q \in Q_1$ we have $\overline{vq} \cap Q = q$. Let a be an arbitrary point of Q, let $\overline{vr}$ be a maximal prolongation of va in Q and suppose that $a \neq r$. Let $\overline{as}$ be a maximal prolongation of $\overline{ar}$ in Q. According to 7.4 we have $s \in \operatorname{Bd} Q$. Take $q \in Q_1 \setminus \overline{as}$ (because of $\dim Q_1 > 1$ such a point q exists). By the convexity of Q, $q, r \in Q$, $v \notin Q$, the maximality of $\overline{vq}$ in Q and by $\overline{vq} \cap Q = q$ we infer that the triple $\{v, q, r\}$ is not linear. By 6.4, $D = C_\varrho(\overline{qr}, v)$ is a disk with the boundary $B = \overline{vr} \cup \overline{rq} \cup \overline{qv}$. We have $a, q \in D$ and, by the CT-condition, $\overline{aq} \subset D$. It can be verified that $\overline{aq} \cap B = \{a, q\}$. A simple proof of this equality can be deduced from the strict convexity of D (see 9.1), because neither v nor r belong to $\overline{aq}$. So, every x belonging to $\overline{aq}$, $a \neq x \neq q$, is an interior point of a convex disk D, and by 7.4 a segment $\overline{vx}$ is prolongable to a point on $\overline{qr}$, in consequence $\overline{vx}$ is not maximal in Q. On the other hand, by an argument analogous to the beginning of the present proof, we can find a point $x \in \overline{aq}$, $a \neq x \neq q$, such that $\overline{vx} \cap \mathrm{Bd}Q = 0$, which contradicts 14.1. We have thus shown that $a \neq r$, which means that $\overline{va}$ is maximal in Q and, by 4.4, $a \in B_\varrho(Q, v)$.

§ 15. Construction of a convex n-cell. For condition (C) below see formula (4) in § 7.

15.1. *Let* $\langle X, \varrho\rangle$ *be a compact* SC-WR-CT-*space*, $\dim X = n$, $n \geqslant 1$. *Then there exists a convex cell* $Q \subset X$ *satisfying the following condition*:

(C) *if* $p \in \mathrm{Int}Q$, $0 < \varepsilon < \varrho(p, \mathrm{Bd}Q)$, *then* $\overline{B}(p, \varepsilon) \subset Q$.

Proof. Take two arbitrary points $a, b \in X$, $a \neq b$ and put $Q_1 = \overline{ab}$. If Q_1 does not satisfy (C), we find a point $p \in \mathrm{Int}Q_1$, $0 < \varepsilon < \varrho(p, \mathrm{Bd}Q_1)$, and a point $v \in \overline{B}(p, \varepsilon)\backslash Q_1$. Then the triple $\{a, b, v\}$ is evidently not linear and, by 6.4 and by CT-condition, we infer that $Q_2 = C_\varrho(\overline{ab}, v)$ is a convex disk. Suppose that $Q_m \subset X$ is a convex cell of maximal dimension m. Evidently, we have $2 \leqslant m \leqslant n$. If Q_m satisfies (C), we take $Q = Q_m$ and the proof is finished; otherwise, we find a point $p \in \mathrm{Int}Q_m$, $0 < \varepsilon < \varrho(p, \mathrm{Bd}Q_m)$, and a point $v \in \overline{B}(p, \varepsilon)\backslash Q_m$. We can see that $\overline{vp} \cap \mathrm{Bd}Q_m = 0$, whence, by 14.2, $B_\varrho(Q_m, v) = Q_m$. By 6.2, the ϱ-cone $Q_{m+1} = C_\varrho(Q_m, v)$ is an $(m+1)$-cell, by 10.3 the cell Q_{m+1} is convex and this contradicts the maximality of m.

In 7.5 it has been shown that every n-cell has property (C). We have to show that the converse is also true, namely the homogeneity of the space X stated in 5.5 implies that

15.2. *If* $\langle X, \varrho\rangle$ *is a compact* SC-WR-space *and a cell* $Q, Q \subset X$, *satisfies* (C), *then* $\dim X = \dim Q$.

Joining 15.1 with 15.2, we get

15.3. *A compact* SC-WR-CT-*space* $\langle X, \varrho\rangle$ *of finite dimension* n *contains a convex* n-*cell*.

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