Chains of simple closed curves and a dogbone space

by

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1. Introduction. R. H. Bing in [4] presented an example of an upper semicontinuous decomposition of E^3 into points and tame arcs, Bing's dogbone space, that is not topologically E^3. In [2], a second dogbone space, resulting from a simpler construction than that of Bing's dogbone space, was shown to be topologically different from E^3 but the proof could not be easily modified to apply to a third dogbone space, also presented in [2], resulting from an apparently minor change in the construction. In this paper, we prove some theorems about linking simple closed curves and use them to show that this third dogbone space is not topologically E^3.

It will be assumed where necessary or convenient that all embedded complexes are triangulated and polyhedral and any two are in relative general position and all homeomorphisms are piecewise linear.

The standard definitions and basic results employed will be those of Hocking and Young [6].

After Casler [5], if N is a positive integer, $N\alpha$ will denote a sequence of positive integers $J(1), \ldots, J(N)$, and if r is a positive integer, the sequence $J(1), \ldots, J(N), r$ will be denoted by $N\alpha, r$. If $N = 0$, $N\alpha = 0$ and $N\alpha, r = r$.

2. Chains of simple closed curves. The concept of linking of simple closed curves will be that of [3], namely, two simple closed curves X_1 and X_2 link if and only if there is a two complex Y_1 with boundary X_1 and X_2 intersects Y_1 an odd number of times.

A simple chain ζ is a collection $L_1, \ldots, L_N$, $N \geqslant 3$, of simple closed curves which can be numbered so that L_1 links only L_2, L_N links only L_{N-1} and if $i \neq 1, N$, L_i links only L_{i-1} and L_{i+1}. A closed chain ζ is a collection $L_1, \ldots, L_N$, $N \geqslant 3$, of simple closed curves which can be numbered so that each L_i links only L_{i-1} and L_{i+1}, where subscripts are taken modulo N. A simple closed curve in a chain is called a *link*.

The following is well-known:

THEOREM 1. *Suppose I^3 is a topological cube in E^3 and f is a homeomorphism of S^2 into E^3. Then each component of $I^3 \cap f(S^2)$ separates I^3 into exactly two components.*

We paraphrase Theorem 3 of [4] by Bing:

THEOREM 2. *Suppose L_1 and L_2 are two linking simple closed curves in the interior of a topological cube I^3 in E^3 and f is a homeomorphism of S^2 in E^3. Then, for each component M of $I^3 \cap f(S^2)$, there is a component of $I^3 - M$ that intersects both L_1 and L_2.*

We prove:

THEOREM 3. *Suppose ζ is a simple chain in the interior of a topological cube I^3 in E^3, f is a homeomorphism of S^2 into E^3, and some component M of $I^3 \cap f(S^2)$ separates two links of ζ. Then some link of ζ intersects M.*

Proof. Suppose M, a component of $I^3 \cap f(S^2)$ separates links L_i and L_{i+j} of ζ. Denote the two components of $I^3 - M$ by A and B with $L_i \subset A$ and $L_{i+j} \subset B$. Let T be the least integer such that $i < T \leqslant i+j$ and $L_T \cap B \neq \emptyset$. Then $L_{T-1} \subset A$. By Theorem 2, $L_T \cap A \neq \emptyset$. Thus, L_T intersects M and the proof of Theorem 3 is completed.

Since a closed chain $\zeta = \{L_1, \ldots, L_N\}$ may be expressed for each integer J as the sum of two simple chains $\{L_1, \ldots, L_J\}$ and $\{L_J, \ldots, L_N, L_1\}$, we apply Theorem 3 twice and obtain:

THEOREM 4. *Suppose ζ is a closed chain in the interior of a topological cube I^3 in E^3, f is a homeomorphism of S^2 into E^3 and some component M of $I^3 \cap f(S^2)$ separates two links of ζ. Then two links of ζ intersect M.*

The proof of the following is inspired by Theorem 5 of [4] by Bing:

THEOREM 5. *Suppose ζ is a closed chain in the interior of a topological cube I^3 in E^3, f is a homeomorphism of S^2 in E^3, $\{U_t\}, 1 \leqslant t \leqslant T$, is the set of components of $I^3 - f(S^3)$, and $L_i \in \zeta$. Then some U_t intersects L_i and two other elements of ζ.*

Proof. Let Z be a continuum such that

(a) Z is composed of closures of elements of $\{U_t\}, 1 \leqslant t \leqslant T$,

(b) Z intersects L_i and two other elements of ζ,

(c) no proper subcontinuum of Z satisfies (a) and (b).

We show Z contains exactly one element of $\{U_t\}, 1 \leqslant t \leqslant T$. For, suppose Z contains two elements of $\{U_t\}, 1 \leqslant t \leqslant T$. Then, Z is the sum of two proper subcontinua Z_1 and Z_2, both composed of closures of elements of $\{U_t\}, 1 \leqslant t \leqslant T$, and $Z_1 \cap Z_2 = M$ for some component M of $I^3 \cap f(S^2)$. Suppose Z intersects L_i, L_n and L_m. We show a contradiction when we show that the assumption that M does not separate any pair of L_i, L_n and L_m violates (c) and the assumption that M separates some pair of L_i, L_n and L_m violates (c).

Suppose M does not separate any pair of L_i, L_n and L_m. Then, each of L_i, L_n and L_m intersects M or some one of L_i, L_n and L_m does not intersect M. If each of L_i, L_n and L_m intersects M, then each of L_i, L_n and L_m intersects both Z_1 and Z_2, a violation of (c). If some one of L_i, L_n and L_m, say L_i, does not intersect M, then L_i intersects exactly one of Z_1 and Z_2, say Z_1. Then, each of L_n and L_m must intersect Z_1 since, if, say L_n intersects Z_2 only, L_n would not intersect M and M would separate L_i and L_n. Thus, each of L_i, L_n and L_m would intersect Z_1, a violation of (c). Thus, the assumption that M does not separate any pair of L_i, L_n and L_m violates (c).

Suppose M separates some pair of L_i, L_n and L_m. If M separates L_i and, say, L_n, by Theorem 4 two links L_p and L_q of ζ intersect M and $L_i \neq L_p, L_q$ since L_i does not intersect M. Thus, L_i, L_p and L_q intersect one of Z_1 and Z_2, a violation of (c). If M separates L_n and L_m, by Theorem 4 two links L_p and L_q of ζ intersect M and one of them, say L_p, is not L_i, then, L_i, L_p and one of L_n and L_m would intersect one of Z_1 and Z_2, a violation of (c). Thus, the assumption that M separates some pair of L_i, L_n and L_m violates (c).

Thus, the promised contradiction has been demonstrated and the proof of Theorem 5 is complete.

Theorem 5 is the best result obtainable since for every integer $N \geqslant 3$, it is possible to construct a closed chain ζ of N elements in the interior of a topological cube I^3 in E^3 and a homeomorphism f of S^2 into E^3 such that every component of $I^3 \cap f(S^2)$ intersects at most three elements of ζ.

A result needed later is:

THEOREM 6. *Suppose ζ is a closed chain in the interior of a topological cube I^3 in E^3 and f is a homeomorphism of S^2 into E^3. Then either*

(i) *some component of $I^3 \cap f(S^2)$ separates two elements of ζ, or,*

(ii) *some component of $I^3 - f(S^2)$ intersects each element of ζ.*

Proof. We suppose (i) is false and show (ii) is true. If (i) is false, then no component M of $I^3 \cap f(S^2)$ separates any two elements of ζ and, hence, for any two elements of ζ, there is a component of $I^3 - f(S^2)$ intersecting both. Thus, we complete the proof of Theorem 6, by applying the following theorem by Bing [4, Theorem 5]:

Suppose U is the interior of a topological cube, Y is a collection of bounded continua in U, and M is a compact 2-manifold with boundary such that for each pair of elements of Y, there is a component of $U - M$ intersecting both of these elements. Then there is a component of $U - M$ intersecting each element of Y.

3. Topological Figure Eights and Property R. An arc l is the image of the unit interval $I = [0, 1]$ under a homeomorphism which will also be denoted by l. The end-points of an arc l are $l(0)$ and $l(1)$ and l may be written $l(0)l(1)$. A p-od k is the union of p arcs $\{l_i\}$ such that if $i \neq j$, $l_i \cap l_j = l_i(0) = l_j(0)$; the center of k is $l_1(0)$ and the set of end-points of k is $\{l_i(1)\colon i = 1, \ldots, p\}$.

Suppose l is an arc and A and B are sets. The integer N is an Intersection Number of l with respect to A and B if and only if there are $N+1$ points $v_0, \ldots, v_N$ in I such that $0 = v_0 < \ldots < v_N = 1$ and for each K, $l([V_K, V_{K+1}])$ intersects at most one of A and B.

A topological figure eight has Property R with respect to sets A and B if and only if for every two points p and q in opposite loops there is an arc l in it from p to q and 2 is an Intersection Number of l with respect to A and B.

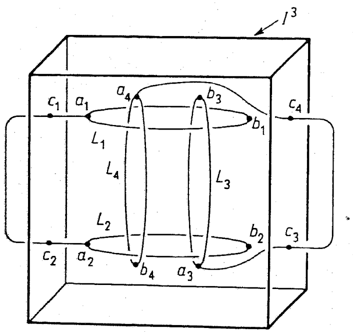


Fig. 1

For the remainder of this section, we adopt the notation of Figure 1. As in Figure 1, let I^3 be a topological cube in E^3 and $L_1, \ldots, L_4$ a collection of simple closed curves in Interior (I^3) linked as shown. For each $i = 1, \ldots, 4$, L_i is the sum of two arcs from a_i to b_i which intersect only at their end-points; to distinguish these arcs, we arbitrarily designate one $+a_ib_i$ and the other $-a_ib_i$; a_ic_i is an arc with c_i only on Boundary (I^3). The arcs c_1c_2 and c_3c_4 are in the complement of Interior (I^3) with end-points only on Boundary (I^3).

The main result of this section is

THEOREM 7. *Suppose f is a homeomorphism of S^2 into E^3, A and B are closed disjoint subsets of $f(S^2)$, $I^3 \cap f(S^2) \subset A \cup B$, for each $i = 1, \ldots, 4$, each arc $\pm a_ib_i$ intersects at most one of A and B, and $f(S^2) \cap (c_1c_2 \cup c_3c_4 \cup \bigcup \sum a_ic_i) = \emptyset$. Then there is a topological figure eight Φ in Interior $(I^3) \cup c_1c_2 \cup c_3c_4$ such that c_1c_2 and c_3c_4 are in opposite loops of Φ and Φ has Property R with respect to A and B.*

Proof. From Theorem 6 we have

(i) some component of $I^3 \cap f(S^2)$ separates two of $L_1, \ldots, L_4$ or

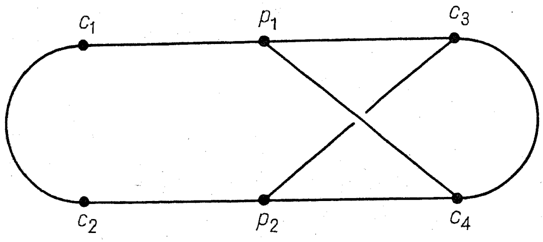
(ii) some component of $I^3 - f(S^2)$ intersects each of $L_1, \ldots, L_4$.

We begin the argument for (i) by supposing that some component M of $I^3 \cap f(S^2)$ separates two of $L_1, \ldots, L_4$. The component M cannot separate L_1 from L_3 or L_4 nor L_2 from L_3 or L_4. Thus, we assume that M separates L_1 from L_2. Then, by Theorem 5, some component U_1 of $I^3 - f(S^2)$ intersects L_1, L_3 and L_4 and some component U_2 of $I^3 - f(S^2)$ intersects L_2, L_3 and L_4.

Select a point p_1 in U_1 and for $i = 1, 3, 4$, construct an arc p_1c_i by constructing an arc in U_1 from p_1 to L_i; then along L_i to a_i so as to intersect at most one of A and B, and finally along a_ic_i. There results a 3-od k_1 in I^3 such that p_1 is the center of k_1, the end-points of k_1 are c_1, c_3 and c_4, and each arc p_1c_i, $i = 1, 3, 4$ intersects at most one of A and B. Similarly, select a point p_2 in U_2 and construct a 3-od k_2 with center p_2 and end-points c_2, c_3 and c_4 such that each arc p_2c_i, $i = 2, 3, 4$, in k_2 intersects at most one of A and B. Let $K = k_1 \cup k_2 \cup c_1c_2 \cup c_3c_4$.

A copy of K is shown in Figure 2a (see p. 136). We show how to construct the desired figure eight Φ by selecting, except for one case, arcs p_ic_j in K or arcs each of which are so close to some arc p_ic_j that the selected arc intersects A or B only if p_ic_j intersects A or B. It is always true that $(c_1c_2 \cap c_3c_4) \cap (A \cup B) = \emptyset$. The cases where some p_ic_j does not intersect $A \cup B$ may be neglected. Thus, we have six arcs of the form p_ic_j, each of which intersects at most one of A and B, a total of 64 cases. However, we may assume p_1c_1 intersects only A and appeal to symmetry in the cases where p_1c_1 intersects B. Further, if p_1c_1 intersects A, the problem is not simplified if we assume p_2c_2 intersects B. Thus, we need consider only the 16 cases listed in Figure 2b (see p. 136), where each row is a case and the letter A or B in each column designates which of the sets A or B each p_ic_j intersects. The solutions, the desired figure eight Φ with Property R, are shown in Figure 3 (see p. 137). Except for case 13, the solutions are obtainable from arcs in K or arcs near K. To solve case 13, we use a theorem by Bing [4, Theorem 6], paraphrased for our purposes;

Suppose A and B are two mutually exclusive closed subset of a topological cube I^3 and $c_3p_1c_4$ and $c_3p_2c_4$ are homotopic arcs in I^3 such that $c_3p_1c_4 \cap A = c_3p_2c_4 \cap B = \emptyset$. Then, there is an arc l in I^3 with end-points c_3 and c_4 such that $l \cap (A \cup B) = \emptyset$.



a

	p_1c_3	p_1c_4	p_2c_3	p_2c_4
1	A	A	A	A
2	A	A	A	B
3	A	A	B	A
4	A	A	B	B
5	A	B	A	A
6	A	B	A	B
7	A	B	B	A
8	A	B	B	B
9	B	A	A	A
10	B	A	A	B
11	B	A	B	A
12	B	A	B	B
13	B	B	A	A
14	B	B	A	B
15	B	B	B	A
16	B	B	B	B

b

Fig. 2

The arc l allows the solution of case 13 and the argument when some component M of $I^3 \cap f(S^2)$ separates L_1 and L_2 is complete. The argument when some component of $I^3 \cap f(S^2)$ separates L_3 and L_4 follows by symmetry, thus completing the argument for (i).

The argument for (ii) follows readily since it may be assumed that the centers p_1 and p_2 of the 3-ods k_1 and k_2 of (i) are in the same component of $I^3 - f(S^2)$. Thus, the proof of Theorem 7 is complete.

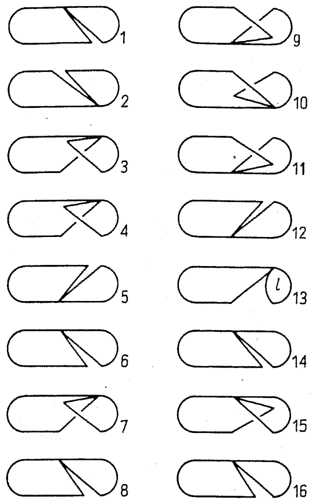


Fig. 3

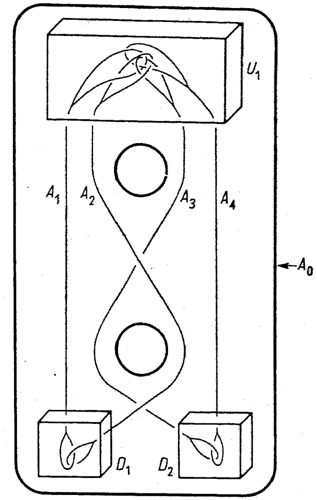


Fig. 4

4. A dogbone space that is not topologically E^3. To construct the dogbone space of this paper, let A_0 be a solid double torus in E^3, as in Figure 4. Embed a cube U_1 in the top of A_0 and cubes D_1 and D_2 in the bottom of A_0. Then, embed solid double tori $A_1, \ldots, A_4$, linked as indicated, in A_0; although each of $A_1, \ldots, A_4$ is shown as a finite graph, it is topologically equivalent to A_0. For each $i = 1, \ldots, 4$, Closure $\big(A_i - (U_1 \cup D_1 \cup D_2)\big)$ is a topological cube and the intersection of Interior (A_i) with any horizontal plane is an open disk or the sum of two disjoint open disks.

For each $i = 1, \ldots, 4$, cubes $U_{i,1}, D_{i,1}$ and $D_{i,2}$ and solid double tori $A_{i,1}, \ldots, A_{i,4}$ are embedded in A_i such that there is a homeomorphism of E^3 onto itself which is the identity on the compliment of some open set containing A_0 and takes A_0 onto A_i, U_1 onto $U_{i,1}$ and D_j onto $D_{i,j}$, $j = 1, 2$. Let this process be continued; succeeding steps of the construction may be described inductively.

Let $M = A_0 \cap \sum A_i \cap \sum A_{i,j} \cap \sum A_{i,j,k} \cap \ldots$ Let G be the set whose elements are components of M and one-point subsets of $E^3 - M$. Then, G is an upper semicontinuous decomposition of E^3 into tame arcs and one-point sets. Let E^3/G denote the associated decomposition space, the dogbone space of this paper. We show:

THEOREM 8. *E^3/G is not topologically E^3.*

Proof. To assist in the proof of Theorem 8, we state some definitions and prove some lemmas.

Let C denote $U_1 \cup D_1 \cup D_2 \cup \sum A_i$, $i = 1, \ldots, 4$. Then, C is a topological cube with handles. As in Figure 5, let Γ_0 be a central curve of C consisting of points u_1, d_1 and d_2 and arcs $a_1, \ldots, a_4$, where the end-points of a_i are u_1 and d_j if A_i intersects U_1 and D_j. Similarly, for a fixed sequence $N\alpha$, $U_{N\alpha,1} \cup D_{N\alpha,1} \cup D_{N\alpha,2} \cup \sum A_{N\alpha,i}$, $i = 1, \ldots, 4$, is a cube with handles with central curve $\Gamma_{N\alpha}$.

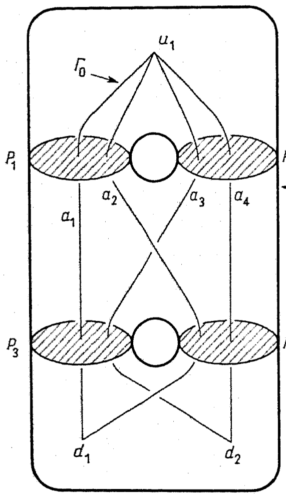


Fig. 5

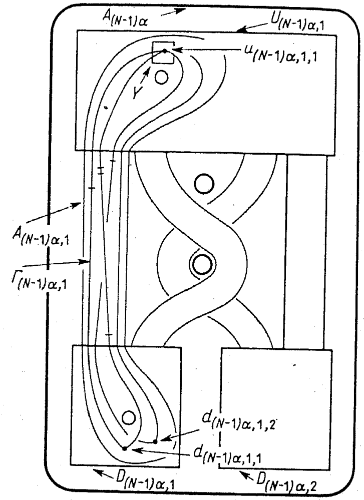


Fig. 6

Also in Figure 5, let $P_1, \ldots, P_4$ be disks in A_0 such that for each i, $P_i \cap \mathrm{Boundary}(A_0) = \mathrm{Boundary}(P_i)$. We may regard $P_1 \cup P_2 \cup P_3 \cup P_4$ as the intersection of A_0 with a homeomorphic image of S^2.

The statement and proof of the following lemma is identical to that of the proof of Lemma 1 for Theorem 5 of [2].

LEMMA 1. *Suppose g is a continuous function of A_0 into A_0 which is homotopic to the identity by a homotopy G which is fixed on Boundary(A_0). Then, for some $i = 1, \ldots, 4$, $g(a_i)$ intersects both $P_1 \cup P_2$ and $P_3 \cup P_4$.*

We prove:

LEMMA 2. *Suppose N is a positive integer and F is a homeomorphism of Boundary$(A_0) \cup \sum P_i$, $i = 1, \ldots, 4$, into A_0 which satisfies*

(i) F is the identity on Boundary(A_0),

(ii) each $a_{N\alpha,i}$ in each $\Gamma_{N\alpha}$ intersects at most one on $F(P_1 \cup P_2)$ and $F(P_3 \cup P_4)$.

Then there is a homeomorphism h of Boundary$(A_0) \cup \sum P_i$, $i = 1, \ldots, 4$, into A_0 which satisfies

(i) h is the identity on Boundary (A_0),

(ii) each $a_{(N-1)\alpha,i}$ in each $\Gamma_{(N-1)\alpha}$ intersects at most one of $h(P_1 \cup P_2)$ and $h(P_3 \cup P_4)$.

Proof. Suppose N is a positive integer and F is a homeomorphism which satisfy the hypotheses of the lemma. Let $(N-1)\alpha$ be a fixed sequence. The solid double torus $A_{(N-1)\alpha}$ is shown in Figure 6. For clarity, only the details of $A_{(N-1)\alpha,1}$ and $\Gamma_{(N-1)\alpha,1}$ are shown and possible intersections of $F(P_1 \cup P_2 \cup P_3 \cup P_4)$ with $a_{(N-1)\alpha,i}$, $i = 1, \ldots, 4$, are indicated. It may be assumed that $F(P_1 \cup P_2 \cup P_3 \cup P_4)$ does not intersect $u_{(N-1)\alpha,1,1} \cup d_{(N-1)\alpha,1,1} \cup d_{(N-1)\alpha,1,2}$ since $F(P_1 \cup P_2 \cup P_3 \cup P_4)$ could be adjusted in a neighborhood of, say, $u_{(N-1)\alpha,1,1}$ without adding intersections to any arc $a_{(N-1)\alpha,1,i}$. Thus, a cube Y may be constructed in $A_{(N-1)\alpha,1}$ such that Y contains $u_{(N-1)\alpha,1,1}$, $Y \cap \Gamma_{(N-1)\alpha,1}$ is a 4-od and Y does not intersect $F(P_1 \cup P_2 \cup P_3 \cup P_4)$. Replace $Y \cap \Gamma_{(N-1)\alpha,1}$ by two 3-ods with a single common end-point, expand Y by a homeomorphism h_1 of E^3 onto E^3 which is the identity on the complement of Interior$(A_{(N-1)\alpha,1})$ and arrive at the situation of Figure 7 (see p. 140).

If a cube Y' similar to Y is constructed in $A_{(N-1)\alpha,2}$, $Y' \cap \Gamma_{(N-1)\alpha,2}$ is replaced by two 3-ods and Y' is expanded by a homeomorphism h_2 of E^3 onto E^3 which is the identity on the complement of Interior$(A_{(N-1)\alpha,2})$, there results four simple closed curves which link in $D_{(N-1)\alpha,1}$ as shown in Figure 8 (see p. 140). For $i = 1, 2$ each pair of simple closed curves in $A_{(N-1)\alpha,i}$ is connected by an arc in $A_{(N-1)\alpha,i}$ which does not intersect $h_2h_1F(P_1 \cup P_2 \cup P_3 \cup P_4)$; for each i, let l_i denote the closure of each arc in the complement of $D_{(N-1)\alpha,1}$. Each simple closed curve in $D_{(N-1)\alpha,1}$ is the sum of two arcs each of which intersects at most one of $h_2h_1F(P_1 \cup P_2)$ and $h_2h_1F(P_3 \cup P_4)$. We apply Theorem 7 to obtain a topological figure eight Φ_1, shown in Figure 9 (see p. 140), composed of a 4-od k_1 and the arcs l_1 and l_2 such that l_1 and l_2 are in opposite loops of Φ_1 and Φ_1 has Property R with respect to $h_2h_1F(P_1 \cup P_2)$ and $h_2h_1F(P_3 \cup P_4)$. By a homeomorphism h_3 of E^3 onto E^3 which is the identity on the complement of a small neighborhood W_1 of $D_{(N-1)\alpha,1}$, each component of $h_2h_1F(P_1 \cup P_2 \cup P_3 \cup P_4)$ may be pushed along the arc of k_1 it intersects to the complement of $D_{(N-1)\alpha,1}$ so that $k_1 \cap h_3h_2h_1F(P_1 \cup P_2 \cup P_3 \cup P_4) = \emptyset$ and Φ_1 has Property R with respect to $h_3h_2h_1F(P_1 \cup P_2)$ and $h_3h_2h_1F(P_3 \cup P_4)$.

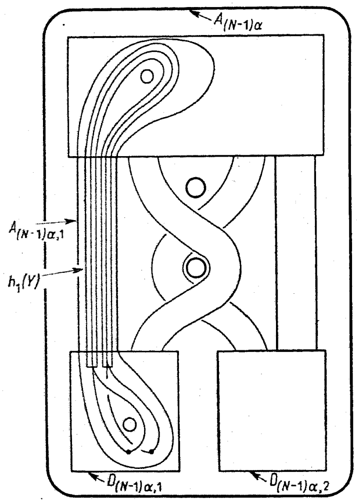


Fig. 7

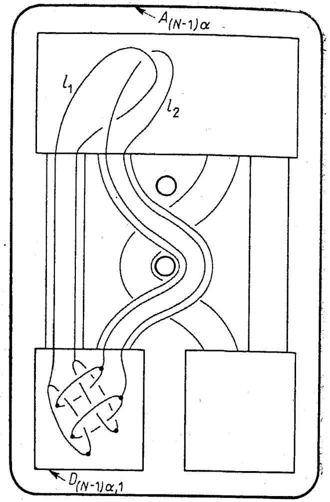


Fig. 8

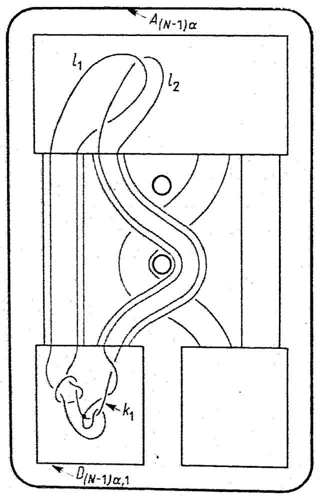


Fig. 9

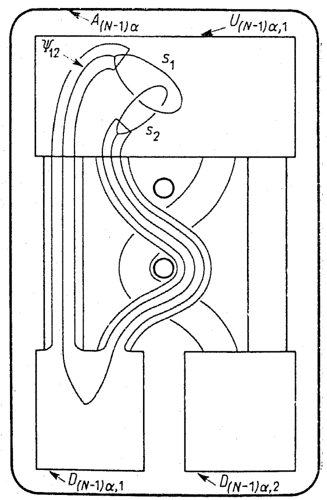


Fig. 10

The 4-od k_1 is contained in $D_{(N-1)\alpha,1}$ and has endpoints only on Boundary$(D_{(N-1)\alpha,1})$. Let W_2 be a neighborhood of $D_{(N-1)\alpha,1}$ contained in W_1. The cutting and sewing process of [1] may be applied which results in a homeomorphism h_4 of $P_1 \cup P_2 \cup P_3 \cup P_4$ into A_0 such $D_{(N-1)\alpha,1} \cap h_4(P_1 \cup P_2 \cup P_3 \cup P_4) = \emptyset$ and for each i, h_4 is the identity on Boundary(P_i), $h_4(\text{Interior}(P_i)) \subset \text{Interior}(A_0)$, $h_4(P_i) - W_2 \subset h_3h_2h_1F(P_i)$ and Φ_1 has Property R with respect to $h_4(P_1 \cup P_2)$ and $h_4(P_3 \cup P_4)$. An important point is that for each sequence $(N-1)\beta, j \neq (N-1)\alpha, 1$ or $(N-1)\alpha, 2$, $h_4(P_i)$, $i = 1, \ldots, 4$, intersects an arc $a_{(N-1)\beta,j,c}$ in $\Gamma_{(N-1)\beta,j}$ only if $F(P_i)$ intersects $a_{(N-1)\beta,j,c}$ since $h_4(P_i) - W_2 \subset h_3h_2h_1F(P_i)$ and $h_3h_2h_1$ is the identity on the complement of $A_{(N-1)\alpha,1} \cup A_{(N-1)\alpha,2}$. Extend h_4 to a homeomorphism of Boundary$(A_0) \cup \sum P_i$, $i = 1, \ldots, 4$, by defining h_4 as the identity on Boundary(A_0).

Let h_5 be a homeomorphism of E^3 onto E^3 which is the identity on the complement of Interior$(A_{(N-1)\alpha,1} \cup A_{(N-1)\alpha,2} \cup D_{(N-1)\alpha,1})$ and, as shown in Figure 10, expands Interior $(D_{(N-1)\alpha,1})$ so that $h_5(\text{Interior}(D_{(N-1)\alpha,1}))$ contains $(a_{(N-1)\alpha,1} \cup a_{(N-1)\alpha,2}) - U_{(N-1)\alpha,1}$. The closure of $h_5(\Phi_1 - D_{(N-1)\alpha,1})$ is composed of two arcs, $h_5(l_1)$ and $h_5(l_2)$. For each $i = 1, 2$, extend $h_5(l_i)$ to a point in the interior of the component of $U_{(N-1)\alpha,1} \cap h_5(D_{(N-1)\alpha,1})$ it intersects and from this point construct an arc in $U_{(N-1)\alpha,1} \cup h_5(D_{(N-1)\alpha,1})$ to $\Gamma_{(N-1)\alpha} \cap$ Boundary$(U_{(N-1)\alpha,1})$. Thus, a finite graph Ψ_{12} composed of two simple closed curves s_1 and s_2 joined by a connecting arc has been constructed in $A_{(N-1)\alpha,1} \cup A_{(N-1)\alpha,2} \cup D_{(N-1)\alpha,1}$. The simple closed curves s_1 and s_2 are linked and each links $A_{(N-1)\alpha,3}$ and $A_{(N-1)\alpha,4}$ in Interior$(U_{(N-1)\alpha,1})$. That part of Ψ_{12} in the complement of $U_{(N-1)\alpha,1}$, which is also that part of the connecting arc in the complement of $U_{(N-1)\alpha,1}$ is $(a_{(N-1)\alpha,1} \cup a_{(N-1)\alpha,2}) - U_{(N-1)\alpha,1}$. The connecting arc does not intersect $h_5h_4(P_1 \cup P_2 \cup P_3 \cup P_4)$. Since $h_5(\Phi_1)$ has Property R with respect to $h_5h_4(P_1 \cup P_2)$ and $h_5h_4(P_3 \cup P_4)$, if p and q are points in, respectively, s_1 and s_2, there is an arc pq in Ψ_{12} such that 2 is an Intersection Number of pq with respect to $h_5h_4(P_1 \cup P_2)$ and $h_5h_4(P_3 \cup P_4)$. If $(N-1)\beta, j \neq (N-1)\alpha, 1$ or $(N-1)\alpha, 2$, $h_5h_4(P_i)$ intersects an arc $a_{(N-1)\beta,j,c}$ in $\Gamma_{(N-1)\beta,j}$ only if $F(P_i)$ intersects $a_{(N-1)\beta,j,c}$, since h_5 is the identity on the complement of Interior$(A_{(N-1)\alpha,1} \cup A_{(N-1)\alpha,2} \cup D_{(N-1)\alpha,1})$.

Thus far, the definition of homeomorphisms and construction has been done relative to $A_{(N-1)\alpha,1}$, $A_{(N-1)\alpha,2}$ and $D_{(N-1)\alpha,1}$. A similar definition of homeomorphisms and construction is to be done relative to $A_{(N-1)\alpha,3}$, $A_{(N-1)\alpha,4}$ and $D_{(N-1)\alpha,2}$ resulting, as shown in Figure 11, in a homeomorphism h_6 of Boundary$(A_0) \cup \sum P_i$, $i = 1, \ldots, 4$, into A_0 which is the identity on Boundary(A_0), and a finite graph Ψ_{34} in $A_{(N-1)\alpha,3} \cup A_{(N-1)\alpha,4} \cup D_{(N-1)\alpha,2}$. In the complement of $A_{(N-1)\alpha,3} \cup A_{(N-1)\alpha,4} \cup D_{(N-1)\alpha,2}$, for each i, $h_6(P_i)$ is contained in $h_5h_4(P_i)$. Thus, for $(N-1)\beta, j \neq (N-1)\alpha, n$, $n = 1, \ldots, 4$, each arc $a_{(N-1)\beta,j,c}$ in each $\Gamma_{(N-1)\beta,j}$ intersects $h_6(P_i)$ only if $F(P_i)$ intersects $a_{(N-1)\beta,j,c}$. The finite graph Ψ_{34} is composed of two simple closed curves s_3 and s_4 joined by a connecting arc. The connecting arc does not intersect $h_6(P_1 \cup P_2 \cup P_3 \cup P_4)$. If p and q are points in, respectively, s_3 and s_4 there is an arc pq in Ψ_{34} such that 2 is an Intersection Number of pq with respect to $h_6(P_1 \cup P_2)$ and $h_6(P_3 \cup P_4)$. All four simple closed curves in Ψ_{12} and Ψ_{34} are linked in Interior $(U_{(N-1)\alpha,1})$. The sum of Ψ_{12} and Ψ_{34} in the complement of $U_{(N-1)\alpha,1}$, which is also

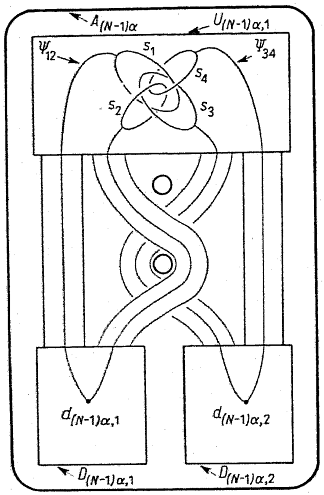


Fig. 11

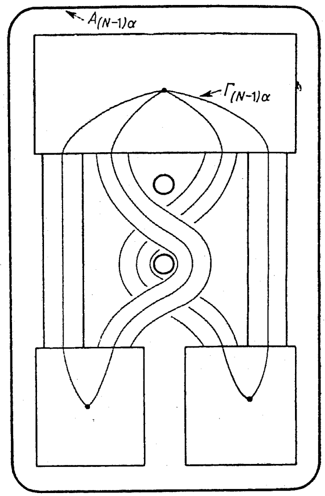


Fig. 12

the sum of the connecting arcs in the complement of $U_{(N-1)\alpha,1}$, is $\Gamma_{(N-1)\alpha} - U_{(N-1)\alpha,1}$.

Since the four simple closed curves in Ψ_{12} and Ψ_{34} are linked in $U_{(N-1)\alpha,1}$, by Theorems 3 and 5 of [4], there is a component V of $U_{(N-1)\alpha,1} - h_6(P_1 \cup P_2 \cup P_3 \cup P_4)$ which intersects each simple closed curve. In V, select a point p_2 and for $i = 1, \dots, 4$, construct an arc $p_2 r_i$ to a point r_i in s_i. Then, for each Ψ_{ij}, $ij = 12, 34$, there is an arc $r_i r_j$ in Ψ_{ij} such that 2 is an Intersection Number of $r_i r_j$ with respect to $h_6(P_1 \cup P_2)$ and $h_6(P_3 \cup P_4)$ and we select a point r_{ij} in $r_i r_j$ such that each of $r_i r_{ij}$ and $r_{ij} r_j$ intersects at most one of $h_6(P_1 \cup P_2)$ and $h_6(P_3 \cup P_4)$. The sum of the arcs $\pm p_2 r_{ij}$, $ij = 12, 23$, is a figure eight, Φ_2. By a homeomorphism h_7 of E^3 onto E^3 which is the identity on the complement of $U_{(N-1)\alpha,1} \cup \cup \sum A_{(N-1)\alpha,n}$, $n = 1, \dots, 4$, push each of r_{12} and r_{34} along the arc of $\pm p_2 d_{(N-1)\alpha,m}$, $m = 1, 2$, they intersect to, respectively, $d_{(N-1)\alpha,1}$ and $d_{(N-1)\alpha,1}$ and push the intersections of Φ_2 and $h_6(P_1 \cup P_2 \cup P_3 \cup P_4)$ along

the arcs they intersect to the complement of $U_{(N-1)\alpha,1}$. Then, Φ_2 may be regarded as the union of four arcs $\pm p_2 d_{(N-1)\alpha,m}$, $m = 1, 2$, and each arc intersects at most one of $h_7 h_6(P_1 \cup P_2)$ and $h_7 h_6(P_3 \cup P_4)$. Since $\Phi_2 \cap U_{(N-1)\alpha,1}$ is a 4-od k_2 with center p_2 and end-points only on Boundary $(U_{(N-1)\alpha,1})$ and $k_2 \cap h_7 h_5(P_1 \cup P_2 \cup P_3 \cup P_4) = \emptyset$, we apply the cutting and sewing process of [1] in a small neighborhood W_3 of $U_{(N-1)\alpha,1}$ and obtain a homeomorphism h_8 of Boundary $(A_0) \cup \sum P_i$, $i = 1, \dots, 4$, into A_0 which is the identity on Boundary(A_0), each arc $\pm p_2 d_{(N-1),j}$, $j = 1, 2$, in Φ_2 intersects at most one of $h_8(P_1 \cup P_2)$ and $h_8(P_3 \cup P_4)$, $U_{(N-1)\alpha,1} \cap h_8(\text{Boundary}(A_0) \cup P_1 \cup P_2 \cup P_3 \cup P_4) = \emptyset$ and for $i = 1, \dots, 4$, $h_8(P_i) - W_2 \subset h_7 h_6(P_i)$. Since $U_{(N-1)\alpha,1} \cap h_8(\text{Boundary}(A_0) \cup P_1 \cup P_2 \cup \cup P_3 \cup P_4) = \emptyset$, we may replace k_2 by $\Gamma_{(N-1)\alpha} \cap U_{(N-1)\alpha,1}$, as shown in Figure 12. Thus, h_8 is a homeomorphism of Boundary$(A_0) \cup \sum P_i$, $i = 1, \dots, 4$, into A_0 which is the identity on Boundary(A_0), each $a_{(N-1)\alpha,j}$ in $\Gamma_{(N-1)\alpha}$ intersects at most one of $h_8(P_1 \cup P_2)$ and $h_8(P_3 \cup P_4)$, and for $i = 1, \dots, 4$, $h_8(P_i) - A_{(N-1)\alpha} \subset F(P_i)$.

To complete the proof of Lemma 2, we note that the construction and definition of homeomorphisms leading to the definition of the homeomorphism h_8 has been done relative to $A_{(N-1)\alpha}$. If a similar construction and definition of homeomorphisms were done for each $A_{(N-1)\beta}$, $(N-1)\beta \neq (N-1)\alpha$, the result would be a homeomorphism h satisfying the conclusions of Lemma 2. Thus, the proof of Lemma 2 is completed.

We now prove Theorem 8 by showing that the assumption that E^3/G is topologically E^3 leads to a contradiction.

For, suppose E^3/G is topologically E^3. Then, we have the following two definitions of the shrinking number L of E^3/G which were shown to be equivalent in [2]:

Definition. If E^3/G is topologically E^3, the shrinking number L of E^3/G is the least integer such that there is a homeomorphism g of E^3 onto E^3 which satisfies

(i) g is isotopic to the identity by an isotopy which is fixed on the complement of Interior(A_0), and

(ii) each $g(a_{L\alpha,i})$ in each $g(\Gamma_{L\alpha})$ intersects at most one of $P_1 \cup P_2$ and $P_3 \cup P_4$.

Definition. If E^3/G is topologically E^3, the shrinking number L of E^3/G is the least integer such that there is a homeomorphism F of Boundary$(A_0) \cup \sum P_i$, $i = 1, \dots, 4$, into A_0 which satisfies

(i) F is the identity on Boundary(A_0), and

(ii) each $a_{L\alpha,i}$ in each $\Gamma_{L\alpha}$ intersects at most one of $F(P_1 \cup P_2)$ and $F(P_3 \cup P_4)$.

By the first of the above definitions and Lemma 1, L is not zero since there is no homeomorphism g of E^3 onto E^3 which is isotopic to the

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identity by an isotopy which is fixed on the complement of Interior(A_0) and such that for each $i = 1, \ldots, 4$, $g(a_i)$ intersects at most one of $P_1 \cup P_2$ and $P_3 \cup P_4$. By the second of the above definitions and Lemma 2, L cannot be greater than zero since if L is any integer greater than zero and F is a homeomorphism satisfying the requirements of the second of the above definitions, there is a homeomorphism h such that $L-1$ and h also satisfy the requirements of the second of the above definitions. The contradiction that L is not zero nor greater than zero completes the proof of Theorem 8.

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Some characterizations of paracompactness in k-spaces *

by

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1. Introduction. Paracompact spaces and k-spaces both have the distinction of being simultaneous generalizations of metric and compact spaces. The purpose of this paper is to present some of the interactions between these seemingly unrelated notions. Throughout this paper the underlying topological structures will be the k-spaces (e.g. first countable, Fréchet, sequential, locally compact, k'-space, and k-space). Specifically, for spaces within the class of k-spaces, those with the paracompact property are characterized. For this purpose, four generalizations of paracompactness are introduced. These generalizations are defined in terms of refinements which have some finiteness condition on the elements of a given collection of subsets. With the additional structure of the k-spaces these refinements have the properties required for the characterizations. These characterizations are given in § 3 and are summarized in the implication diagram which appears in Figure 3.2.

The fundamental notions used in this study are developed in § 2. Applications of these concepts to metrizability of spaces are given in § 4. Some examples are presented in § 5. The term "space" will mean a Hausdorff topological space and the term "family" will mean a family of subsets.

2. Preliminaries. The fundamental notions involved in this work will be developed in the general setting of F-hereditary collections and weak topology in the sense of Whitehead. A family $\mathcal{K} = \{K_\alpha\colon \alpha \in A\}$ in a space X is said to be an *F-hereditary collection* provided: (i) $\mathcal{K}$ is a covering of X and (ii) for each closed set $F \subset X$, $F \cap K_\alpha \in \mathcal{K}$ for each $\alpha \in A$. Some mapping properties of collections with property (ii) were investigated by Rønnow [11]. For all F-hereditary collections of interest, the singletons are in $\mathcal{K}$, and (i) is satisfied. For instance, the collection of all compact

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