Elementary arcs in 3-space that can be realized by squeezing 3-cells

by

R. J. Daverman* (Knoxville, Tenn.)

1. Introduction. If C is a 3-cell in Euclidean 3-space E^3, then it follows from [7, Theorem 3] that there exists a map s of E^3 onto itself squeezing C to an arc. This paper deals with the converse question asking whether each arc in E^3 can be realized by squeezing some 3-cell. The main result here, Theorem 1, indicates that if A is such an arc which is locally tame modulo one point and which satisfies a finiteness condition at this point, then A lies in the boundary of a disk in E^3. Consequently, many almost tame arcs in E^3 cannot be realized by squeezing 3-cells.

Let Δ_3 denote the set $\{(x,y,z)\in E^3|\ x^2+y^2+z^2\leqslant 1\}$ and Δ_1 the set $\{(x,0,0)\in \Delta_3|-1\leqslant x\leqslant 1\}$. Define π as the projection of Δ_3 onto Δ_1 sending (x,y,z) to $(x,0,0)$.

Let C be a 3-cell in the interior of a 3-manifold M. A map s of M onto itself is said to *squeeze C to an arc A* if and only if there exist homeomorphisms h_3 of Δ_3 onto C and h_1 of Δ_1 onto $A = s(C)$ such that $sh_3 = h_1\pi$ and s takes $M-C$ homeomorphically onto $M-s(C)$. In addition, we say that *an arc A in the interior of M can be realized by squeezing a 3-cell* if and only if there exist a 3-cell C in M and a map s squeezing C to A.

Although we could define similarly the concept of a map of E^3 to itself squeezing a disk onto an arc, it would not enlarge the context of the question at hand, because any disk in the interior of M can be thickened to a 3-cell that flattens back onto the same disk [7, Theorem 9]. Thus, an arc in the interior of M can be realized by squeezing a 3-cell if it can be realized by squeezing a disk.

Let A be an arc in E^3 locally tame modulo an interior point p. The *local enveloping genus of A at p*, written $\mathrm{LEG}(A,p)$, is the smallest non-negative integer r such that there exist arbitrarily small neighborhoods of p, each of which is bounded by a sphere with r handles that intersects A in precisely two points. If no such integer r exists, we write

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$\mathrm{LEG}(A, p) = \infty$. In the special case that $\mathrm{LEG}(A, p) = 0$, we say that A is *locally peripherally unknotted at* p.

An arc A in E^3 is *locally unknotted at a point* $p \in A$ if and only if there exist a disk D in E^3 and a neighborhood N of p such that $N \cap A$ is contained in the boundary of D.

L. V. Keldyš [13] has shown that for any locally unknotted arc A in E^3 there exists a pseudo-isotopy of E^3 translating a straight line segment homeomorphically onto A. The property considered in this paper is almost as strong, for if A is an arc in E^3 that can be realized by squeezing a 3-cell, it follows from [16, Lemma 6] and [19] that there exists a pseudo-isotopy of E^3 (not necessarily satisfying all the conditions of [13]) translating the 3-cell onto A. From this one can easily construct a pseudo-isotopy translating a straight line segment homeomorphically onto A. Loosely speaking, then, the results given in Section 5 represent converses, in severely limited cases, to Keldyš' result.

If C is an n-cell, then $\mathrm{Int}\, C$ and $\mathrm{Bd}\, C$ denote the interior and boundary of C, respectively. The symbol Cl denotes the topological closure operator.

The attack on Theorem 1 involves three distinct procedures. First, we show that any 3-cell which squeezes to an almost tame arc can be altered so that the 3-cell is locally tame modulo the preimage of the wild point of the arc. Secondly, we show that the finiteness of the local enveloping genus implies that this preimage contains at most one troublesome point. Thirdly, beginning with a singular disk attached to the 3-cell, we enact the map approximations and disk tradings typical of 3-space topology to define a set whose image under the squeezing map is the desired disk.

2. Improving the 3-cell.

LEMMA 1. *Let C be a 3-cell in E^3, s a map of E^3 onto itself squeezing C to an arc that is locally tame modulo its endpoints e_1 and e_2, and U an open subset of E^3 containing $C - s^{-1}(e_1 \cup e_2)$. Suppose g_3 is a homeomorphism of Δ_3 onto a 3-cell C' in $s^{-1}(e_1 \cup e_2) \cup \mathrm{Int}\, C$ such that for each $x \in \mathrm{Int}\, s(C)$ there exists $x' \in \mathrm{Int}\, \Delta_1$ satisfying $g_3\pi^{-1}(x') = s^{-1}(x) \cap C'$. Then there exists a map s' squeezing C' onto $s(C)$ and satisfying $s'|C' \cup (E^3 - U) = s|C' \cup$ $\cup (E^3 - U)$.*

Proof. Let K be a 3-cell in $s(U)$ obtained by thickening $s(C)$; that is, K is obtained so that there exists a homeomorphism of (Δ_3, Δ_1) onto $(K, s(C))$.

Since $s^{-1}|\mathrm{Bd}\, K$ is 1-1, $s^{-1}(\mathrm{Bd}\, K)$ must be a 2-sphere. Applying 1, Theorem 1] we find a homeomorphism G of E^3 onto itself taking $s^{-1}(K)$ onto K and satisfying $G|s^{-1}(E^3 - K) = s|s^{-1}(E^3 - K)$. For simplicity we suppress the homeomorphism G and assume that $K = s^{-1}(K)$ and $s|E^3 - K$ $=$ identity.

It follows from [9, 12] that $\mathrm{Bd}\, C'$ is locally tame modulo $\{e_1, e_2\}$. (For applications of this section this argument can be avoided by hypothesizing that $\mathrm{Bd}\, C'$ be locally tame modulo $\{e_1, e_2\}$.) Thus, it is easy to construct a map f of E^3 onto itself such that

(1) f squeezes C' to an arc,

(2) $f|E^3 - K =$ identity,

(3) there exists a homeomorphism g_1 of Δ_1 onto $f(C')$ such that $g_1\pi = fg_3$.

Note that the arc $f(C')$ is locally tame modulo its endpoints e_1 and e_2 [5, Theorem 1]. The crucial fact to be established is that the pair $(K, f(C'))$ is homeomorphic to the standard pair (Δ_3, Δ_1).

In this paragraph we prove that $\pi_1(K - f(C'))$ is infinite cyclic. Note that $\pi_1(K - C)$ is infinite cyclic; therefore, a loop in $K - C$ is null homotopic if and only if it is null homologous. First, let L be a loop in $K - C$ that is null homotopic in $K - C'$. It is a simple matter to show that L is null homotopic in K-$\mathrm{Int}\, C$, and it then follows from [4, Theorem 6.3] that L is homologous in $K - C$ to the sum of finitely many small loops near $\mathrm{Bd}\, C$. Since C is a cell, these small loops can be obtained to be homologously trivial in $K - C$. As a consequence, L is both null homologous and null homotopic in $K - C$. Secondly, let L be a loop in $K - C'$. Then L can be deformed in $K - C'$ to a loop in K-$\mathrm{Int}\, C$, and afterwards it can be pushed off $\mathrm{Bd}\, C$ to a loop in $K - C$. These two properties imply that $\pi_1(K - C')$ is isomorphic to $\pi_1(K - C)$. But $K - f(C')$ and $K - C'$ are homeomorphic, so $\pi_1(K - f(C'))$ is infinite cyclic.

Now we show that $f(C')$ can be straightened in K at its endpoints. Let H be an embedding of K in the 3-sphere S^3 such that the closure K^* of $S^3 - H(K)$ is a 3-cell. Extend $H(f(C'))$ to a simple closed curve J such that $J \cap K^*$ is a tame, unknotted spanning arc of K^*. Using an argument like the one given in the preceding paragraph, we find that $S^3 - J$ has uniformly abelian local fundamental groups at the points $J \cap \mathrm{Bd}\, K^*$. Hence, J is tame [14, Corollary to Theorem 1].

Finally, it follows from [17, Theorem 2] that $J \cup \mathrm{Bd}\, K^*$ is tame. In particular, $H(f(C') \cup \mathrm{Bd}\, K)$ is tame. As a consequence, the pair $(K, f(C'))$ is homeomorphic to (Δ_3, Δ_1), and, therefore, $(K, f(C'))$ is homeomorphic to $(K, s(C))$. It is now a simple matter to obtain a homeomorphism f' of E^3 onto itself such that

(4) $f'|E^3 - K =$ identity,

(5) $f'(f(C')) = s(C)$,

(6) $f'f|C' = s|C'$.

The required map s' is defined as $s' = f'f$.

LEMMA 2. *Let C be a 3-cell in E^3 and s a map of E^3 onto itself squeezing C to an arc $s(C)$ that is locally tame modulo an interior point p. Then there exists a 3-cell C' contained in C and there exists a map t of E^3 onto itself squeezing C' onto $s(C)$ such that $\operatorname{Bd} C'$ is locally tame modulo $\operatorname{Bd} C' \cap t^{-1}(p)$.*

Proof. Let h_3 denote the homeomorphism of Δ_3 onto C and h_1 the homeomorphism of Δ_1 onto $s(C)$ such that $h_1\pi = sh_3$. Define a homeomorphism g_3 of Δ_3 into $h_3(\Delta_1 \cup \operatorname{Int}\Delta_3) \cup s^{-1}(p)$ such that $g_3(\Delta_3)$ contains $s^{-1}(p) \cup h_3(\Delta_1)$ and $h_1\pi = sg_3$.

Since the disk $s^{-1}(p)$ is cellular [16, Lemma 6], there exists a map w of E^3 onto itself whose only inverse set is $s^{-1}(p)$. The map sw^{-1} transforms each of the two maximal 3-cells in $wg_3(\Delta_3)$ onto a subarc of $s(C)$ having p as an endpoint. Hence, it follows from two applications of Lemma 1 that there exists a map s' squeezing the set $wg_3(\Delta_3)$ onto $s(C)$. Define t as $s'w$. The 3-cell $C' = g_3(\Delta_3)$ is locally tame modulo $g_3(\operatorname{Bd}\Delta_1) \cup s^{-1}(p)$ (see the proof of Lemma 1). By hypothesis, however, $E^3 - s(C)$ is $1-\mathrm{LC}$ at each endpoint of $s(C)$, so both $E^3 - C$ and $E^3 - C'$ are $1-\mathrm{LC}$ at the points $h_3(\operatorname{Bd}\Delta_1) = g_3(\operatorname{Bd}\Delta_1)$. Consequently, Theorem 6 of [3] implies that $\operatorname{Bd} C'$ is locally tame modulo $\operatorname{Bd} C' \cap t^{-1}(p)$.

3. Isolating the bad point. It would be interesting to know whether the hypothesis $\mathrm{LEG}(s(C), p) < \infty$ in the following lemma is truly necessary. If not, the corresponding hypothesis about the local enveloping genus could be eliminated from each of the results stated in Section 5.

LEMMA 3. *Suppose C is a 3-cell in E^3, s a map of E^3 onto itself squeezing C to an arc, and p an interior point of $s(C)$ such that $\operatorname{Bd} C$ is locally tame modulo the simple closed curve $J = \operatorname{Bd} C \cap s^{-1}(p)$ and $\mathrm{LEG}(s(C), p) = n < \infty$. Then there exists a point $b \in J$ such that for each neighborhood U of J there exists an open set V containing $J - \{b\}$ such that every loop in $V - C$ is null homotopic in $U - C$.*

Proof. The argument parallels that of [6, Theorems 2 and 3]. Let W be a neighborhood of p. By hypothesis there exists a sphere with n handles H such that H separates p from $E^3 - W$ and H intersects $s(C)$ in two points z_1 and z_2. Thus, the set $s^{-1}(H)$ intersects C in the disks $s^{-1}(z_1)$ and $s^{-1}(z_2)$. Note that $s^{-1}(H)$ may fail to be a 2-manifold at points of $\operatorname{Bd} s^{-1}(z_i)$ $(i = 1, 2)$.

Because $s(C)$ is locally tame away from p, we can perform isotopies only moving points near $s(C)$ to show that if A is a subarc of $\operatorname{Int} s(C)$ containing p in its interior and W is a neighborhood of $s^{-1}(A)$, then there exists a sphere with n handles H such that $H \cap s(C) = \operatorname{Bd} A$ and $W \supset s^{-1}(H)$.

Using the above we can reapply the techniques of [6, Theorem 2], in spite of the fact that the sets $s^{-1}(H)$ are not exactly closed 2-manifolds, to prove that there exists a finite set Q in J such that, for each neighborhood U of J, each point of $J - Q$ has a neighborhood N such that every loop in $N - C$ is null homotopic in $U - C$. The preimages under s of these spheres with handles provide the means for shrinking near J, without allowing the image to stretch out towards $s^{-1}(\operatorname{Bd} s(C))$.

Let U be a neighborhood of J. Since $s^{-1}(p)$ is cellular [16, Lemma 6], C is cellular [19]; hence, J contains at most one point b such that $\operatorname{Bd} C$ fails to be pierced by a tame arc at b [15, Theorem 2]. Find a neighborhood V of $J - \{b\}$ such that each loop in $V - C$ is null homotopic in $U - (\{b\} \cup \operatorname{Int} C)$. With this, the argument of [6, Theorem 3] can be applied to finish the proof.

Remark. If the arc $s(C)$ of Lemma 3 is locally peripherally unknotted, then one can show by appealing to [6, Theorem 6] that the exceptional point b can be deposed. Accordingly, the conclusion of the lemma must be changed to read: *Then, for each neighborhood U of J, each point of J has a neighborhood N such that every loop in $N - C$ is null homotopic in $U - C$.*

4. Obtaining the disk. The proof of the following lemma could be shortened considerably if the simple closed curve J pierced a disk at some point, for in that case we could readily produce a disk D attached to C such that the image $s(D)$ has the desired properties.

LEMMA 4. *Suppose C is a 3-cell in E^3, s is a map of E^3 onto itself squeezing C to an arc, and p is an interior point of $s(C)$ such that $\operatorname{Bd} C$ is locally tame modulo $J = \operatorname{Bd} C \cap s^{-1}(p)$. Suppose further that there exists an interval T in J such that for each neighborhood U of J there exists a neighborhood V of T such that every loop in $V - C$ is null homotopic in $U - C$. Then $s(C)$ lies in the boundary of a disk in E^3.*

Proof. Let $W_1, \ldots, W_k, \ldots$ be a decreasing sequence of open sets whose intersection equals J. Let B denote an (abstract) disk, R a subarc of $\operatorname{Bd} B$, and R' the closure of $\operatorname{Bd} B - R$.

Use Theorem 6.3 of [4] to obtain a map f of B into E^3 such that

(1) $sf|R$ is a homeomorphism of R onto $s(C)$,

(2) $f(B - R) \subset T \cup (E^3 - C)$,

(3) $f^{-1}(T \cap f(B))$ is a 0-dimensional subset X of B,

(4) $f|B - X$ is one-one.

It follows from standard methods in plane topology that there exists a null sequence of disks $D_1, \ldots, D_k, \ldots$ in $\operatorname{Int} B$ such that

(5) $\operatorname{Bd} D_i \cap X = \emptyset$,

(6) $f(D_i) \subset W_i$,

(7) $D_i \cap D_{i+1}$ is a subarc of $\operatorname{Bd} D_i \cap \operatorname{Bd} D_{i+1}$,

(8) $D_i \cap D_j = \emptyset$ if $|i-j| > 1$,
(9) $(X \cap R) \cup (\bigcup D_i)$ is a disk,
(10) $X \subset Cl(\bigcup D_i)$.

Extend the disk D_1 so that $D_1 \cap \mathrm{Bd}\, B$ is a subarc of $\mathrm{Bd}\, B - R$. Define $F = Cl f(B - \bigcup D_i)$. Then F is the union of two disks F_1 and F_2 whose only intersection is the point $f(X \cap R) \subset T$.

Now we begin to replace the sets $f(D_i)$ with disks. Let U_1 be a neighborhood of T such that each simple closed curve in $U_1 \cap F_i$ bounds a disk in $W_1 \cap F_i$ $(i = 1, 2)$. According to the hypothesis of this lemma, we can redefine the map $f|D_1$ on very small subdisks of D_1 containing $D_1 \cap X$, thereby obtaining a map $f'\colon D_1 \to E^3$ such that

(11) $f'(D_1) \cap \big(C \cup f(R')\big) = \emptyset$,
(12) $f'(D_1) \subset f(D_1) \cup U_1$,
(13) $f'|\mathrm{Bd}\, D_1 = f|\mathrm{Bd}\, D_1$,
(14) f' is a homeomorphism in a neighborhood of $\mathrm{Bd}\, D_1$.

Thus, it follows from Dehn's Lemma [18] that there exists a disk E_1 such that

(15) $\mathrm{Bd}\, E_1 = f(\mathrm{Bd}\, D_1)$,
(16) $E_1 \subset f(D_1) \cup U_1$,
(17) $\mathrm{Int}\, E_1 \cap \big(C \cup f(R')\big) = \emptyset$.

The only undesirable property is that $\mathrm{Int}\, E_1$ may meet F, so we adjust E slightly to produce 1-manifolds as the components of $\mathrm{Int}\, E_1 \cap F$, and then we either trade disks or perform isotopies on arcs of intersection, pushing the latter towards the hole between the components of F, to remove all intersections. This leaves us with a disk E_1 such that

(15′) $\mathrm{Bd}\, E_1 = f(\mathrm{Bd}\, D_1)$,
(16′) $E_1 \subset f(D_1) \cup W_1$,
(17′) $E_1 \cap C = \emptyset$,
(18) $E_1 \cap F \subset \mathrm{Bd}\, E_1 \cap \mathrm{Bd}\, F$.

From Conditions 6 and 17′ we see that E_1 intersects only finitely many of the sets $f(D_i)$. We could collect the associated D_i's in a larger disk, but to prevent notational complications, we simply assume that $E_1 \cap f(D_i) = \emptyset$ $(i > 2)$. This implies that $\mathrm{Int}\, E_1 \cap f(\mathrm{Bd}\, D_2) = \emptyset$. Adjust $f|\mathrm{Int}\, D_2$ slightly so that the components of $E_1 \cap f(\mathrm{Int}\, D_2)$ are simple closed curves, and define Y_2 as the union of the disks of D_2 bounded by the preimages of these curves. We redefine f on Y_2 so that $f(D_2) \cap E_1 = \emptyset$, $f|\big(D_2 - (X - Y_2)\big)$ is one-one, and $f(Y_2) \subset W_1$.

Let U_2 be a neighborhood of T such that each simple closed curve in $U_2 \cap F_i$ bounds a disk in $W_2 \cap F_i$ $(i = 1, 2)$. Cover the points of $X \cap (D_2 - Y_2)$ with pairwise disjoint disks $G_{21}, \ldots, G_{2n}$ in $\mathrm{Int}\, D_2 - Y_2$.

According to the hypothesis of this lemma, we can redefine the map f on very small subsets of G_{2i}, thereby obtaining a map $f'\colon D_2 \to E^3$ such that

(19) $f'|D_2 - \cup G_{2i} = f|D_2 - \cup G_{2i}$,
(20) $f'(G_{2i}) \cap \big(C \cup E_1 \cup f(R') \cup f(D_2 - \cup G_{2i})\big) = \emptyset$,
(21) $f'(G_{2i}) \subset f(G_{2i}) \cup U_2$,
(22) $f'|\mathrm{Bd}\, G_{2i} = f|\mathrm{Bd}\, G_{2i}$,
(23) f' is a homeomorphism in a neighborhood of $\mathrm{Bd}\, G_{2i}$ $(i = 1, \ldots, n)$.

Thus, it follows from Dehn's Lemma [18] that $f|D_2$ can be replaced with a homeomorphism $f^*\colon D_2 \to E^3$ such that f^* has the same properties as f' listed in Conditions 19–23 above.

By removing intersections between F and $f^*(D_2)$ as before, we obtain a disk E_2 such that

(24) $\mathrm{Bd}\, E_2 = f(\mathrm{Bd}\, D_2)$,
(25) $E_2 \subset f(D_2) \cup W_1$,
(26) $E_2 \cap C = \emptyset$,
(27) $E_2 \cap F \subset \mathrm{Bd}\, E_2 \cap \mathrm{Bd}\, F$,
(28) $E_2 \cap f(D_i)$ $(i > 2)$ is contained in the union of subdisks of E_2, each of which is contained in W_2.

Repeating the procedure outlined in the three preceding paragraphs, making certain to use the disks of Condition 28 in the initial step of each repetition, we obtain disks $E_3, \ldots, E_k, \ldots$ such that

(29) $\mathrm{Bd}\, E_k = f(\mathrm{Bd}\, D_k)$,
(30) $E_k \subset f(D_k) \cup W_{k-1}$,
(31) $E_k \cap C = \emptyset$,
(32) $E_k \cap F \subset \mathrm{Bd}\, E_k \cap \mathrm{Bd}\, F$.

To complete the proof, let $F^* = F \cup (\bigcup E_k)$. Although F^* itself may fail to be compact, the construction guarantees that $s(F^*)$ is the desired disk.

5. The main results.

THEOREM 1. *Suppose the arc A in E^3 can be realized by squeezing a 3-cell, A is locally tame modulo an interior point p, and* $\mathrm{LEG}(A, p) < \infty$. *Then A lies in the boundary of a disk in E^3.*

Proof. By Lemma 2 there exist a 3-cell C in E^3 and a map s squeezing C to A such that $\mathrm{Bd}\, C$ is locally tame modulo $\mathrm{Bd}\, C \cap s^{-1}(p)$. Since $\mathrm{LEG}(A, p) < \infty$, it follows from Lemmas 3 and 4 that A lies in the boundary of a disk.

COROLLARY 1. *Suppose A is the union of two arcs A_1 and A_2 in E^3 that intersect only in a common endpoint p, A is locally tame modulo p,*

A_1 *is tame*, $\mathrm{LEG}(A, p) < \infty$, *and* A *can be realized by squeezing some 3-cell. Then* A *is tame.*

Proof. Theorem 1 implies that A is contained in the boundary of a disk D. It follows from [2, Theorem 8] that D can be amended so that it is locally tame modulo p. Since D is locally tame modulo the tame arc A_1, D is tame [8, Theorem 1], and this implies that A is tame.

COROLLARY 2. *Let* A *be an arc in* E^3 *locally tame modulo an interior point* p. *Then* A *is tame if and only if* A *can be realized by squeezing a 3-cell and* A *is locally peripherally unknotted at* p.

Proof. One implication is obvious, and the other is an immediate consequence of Theorem 1 and the characterization of tame arcs given by Theorem VI of [11].

COROLLARY 3. *No Wilder arc (see [10]) in* E^3 *can be realized by squeezing a 3-cell.*

THEOREM 2. *If the arc* A *in* E^3 *can be realized by squeezing a 3-cell and* p *is an isolated wild point of* A *such that* $\mathrm{LEG}(A, p) < \infty$, *then* A *is locally unknotted at* p.

Proof. If p is an endpoint of A, the theorem is a rather widely known piece of folklore, which we have already presumed in the proof of Lemma 1, and which we do not prove.

If p is an interior point of A, let B be a subarc of A such that B is locally tame modulo p. With the methods of [1] used to establish Lemma 2, one can find a 3-cell that squeezes onto B, in which case Theorem 2 follows from Theorem 1.

THEOREM 3. *Suppose the arc* A *in* E^3 *can be realized by squeezing a 3-cell and* A *is locally tame modulo a finite set of points* $p_1, \ldots, p_n$ *such that* $\mathrm{LEG}(A, p_i) < \infty$ *for those* p_i*'s in* $\mathrm{Int}\,A$. *Then* A *lies in the boundary of a disk in* E^3.

Proof. Simply piece together those local disks promised by Theorem 2 to determine the required disk.

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THE UNIVERSITY OF TENNESSEE

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