[4] R. Montague, *Theories incomparable with respect to relative interpretability*, J. Symb. Logic 27 (1962), pp. 195–211.
[5] M. B. Pour-El, *Effectively extensible theories*, J. Symb. Logic 33 (1968), pp. 56–68,
[6] B. Rosser, *Extensions of some theorems of Gödel and Church*, J. Symb. Logic 1 (1936), pp. 87–91.

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Shapes of compacta and ANR-systems

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1. Introduction. It is well-known that local difficulties prevent a successful application of homotopy notions to arbitrary compacta. In an attempt to remedy this K. Borsuk introduced a theory of shapes of metric compacta [3, 4]. In this paper we give an alternate description of shapes and at the same time we generalize the theory to the non-metric case. Our approach is based on inverse systems of ANR's, one advantage of which is that it is more categorical. The actual proof that the two approaches are equivalent on metric compacta is given in a sequel to this paper [11]. As an application of our method we classify all P-adic solenoids and all (n-sphere)-like continua as to their shape. It is also shown that the shape classification of 0-dimensional compacta agrees with their topological classification. The theory is presented in detail only in the absolute case, while for the relative case, i.e. the case of pairs of spaces, we content ourselves with indicating the appropriate changes.

2. Category of ANR-systems. A directed set $(A, \leqslant)$ is said to be closure-finite provided for every $\alpha \in A$ the set of all predecessors of α is finite. Note that the natural numbers N with the usual ordering form a closure-finite directed set. Another example is the set $F(\Omega)$ of all (non-empty) finite subsets of a given set Ω ordered by inclusion ($\alpha \leqslant \alpha'$ if and only if $\alpha \subset \alpha'$). For $\alpha \in (A, \leqslant)$ we define the rank $r(\alpha)$ as the maximal cardinal of a chain (linearly ordered set) in A having α for its terminal point. If $(A, \leqslant)$ is closure-finite, each α has a finite number of predecessors and therefore $r(\alpha)$ is a well-defined natural number. Note that in the case of $F(\Omega)$ the rank $r(\alpha)$ is just the cardinal of α and in the case of the integers N the rank $r(n) = n$.

By an ANR in this paper we mean a compact absolute neighborhood retract for metric spaces (see [2], p. 100). We shall now introduce

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our basic notion of the category of ANR-systems. The objects of the category, called ANR-*systems* or just systems, are inverse systems $\underline{X} = \{X_\alpha, p_{\alpha\alpha'}, A\}$ where all X_α are ANR's and A is a closure-finite directed set of indices; the bonding maps $p_{\alpha\alpha'}\colon X_{\alpha'} \to X_\alpha$, $\alpha \leqslant \alpha'$, are continuous maps (not necessarily onto). A morphism in this category $\underline{f}\colon \underline{X} \to \underline{Y} = \{Y_\beta, q_{\beta\beta'}, B\}$, called a *map of* ANR-*systems* or just a map of systems, consists of an increasing function $f\colon B \to A$ and a collection $\{f_\beta, B\}$ of maps $f_\beta\colon X_{f(\beta)} \to Y_\beta$ such that for $\beta \leqslant \beta'$ we have

(1) $$f_\beta p_{f(\beta)f(\beta')} \simeq q_{\beta\beta'} f_{\beta'},$$

i.e. the diagram

$$\begin{array}{ccc} X_{f(\beta)} & \xleftarrow{p} & X_{f(\beta')} \\ {\scriptstyle f_\beta}\downarrow & & \downarrow{\scriptstyle f_{\beta'}} \\ Y_\beta & \xleftarrow[q]{} & Y_{\beta'} \end{array}$$

commutes up to homotopy.

The identity map $\underline{1}_{\underline{X}}\colon \underline{X} \to \underline{X}$ is given by $1(\alpha) = \alpha$, $1_\alpha = 1_{X_\alpha}$. The composition of maps $\underline{f}\colon \underline{X} \to \underline{Y}$, $\underline{g}\colon \underline{Y} \to \underline{Z} = \{Z_\gamma, r_{\gamma\gamma'}, C\}$ is the map $\underline{h} = \underline{g}\underline{f}\colon \underline{X} \to \underline{Z}$ defined as follows. $h\colon C \to A$ is the composition of $g\colon C \to B$ with $f\colon B \to A$. The function h is clearly increasing. For $h_\gamma\colon X_{h(\gamma)} \to Z_\gamma$ we take the composition $g_\gamma f_{g(\gamma)}$. The function h and the collection $\{h_\gamma, C\}$ form a mapping of systems $\underline{h}\colon \underline{X} \to \underline{Z}$. Indeed, for $\beta = g(\gamma)$, $\beta' = g(\gamma')$, $\gamma \leqslant \gamma'$, we have

$$f_\beta p_{f(\beta)f(\beta')} \simeq q_{\beta\beta'} f_{\beta'} \quad \text{and} \quad g_\gamma q_{g(\gamma)g(\gamma')} \simeq r_{\gamma\gamma'} g_{\gamma'}.$$

This implies

$$g_\gamma f_{g(\gamma)} p_{fg(\gamma)fg(\gamma')} \simeq g_\gamma q_{g(\gamma)g(\gamma')} f_{g(\gamma')} \simeq r_{\gamma\gamma'} g_{\gamma'} f_{g(\gamma')},$$

which is

$$h_\gamma p_{h(\gamma)h(\gamma')} \simeq r_{\gamma\gamma'} h_{\gamma'}.$$

It is readily seen that composition of maps of systems is associative and that the map $\underline{1}_{\underline{X}}$ acts as a unit. Therefore, we have the following

THEOREM 1. ANR-*systems and maps of systems form a category.*

In the relative case an ANR-system $(\underline{X}, \underline{X}_0)$ consists of an inverse system of pairs $\{(X, X_0)_\alpha, p_{\alpha\alpha'}, A\}$ over a closure-finite directed set A and each $(X, X_0)_\alpha = (X_\alpha, X_{0\alpha})$ is a pair of ANR's, i.e. both X_α and $X_{0\alpha}$ are ANR's and $X_{0\alpha} \subset X_\alpha$; the bonding maps $p_{\alpha\alpha'}$ are maps of pairs, i.e. $p_{\alpha\alpha'}(X_{0\alpha'}) \subset X_{0\alpha}$. A morphism is defined as in the absolute case the only difference being that the maps f_β are maps of pairs $f_\beta\colon (X, X_0)_{f(\beta)} \to (Y, Y_0)_\beta$ and the homotopy in (1) is a homotopy of pairs. Identity and composition are defined as in the absolute case. We thus obtain the category of ANR-systems for pairs.

3. Homotopy of maps of systems. Two maps of systems $\underline{f}, \underline{g}\colon \underline{X} \to \underline{Y}$ are said to be *homotopic*, written $\underline{f} \simeq \underline{g}$, provided for every $\beta \in B$ there is an index $\alpha \in A$, $\alpha \geqslant f(\beta), g(\beta)$, such that

(1) $$f_\beta p_{f(\beta)\alpha} \simeq g_\beta p_{g(\beta)\alpha}.$$

The set of all such α's is denoted by $(f, g)(\beta)$. Clearly, if $\alpha \in (f, g)(\beta)$ and $\alpha' \geqslant \alpha$, then $\alpha' \in (f, g)(\beta)$.

THEOREM 2. *The homotopy relation $\simeq$ on maps of systems is an equivalence relation.*

Only transitivity requires proof. Assume that $\underline{f}, \underline{g}, \underline{h}\colon \underline{X} \to \underline{Y}$ are maps of systems and that $\underline{f} \simeq \underline{g}$, $\underline{g} \simeq \underline{h}$. Then for $\alpha \in (f, g)(\beta)$ and $\alpha' \in (g, h)(\beta)$ we have

$$f_\beta p_{f(\beta)\alpha} \simeq g_\beta p_{g(\beta)\alpha} \quad \text{and} \quad g_\beta p_{g(\beta)\alpha'} \simeq h_\beta p_{h(\beta)\alpha'}.$$

For $\alpha'' \geqslant \alpha, \alpha'$,

$$p_{g(\beta)\alpha} p_{\alpha\alpha''} = p_{g(\beta)\alpha'} p_{\alpha'\alpha''}$$

and we obtain

$$f_\beta p_{f(\beta)\alpha} p_{\alpha\alpha''} \simeq g_\beta p_{g(\beta)\alpha} p_{\alpha\alpha''} = g_\beta p_{g(\beta)\alpha'} p_{\alpha'\alpha''} \simeq h_\beta p_{h(\beta)\alpha'} p_{\alpha'\alpha''},$$

i.e.

$$f_\beta p_{f(\beta)\alpha''} \simeq h_\beta p_{h(\beta)\alpha''},$$

which proves the assertion.

THEOREM 3. *Let $\underline{f}, \underline{f}'\colon \underline{X} \to \underline{Y}$ and $\underline{g}, \underline{g}'\colon \underline{Y} \to \underline{Z}$ be maps of systems. If $\underline{f} \simeq \underline{f}'$ and $\underline{g} \simeq \underline{g}'$, then $\underline{g}\underline{f} \simeq \underline{g}'\underline{f}'$.*

Proof. First we show that $\underline{g}\underline{f} \simeq \underline{g}'\underline{f}$. Since $\underline{g} \simeq \underline{g}'$, for each $\gamma \in C$ there is a $\beta \in B$ such that

(2) $$g_\gamma q_{g(\gamma)\beta} \simeq g'_\gamma q_{g'(\gamma)\beta}.$$

Since the maps f_β form a map of systems we also have

(3) $$f_{g(\gamma)} p_{fg(\gamma)f(\beta)} \simeq q_{g(\gamma)\beta} f_\beta$$

(4) $$f_{g'(\gamma)} p_{fg'(\gamma)f(\beta)} \simeq q_{g'(\gamma)\beta} f_\beta.$$

From (3), (2) and (4) we conclude that

$$g_\gamma f_{g(\gamma)} p_{fg(\gamma)f(\beta)} \simeq g_\gamma q_{g(\gamma)\beta} f_\beta \simeq g'_\gamma q_{g'(\gamma)\beta} f_\beta \simeq g'_\gamma f_{g'(\gamma)} p_{fg'(\gamma)f(\beta)},$$

which proves that $\underline{g}\underline{f} \simeq \underline{g}'\underline{f}$.

Next we show that $\underline{g}'\underline{f} \simeq \underline{g}'\underline{f}'$. Since $f \simeq f'$, for each $\gamma \in C$ there is an $\alpha \in A$ such that

$$f_{g'(\gamma)} p_{fg'(\gamma)\alpha} \simeq f'_{g'(\gamma)} p_{f'g'(\gamma)\alpha}.$$

Composing with g'_γ on the left we obtain the desired homotopy $\underline{g}'\underline{f} \simeq \underline{g}'\underline{f}'$. The proof is completed by applying transitivity of $\simeq$.

In the relative case two maps $\underline{f}, \underline{g}\colon (\underline{X}, \underline{X}_0) \to (\underline{Y}, \underline{Y}_0)$ are said to be homotopic provided for every β there is an index $\alpha \geqslant f(\beta), g(\beta)$ such that (1) holds with $\simeq$ denoting homotopy of pairs. Proofs of Theorems 2 and 3 remain valid in the relative case also.

4. Homotopy type of systems. We say that a map of systems $\underline{f}\colon \underline{X} \to \underline{Y}$ is a *homotopy equivalence* provided there is a map of systems $\underline{g}\colon \underline{Y} \to \underline{X}$ such that $\underline{g}\underline{f} \simeq \underline{1}_{\underline{X}}$ and $\underline{f}\underline{g} \simeq \underline{1}_{\underline{Y}}$. We say that the systems $\underline{X}$ and $\underline{Y}$ are of the same *homotopy type*, and we write $\underline{X} \simeq \underline{Y}$, provided there exists a homotopy equivalence $\underline{f}\colon \underline{X} \to \underline{Y}$.

THEOREM 4. *The relation $\simeq$ on systems is an equivalence relation.*

Proof. Reflexivity and symmetry obviously hold. To prove transitivity assume that $\underline{X} \simeq \underline{Y}$ and $\underline{Y} \simeq \underline{Z}$. There are maps $\underline{f}\colon \underline{X} \to \underline{Y}$ and $\underline{g}\colon \underline{Y} \to \underline{X}$ such that $\underline{g}\underline{f} \simeq \underline{1}_{\underline{X}}$, $\underline{f}\underline{g} \simeq \underline{1}_{\underline{Y}}$, and there are maps $\underline{h}\colon \underline{Y} \to \underline{Z}$ and $\underline{k}\colon \underline{Z} \to \underline{Y}$ such that $\underline{k}\underline{h} \simeq \underline{1}_{\underline{Y}}$, $\underline{h}\underline{k} \simeq \underline{1}_{\underline{Z}}$.

Consider the composite maps $\underline{h}\underline{f}\colon \underline{X} \to \underline{Z}$ and $\underline{g}\underline{k}\colon \underline{Z} \to \underline{X}$. By Theorems 1, 2 and 3,

$$(\underline{h}\underline{f})(\underline{g}\underline{k}) = \underline{h}(\underline{f}\underline{g})\underline{k} \simeq \underline{h}\underline{k} \simeq \underline{1}_{\underline{Z}}, \qquad (\underline{g}\underline{k})(\underline{h}\underline{f}) = \underline{g}(\underline{k}\underline{h})\underline{f} \simeq \underline{g}\underline{f} \simeq \underline{1}_{\underline{X}},$$

which proves that $\underline{X} \simeq \underline{Z}$.

We define the homotopy type $[\underline{X}]$ of a system $\underline{X}$ as the equivalence class of $\underline{X}$ with respect to the relation $\simeq$.

The notions of homotopy equivalence and homotopy type extend to the relative case in the obvious way and Theorem 4 remains valid.

Of special interest (especially in connection with metric compacta) is the case of ANR-*sequences*. These are ANR-systems $\underline{X} = \{X_n, p_{nn'}, N\}$, where the index set $A = N$ is the set of natural numbers. In this case the sequence $\underline{X}$ is completely determined by the bonding maps $p_{n,n+1}\colon X_{n+1} \to X_n$, $n \in N$.

THEOREM 5. *Let $\underline{X}$ and $\underline{X}' = \{X_n, p'_{nn'}, N\}$ be* ANR-*sequences. If $p'_{n,n+1} \simeq p_{n,n+1}$ for every $n \in N$, then $\underline{X} \simeq \underline{X}'$.*

Proof. Indeed, a map $\underline{f}\colon \underline{X} \to \underline{X}'$ is obtained by taking $f(n) = n$, $n \in N$, and $f_n = 1_{X_n}\colon X_n \to X_n$. Clearly, $f_n p_{n,n+1} = p_{n,n+1} \simeq p'_{n,n+1} = p'_{n,n+1} f_{n+1}$, which shows that $\underline{f}$ is a map of systems. Similarly, we define $\underline{g}\colon \underline{X}' \to \underline{X}$ by putting $g(n) = n$, $g_n = 1_{X_n}\colon X_n \to X_n$. Since $\underline{g}\underline{f} = \underline{1}_{\underline{X}}$ and $\underline{f}\underline{g} = \underline{1}_{\underline{X}'}$, it follows that $\underline{X} \simeq \underline{X}'$.

THEOREM 6. *Let $\underline{X} = \{X_n, p_{nn'}, N\}$ be an* ANR-*sequence and let Y_n be an* ANR *of the same homotopy type as X_n for each $n \in N$. Then there is an* ANR-*sequence $\underline{Y} = \{Y_n, q_{nn'}, N\}$ such that $\underline{X} \simeq \underline{Y}$.*

Proof. Let $f_n\colon X_n \to Y_n$ and $g_n\colon Y_n \to X_n$ be maps such that $g_n f_n \simeq 1_{X_n}$, $f_n g_n \simeq 1_{Y}$. We define $q_{n,n+1}\colon Y_{n+1} \to Y_n$ as the composite $q_{n,n+1} = f_n p_{n,n+1} g_{n+1}$ and we define $q_{nn'}$, $n < n'$, as the composite $q_{nn'} = q_{n,n+1} \ldots q_{n'-1,n'}$. Clearly, $\underline{Y} = \{Y_n, q_{nn'}, N\}$ is an ANR-sequence. Now consider the maps $\underline{f}\colon \underline{X} \to \underline{Y}$ and $\underline{g}\colon \underline{Y} \to \underline{X}$ defined by $f(n) = n = g(n)$, and by the maps $f_n\colon X_n \to Y_n$ and $g_n\colon Y_n \to X_n$ respectively. Then $\underline{f}$ and $\underline{g}$ are maps of ANR-sequences, because

$$q_{n,n+1} f_{n+1} = f_n p_{n,n+1} g_{n+1} f_{n+1} \simeq f_n p_{n,n+1}$$

and

$$p_{n,n+1} g_{n+1} \simeq g_n f_n p_{n,n+1} g_{n+1} = g_n q_{n,n+1}.$$

Moreover, $\underline{g}\underline{f} \simeq \underline{1}_{\underline{X}}$ and $\underline{f}\underline{g} \simeq \underline{1}_{\underline{Y}}$, which proves the assertion.

5. Systems associated with spaces. Let X be a compact Hausdorff space. We say that an ANR-system $\underline{X}$ is *associated* with the space X provided $X = \operatorname{Inv}\lim \underline{X}$.

THEOREM 7. *With every compact Hausdorff space X is associated an* ANR-*system $\underline{X} = \{X_\alpha, p_{\alpha\alpha'}, A\}$. This system can be chosen in such a way that all the X_α are polyhedra and the cardinal kA is not greater than the weight of X. With every metric compact space X is associated an* ANR-*sequence $\underline{X}$.*

Proof. We can assume that X is of infinite cardinality, for otherwise, X itself is a polyhedron and the assertion is obviously true. First note that X can be considered as a subset of the cube $I^\Omega = \prod_{\omega \in \Omega} I_\omega$, where each $I_\omega = I$ is the unit segment and Ω is a set of infinite cardinality equal to the weight of X. Let $A = F(\Omega)$ be the set of all (non-empty) finite subsets $\alpha \subset \Omega$ ordered by inclusion. For $\alpha = \{\omega_1, \ldots, \omega_n\}$, $n \in N$, let $I^\alpha = I_{\omega_1} \times \ldots \times I_{\omega_n}$. If $\alpha \leqslant \alpha'$, let $p_{\alpha\alpha'}\colon I^{\alpha'} \to I^\alpha$ be the natural projection. Clearly, $\{I^\alpha, p_{\alpha\alpha'}, A\}$ is an inverse system whose limit is I^Ω. The natural projection $I^\Omega \to I^\alpha$ is denoted by p_α.

For each $\alpha \in A$, we now define, by induction on the rank $r(\alpha)$, a sequence of open sets $U_{\alpha n}$ in I^α such that:

(1) $$p_\alpha(X) = \bigcap_{n \in N} U_{\alpha n},$$

(2) $$U_{\alpha 1} \supset \operatorname{Cl} U_{\alpha 2} \supset U_{\alpha 2} \supset U_{\alpha 3} \supset U_{\alpha 4} \supset \ldots,$$

(3) $$p_{\alpha\alpha'}(U_{\alpha' 1}) \subset U_{\alpha, r(\alpha') - r(\alpha) + 1}, \quad \alpha \leqslant \alpha'.$$

Assume that we have already defined $U_{\alpha n}$ for $r(\alpha) < r$ and that $r(\alpha') = r > 1$. Then for every $\alpha < \alpha'$ we have

$$p_{\alpha\alpha'}(p_{\alpha'}(X)) = p_\alpha(X) \subset U_{\alpha, r(\alpha') - r(\alpha) + 1},$$

and since there are only finitely many such α's, it is possible to find an open neighborhood $U_{\alpha' 1}$ of $p_{\alpha'}(X)$ for which (3) holds. Next, we define $U_{\alpha' 2}, U_{\alpha' 3}, \ldots$ in accordance with (1) and (2).

Clearly, for each $\alpha \in A$ one can find a polyhedron $X_\alpha \subset I^\alpha$ such that

(4) $$\operatorname{Cl} U_{\alpha 2} \subset X_\alpha \subset U_{\alpha 1}.$$

Note that

(5) $$p_{\alpha\alpha'}(X_{\alpha'}) \subset X_\alpha, \qquad \alpha \leqslant \alpha',$$

because, for $\alpha < \alpha'$, (4), (3) and (2) imply

$$p_{\alpha\alpha'}(X_{\alpha'}) \subset p_{\alpha\alpha'}(U_{\alpha'1}) \subset U_{\alpha, r(\alpha')-r(\alpha)+1} \subset U_{\alpha 2} \subset X_\alpha.$$

Therefore, $\{X_\alpha, p_{\alpha\alpha'}, A\}$ is an inverse system of polyhedra and we need only to show that its limit X_∞ equals X.

For every $y \in X_\infty$ and $\alpha \leqslant \alpha'$ we have

(6) $$p_\alpha(y) = p_{\alpha\alpha'} p_{\alpha'}(y) \in p_{\alpha\alpha'}(X_{\alpha'}) \subset U_{\alpha, r(\alpha')-r(\alpha)+1}.$$

Since Ω is infinite, one can find $\alpha' \geqslant \alpha$ of arbitrarily large rank $r(\alpha')$ so that (6), (2) and (1) yield for each $\alpha \in A$

$$p_\alpha(y) \in p_\alpha(X).$$

Consequently, $y \in X$, which shows that $X_\infty \subset X$. Conversely, $X \subset X_\infty$ because for each α we have $p_\alpha(X) \subset U_{\alpha 2} \subset X_\alpha$.

If X is metric, its weight is countable and so is A. Therefore A admits a cofinal sequence which yields an ANR-sequence associated with X. This concludes the proof.

In the relative case Theorem 7 is also valid and asserts that a compact pair (X, X_0) is the limit of some ANR-system for pairs $(\underline{X}, \underline{X_0}) = \{(X, X_0)_\alpha, p_{\alpha\alpha'}, A\}$. The construction follows exactly the one in the absolute case. One has only to interpret (1)–(4) as statements about pairs. Thus, we need for each $\alpha \in A$ and each $n \in N$ a pair of open sets $(U, U_0)_{\alpha n} = (U_{\alpha n}, U_{0\alpha n})$ and a pair of polyhedra $(X, X_0)_\alpha = (X_\alpha, X_{0\alpha})$ satisfying the pair analogues of (1)–(4). For example, in addition to (2) we require that

$$U_{0\alpha 1} \supset \operatorname{Cl} U_{0\alpha 2} \supset U_{0\alpha 2} \supset U_{0\alpha 3} \supset \ldots$$

Remark 1. A similar proof is sketched in ([15], p. 46). On the other hand, the results given in ([5], p. 284) and ([14], Theorem VII.3, p. 303), are inadequate for our purposes since the index set A given there is not closure-finite.

6. Maps of systems associated with maps of spaces. Let $\underline{X}$ and $\underline{Y}$ be ANR-systems associated with spaces X and Y respectively. We say that the map of systems $\underline{f}\colon \underline{X} \to \underline{Y}$ is *associated* with a map $f\colon X \to Y$ provided for every $\beta \in B$ we have

(1) $$f_\beta p_{f(\beta)} \simeq q_\beta f,$$

i.e. the diagram

$$\begin{tikzcd} X_{f(\beta)} \arrow[d, "f_\beta"'] & X \arrow[l, "p"'] \arrow[d, "f"] \\ Y_\beta & Y \arrow[l, "q"] \end{tikzcd}$$

commutes up to homotopy (p_α and q_β are natural projections).

THEOREM 8. *Let $\underline{X}, \underline{Y}, \underline{Z}$ be systems associated with X, Y, Z respectively and let $\underline{f}\colon \underline{X} \to \underline{Y}$, $\underline{g}\colon \underline{Y} \to \underline{Z}$ be maps of systems associated with maps $f\colon X \to Y$, $g\colon Y \to Z$ respectively. Then $\underline{gf}\colon \underline{X} \to \underline{Z}$ is associated with $gf\colon X \to Z$. The map of systems $\underline{1_X}\colon \underline{X} \to \underline{X}$ is associated with the map $1_X\colon X \to X$.*

Proof. By assumption

(2) $$f_\beta p_{f(\beta)} \simeq q_\beta f,$$

(3) $$g_\gamma q_{g(\gamma)} \simeq r_\gamma g.$$

Therefore, for $\beta = g(\gamma)$, (2) and (3) yield

$$g_\gamma f_{g(\gamma)} p_{fg(\gamma)} \simeq g_\gamma q_{g(\gamma)} f \simeq r_\gamma gf,$$

which is the desired result. The second assertion in the theorem follows from the obvious relation

$$1_{X_\alpha} p_\alpha = p_\alpha 1_X.$$

THEOREM 9. *Let $\underline{X}$ and $\underline{Y}$ be ANR-systems associated with X and Y respectively, let $\underline{f}, \underline{g}\colon \underline{X} \to \underline{Y}$ be homotopic maps of systems and $f\colon X \to Y$ a map. If $\underline{f}$ is associated with f, then $\underline{g}$ is also associated with f.*

Proof. By assumption for each $\beta \in B$

$$f_\beta p_{f(\beta)} \simeq q_\beta f.$$

Moreover, $\underline{f} \simeq \underline{g}$ implies the existence of an $\alpha \geqslant f(\beta), g(\beta)$ such that

$$f_\beta p_{f(\beta)\alpha} \simeq g_\beta p_{g(\beta)\alpha}.$$

Consequently,

$$g_\beta p_{g(\beta)} = g_\beta p_{g(\beta)\alpha} p_\alpha \simeq f_\beta p_{f(\beta)\alpha} p_\alpha = f_\beta p_{f(\beta)} \simeq q_\beta f,$$

which shows that $\underline{g}$ is associated with f.

In the relative case we say that $\underline{f}\colon (\underline{X}, \underline{X_0}) \to (\underline{Y}, \underline{Y_0})$ is associated with the map $f\colon (X, X_0) \to (Y, Y_0)$ provided (1) holds for the homotopy of pairs. Theorems 8 and 9 remain valid with the same proofs.

THEOREM 10. *Let $\underline{X}$ and $\underline{Y}$ be ANR-systems associated with X and Y respectively and let $f\colon X \to Y$ be a mapping. Then there exists a mapping of systems $\underline{f}\colon \underline{X} \to \underline{Y}$ associated with f.*

7. Proof of Theorem 10. We first consider a space X^* defined as the disjoint union of $X = \operatorname{Inv}\lim \underline{X}$ and of all X_α, $\alpha \in A$. A basis for the topology of X^* consists of all the open sets U_α from X_α and of the sets

$$U_\alpha^* = \bigcup_{\alpha \leqslant \alpha'} p_{\alpha\alpha'}^{-1}(U_\alpha) \cup p_\alpha^{-1}(U_\alpha) .$$

For every $\alpha \in A$ we also define a map $p_\alpha^*\colon X_\alpha^* \to X_\alpha$, where

$$X_\alpha^* = \bigcup_{\alpha \leqslant \alpha'} X_{\alpha'} \cup X \subset X^*,$$

by putting $p_\alpha^*|X = p_\alpha$, $p_\alpha^*|X_{\alpha'} = p_{\alpha\alpha'}$, $\alpha \leqslant \alpha'$.

LEMMA 1. *X^* is a Hausdorff paracompact space which contains X and X_α with their original topologies. For every open neighborhood U of X in X^* there is an $\alpha \in A$ such that $X_\alpha^* \subset U$. The map $p_\alpha^*\colon X_\alpha^* \to X_\alpha$ is continuous.*

Proof. The first two assertions follow from Theorems 2 and 3 of [9]. The third assertion follows from the fact that for any open set $U_\alpha \subset X_\alpha$ the set $(p_\alpha^*)^{-1}(U_\alpha) = U_\alpha^*$ is open in X_α^*.

LEMMA 2. *For every collection ω of open sets in X^* which covers X, there is an $\alpha \in A$ such that for $\alpha' \geqslant \alpha$ the maps $p_{\alpha'}$ and 1_X are ω-near in X^* (i.e. for every $x \in X$ both points $p_{\alpha'}(x)$ and x belong to some member of ω).*

Proof. Refine ω by a collection covering X and consisting of open sets of the form U_α^*. Since X is compact, a finite collection $U_{\alpha_1}^*, \ldots, U_{\alpha_n}^*$ suffices to cover X. Choose an $\alpha_0 \geqslant \alpha_1, \ldots, \alpha_n$. Let $\alpha' \geqslant \alpha_0$ and let $x \in X$. Then x belongs to some $U_{\alpha_i}^*$, $i \in \{1, \ldots, n\}$, and therefore, $p_{\alpha_i}(x) = p_{\alpha_i\alpha'} p_{\alpha'}(x) \in U_{\alpha_i}$. Consequently, $p_{\alpha'}(x) \in p_{\alpha_i\alpha'}{}^{-1}(U_{\alpha_i}) \subset U_{\alpha_i}^*$. Thus, both points x and $p_{\alpha'}(x)$ lie in any element of ω containing $U_{\alpha_i}^*$.

LEMMA 3. *Let $\underline{X}$ be an ANR-system associated with X, let Y be an ANR and $f\colon X \to Y$ a map. Then there exists an index $\alpha \in A$ and for every $\alpha' \geqslant \alpha$ there exist maps $f^{\alpha'}\colon X_{\alpha'} \to Y$ such that*

$$f \simeq f^{\alpha'} p_{\alpha'} . \tag{1}$$

Proof. First note that X^* is normal (Lemma 1) and therefore $f\colon X \to Y$ admits an extension $f^*\colon U \to Y$ to an open neighborhood U of X in X^* (see e.g. [7], Lemma 5.1, p. 93). Next observe that there is an open covering ε of Y such that any two maps from any space to Y are homotopic in Y provided they are ε-near (see e.g. [7], Theorem 1.1, p. 111). Now apply Lemma 2 to the collection $\omega = f^{*-1}(\varepsilon)$. We can find an index $\alpha \in A$ such that for each $\alpha' \geqslant \alpha$ the maps $p_{\alpha'}$ and 1_X are ω-near in X^* and therefore, $f^* p_{\alpha'}$ and $f^*|X = f$ are ε-near in Y. Consequently, $f^* p_{\alpha'} \simeq f$ in Y, $\alpha \leqslant \alpha'$. By Lemma 1, we can assume that α is so large that $X_{\alpha'} \subset U$ for $\alpha \leqslant \alpha'$. Then f^* is defined on the whole space $X_{\alpha'}$ and we can define a map $f^{\alpha'}\colon X_{\alpha'} \to Y$ as the restriction $f^{\alpha'} = f^*|X_{\alpha'}$, $\alpha \leqslant \alpha'$. Clearly, $f^{\alpha'} p_{\alpha'} \simeq f$ in Y for $\alpha \leqslant \alpha'$ and Lemma 3 is thus established.

LEMMA 4. *Let $\underline{X}$ be an ANR-system associated with X and let Y be an ANR. For some $\alpha \in A$ let $f, g\colon X_\alpha \to Y$ be two maps such that*

$$f p_\alpha \simeq g p_\alpha . \tag{2}$$

Then there is an $\alpha' \in A$, $\alpha \leqslant \alpha'$, such that

$$f p_{\alpha\alpha'} \simeq g p_{\alpha\alpha'} . \tag{3}$$

Proof. Consider the set $X_\alpha^* \subset X^*$ and the map $p_\alpha^*\colon X_\alpha^* \to X_\alpha$. Notice that the set $A_\alpha = \{\alpha' \mid \alpha' \in A, \alpha \leqslant \alpha'\}$ is cofinal in A and therefore, the inverse system $\{X_{\alpha'}, p_{\alpha'\alpha''}, A_\alpha\}$ also has X for its inverse limit. Consider now the set

$$Z_\alpha = (X_\alpha^* \times 0) \cup (X \times I) \cup (X_\alpha^* \times 1) ,$$

which is closed in $X_\alpha^* \times I$, where $I = [0, 1]$. By assumption (2), there is a homotopy $h\colon X \times I \to Y$ connecting $f p_\alpha$ to $g p_\alpha$. We define a map $H\colon Z_\alpha \to Y$ by

$$H(x, t) = h(x, t) , \quad x \in X,\ t \in I , \tag{4}$$

$$H(x, 0) = f p_\alpha^*(x) , \quad x \in X_\alpha^*, \tag{5}$$

$$H(x, 1) = g p_\alpha^*(x) , \quad x \in X_\alpha^* . \tag{6}$$

Observe that (4) yields $H(x, 0) = h(x, 0) = f p_\alpha(x)$, which agrees with the value obtained from (5) for $x \in X$; similarly, for $(x, 1)$, $x \in X$, (4) and (6) agree. Therefore, $H\colon Z_\alpha \to Y$ is continuous.

Since Y is an ANR, H admits an extension H^* to a neighborhood U of Z_α in $X_\alpha{}^* \times I$. From the compactness of $X \times I$ we may conclude that there is an open set V in $X_\alpha{}^*$ such that $X \subset V$ and $V \times I \subset U$. By Lemma 1 applied to $X_\alpha{}^*$ and V, there is an index $\alpha' \in A$, $\alpha \leqslant \alpha'$, such that $X_{\alpha'} \subset V$ and therefore $X_{\alpha'} \times I \subset V \times I$. Clearly, $H^*|X_{\alpha'} \times I$ is a homotopy connecting $f p_{\alpha\alpha'}$ to $g p_{\alpha\alpha'}$ because, for $x \in X_{\alpha'}$, we have

$$H^*(x, 0) = f p_\alpha{}^*(x) = f p_{\alpha\alpha'}(x) ,$$

$$H^*(x, 1) = g p_\alpha{}^*(x) = g p_{\alpha\alpha'}(x) .$$

LEMMA 5. *Let $(A, \leqslant)$ and $(B, \leqslant)$ be two directed closure-finite sets and let $g\colon B \to A$ be a function. Then there exists an increasing function $f\colon B \to A$ such that $g(\beta) \leqslant f(\beta)$ for all $\beta \in B$.*

Proof. We define $f\colon B \to A$ by induction on the rank of the elements of B. If $r(\beta) = 1$, we put $f(\beta) = g(\beta)$. Assume now that we have already defined f for all $\beta \in B$ of rank $r(\beta) < n$ in such a way that $g(\beta) \leqslant f(\beta)$ and that f is increasing. If $r(\beta') = n$, consider the set consisting of $g(\beta')$ and of all $f(\beta)$ for $\beta < \beta'$. Since this set is finite, one can choose $f(\beta')$ in such a way that $g(\beta') \leqslant f(\beta')$ and that $f(\beta) \leqslant f(\beta')$ for all $\beta \leqslant \beta'$. The function f defined in this way has the desired properties.

LEMMA 6. *Let $\underline{X}$ and $\underline{Y}$ be ANR-systems associated with X and Y respectively and let $f: X \to Y$ be a mapping. Then there is an increasing function $f: B \to A$ and for each $\beta \in B$ a map $f_\beta: X_{f(\beta)} \to Y_\beta$ such that*

(7) $$q_\beta f \simeq f_\beta p_{f(\beta)} .$$

Moreover, for each $\beta' \geqslant \beta$ there is an $\alpha' \geqslant f(\beta')$ such that

(8) $$f_\beta p_{f(\beta)\alpha'} \simeq q_{\beta\beta'} f_{\beta'} p_{f(\beta')\alpha'} .$$

Proof. For each $\beta \in B$, we apply Lemma 3 to Y_β and the map $q_\beta f: X \to Y_\beta$. One obtains an index $\alpha = g(\beta) \in A$ and for every $\alpha' \geqslant g(\beta)$ a map $f_\beta^{\alpha'}: X_{\alpha'} \to Y_\beta$ such that

(9) $$q_\beta f \simeq f_\beta^{\alpha'} p_{\alpha'} .$$

By Lemma 5, there is an increasing function $f: B \to A$ such that $g(\beta) \leqslant f(\beta)$ for each $\beta \in B$. Let $f_\beta: X_{f(\beta)} \to Y_\beta$ be the map $f_\beta = f_\beta^{\alpha'}$, where $\alpha' = f(\beta) \geqslant g(\beta)$. Then, by (9), $q_\beta f \simeq f_\beta p_{f(\beta)}$, which is (7).

For $\beta \leqslant \beta'$ we also have

$$q_{\beta'} f \simeq f_{\beta'} p_{f(\beta')} ,$$

which implies

$$q_\beta f = q_{\beta\beta'} q_{\beta'} f \simeq q_{\beta\beta'} f_{\beta'} p_{f(\beta')} .$$

This together with (7) implies

$$f_\beta p_{f(\beta)f(\beta')} p_{f(\beta')} \simeq q_{\beta\beta'} f_{\beta'} p_{f(\beta')} ,$$

which enables us to apply Lemma 4 with α, Y, f, g replaced by $f(\beta')$, $Y_\beta, f_\beta p_{f(\beta)f(\beta')}, q_{\beta\beta'} f_{\beta'}$ respectively. We then conclude that there is an index $\alpha' \in A$, $\alpha' \geqslant f(\beta')$, such that

$$f_\beta p_{f(\beta)f(\beta')} p_{f(\beta')\alpha'} \simeq q_{\beta\beta'} f_{\beta'} p_{f(\beta')\alpha'} ,$$

which is (8).

LEMMA 7. *Let $\underline{X}$ and $\underline{Y}$ be ANR-systems, let $\varphi: B \to A$ be an increasing function and let for every $\beta \in B$, $\varphi_\beta: X_{\varphi(\beta)} \to Y_\beta$ be a map such that for $\beta \leqslant \beta'$ there exists an $\alpha' = \varphi(\beta, \beta') \geqslant \varphi(\beta')$ such that*

(10) $$\varphi_\beta p_{\varphi(\beta)\alpha'} \simeq q_{\beta\beta'} \varphi_{\beta'} p_{\varphi(\beta')\alpha'} .$$

Then there exists a map of systems $\underline{f}: \underline{X} \to \underline{Y}$ such that for every $\beta \in B$, $f(\beta) \geqslant \varphi(\beta)$ and

(11) $$\varphi_\beta p_{\varphi(\beta)} = f_\beta p_{f(\beta)} .$$

Proof. For each $\beta' \in B$ consider the set $\{\varphi(\beta, \beta') |\ \beta \leqslant \beta'\}$. Since B is closure-finite, this set is finite and so one can choose an element $\alpha = f(\beta') \in A$, such that $f(\beta') \geqslant \varphi(\beta, \beta')$ for all $\beta \leqslant \beta'$. By Lemma 5 there is no loss of generality in assuming that $f: B \to A$ is an increasing function. For each $\beta \in B$ we now define $f_\beta: X_{f(\beta)} \to Y_\beta$ by putting $f_\beta = \varphi_\beta p_{\varphi(\beta)f(\beta)}$. Composing on the right with $p_{\alpha' f(\beta')}$ in (10), we obtain

$$\varphi_\beta p_{\varphi(\beta)f(\beta')} \simeq q_{\beta\beta'} \varphi_{\beta'} p_{\varphi(\beta')f(\beta')} ,$$

i.e.

$$f_\beta p_{f(\beta)f(\beta')} \simeq q_{\beta\beta'} f_{\beta'} , \qquad \beta \leqslant \beta' .$$

This proves that $f: B \to A$ and $\{f_\beta, B\}$ form a map of systems $\underline{f}: \underline{X} \to \underline{Y}$. Finally, note that

$$f_\beta p_{f(\beta)} = \varphi_\beta p_{\varphi(\beta)f(\beta)} p_{f(\beta)} = \varphi_\beta p_{\varphi(\beta)} .$$

Proof of Theorem 10. To the maps f_β obtained in Lemma 6 apply Lemma 7. We obtain a map of systems $\underline{f}: \underline{X} \to \underline{Y}$. By (7) and (11) we have for every $\beta \in B$

$$f_\beta p_{f(\beta)} \simeq q_\beta f ,$$

which shows that $\underline{f}$ is associated with f.

Remark 2. A few obvious changes in the proof of Theorem 10 enable us to conclude that every collection of maps $f^\beta: X \to Y_\beta$, for which $f^\beta \simeq q_{\beta\beta'} f^{\beta'}$, $\beta \leqslant \beta'$, admits a mapping of systems $\underline{f}: \underline{X} \to \underline{Y}$ such that $f_\beta p_{f(\beta)} \simeq f^\beta$ for every $\beta \in B$. Theorem 10 follows by taking $f^\beta = q_\beta f$.

Theorem 10 is also valid in the relative case. Only the proofs of Lemmas 3 and 4 require modification.

In Lemma 3 the map $f: (X, X_0) \to (Y, Y_0)$ is extended to a map $f^*: U \to Y$ of a neighborhood U of X in X^* by first extending $f_0 = f|X_0$ to a map $f_0^*: U_0 \to Y_0$ of a neighborhood U_0 of X_0 in $X_0^* = (\bigcup X_{0\alpha}) \cup X_0$ and then extending simultaneously f and f_0^* to f^*. By Lemma 1 there is an $\alpha \in A$ such that, for $\alpha' \geqslant \alpha$, $X_{\alpha'} \subset U$ and $f^*(X_{0\alpha'}) \subset Y_0$, i.e. $f_{\alpha'} = f^*|X_{\alpha'}$ is a map of the pair $(X_{\alpha'}, X_{0\alpha'})$ into the pair (Y, Y_0).

Since Y is an ANR, there is an open covering ε of Y such that for any two ε-near maps $f, g: X \to Y$ and any ε-homotopy $H_0: X_0 \times I \to Y$ connecting f_0 to $g_0 = g|X_0$, there is a homotopy $H: X \times I \to Y$ which extends X_0 and connects f to g (although X is not necessarily metric, the proof in [7], IV, Theorem 1.2, p. 112 applies). Moreover, since Y_0 is also an ANR, there is an open covering ε_0 of Y_0 such that if $f_0, g_0: X_0 \to Y_0$ are ε_0-near maps, then one can find an ε-homotopy $H_0: X_0 \times I \to Y_0$ connecting f_0 to g_0 (see [7], IV, Theorem 1.1, p. 111). Therefore, if the maps f and g are ε-near and their restrictions f_0 and g_0 are ε_0-near, then there is a homotopy $H: (X, X_0) \times I \to (Y, Y_0)$ connecting f to g. Applying Lemma 2, one can achieve that for $\alpha' \geqslant \alpha$, $g = f^* p_{\alpha'}$ and f are ε-near, while $g_0 = f^* p_{\alpha'} | X_0$ and f_0 are ε_0-near and therefore, $f^* p_{\alpha'}$ and f are homotopic as maps of pairs $(X, X_0) \to (Y, Y_0)$.

The only change in the proof of Lemma 4 for the relative case is that the extension of the map H: $Z_\alpha \to Y$ to a neighborhood of Z_α in $X_\alpha^* \times I$ is done in two steps. One first extends the restriction $H_0 = H | Z_{0\alpha}$, where $Z_{0\alpha} = (X_{0\alpha}^* \times 0) \cup (X_0 \times I) \cup (X_{0\alpha}^* \times 1)$, to a map of a neighborhood of $Z_{0\alpha}$ in $X_{0\alpha}^* \times I$ to Y_0 and then one extends to a neighborhood of Z_α in $X_\alpha^* \times I$ to Y.

8. Homotopy of associated maps and systems.

THEOREM 11. *Let $\underline{X}$ and $\underline{Y}$ be* ANR*-systems associated with compact Hausdorff spaces X and Y respectively, and let $\underline{f}, \underline{g}$: $\underline{X} \to \underline{Y}$ be maps of systems associated with maps f, g: $X \to Y$ respectively. Then $f \simeq g$ implies $\underline{f} \simeq \underline{g}$.*

Proof. By assumption, for each $\beta \in B$ we have $f_\beta p_{f(\beta)} \simeq q_\beta f$ and $g_\beta p_{g(\beta)} \simeq q_\beta g$. Since $f \simeq g$, it follows that

$$f_\beta p_{f(\beta)} \simeq g_\beta p_{g(\beta)} . \tag{1}$$

For $\alpha_0 \geqslant f(\beta), g(\beta)$, (1) can be written as

$$f_\beta p_{f(\beta)\alpha_0} p_{\alpha_0} \simeq g_\beta p_{g(\beta)\alpha_0} p_{\alpha_0} , \tag{2}$$

and we can apply Lemma 4 to the maps $f_\beta p_{f(\beta)\alpha_0}, g_\beta p_{g(\beta)\alpha_0}$: $X_{\alpha_0} \to Y_\beta$. We conclude that there is an $\alpha_1 \geqslant \alpha_0$ such that

$$f_\beta p_{f(\beta)\alpha_1} \simeq g_\beta p_{g(\beta)\alpha_1} , \tag{3}$$

which shows that $\underline{f} \simeq \underline{g}$.

THEOREM 12. *Let $\underline{X}$ and $\underline{Y}$ be* ANR*-systems associated with compact Hausdorff spaces X and Y respectively. If X and Y are of the same homotopy type, then $\underline{X}$ and $\underline{Y}$ are also of the same homotopy type, $\underline{X} \simeq \underline{Y}$.*

Proof. By assumption, there are maps f: $X \to Y$, g: $Y \to X$ such that

$$gf \simeq 1_X , \quad fg \simeq 1_Y . \tag{4}$$

By Theorem 10, there exist maps of systems $\underline{f}$: $\underline{X} \to \underline{Y}$, $\underline{g}$: $\underline{Y} \to \underline{X}$ associated with f and g respectively. By Theorem 8, $\underline{g}\underline{f}$: $\underline{X} \to \underline{X}$ is associated with gf: $X \to X$ and $\underline{f}\underline{g}$: $\underline{Y} \to \underline{Y}$ is associated with fg: $Y \to Y$. Moreover, $\underline{1}_X$: $\underline{X} \to \underline{X}$ is associated with 1_X: $X \to X$ and $\underline{1}_Y$: $\underline{Y} \to \underline{Y}$ with 1_Y: $Y \to Y$. Therefore, by Theorem 11, (4) implies

$$\underline{g}\underline{f} \simeq \underline{1}_X , \quad \underline{f}\underline{g} \simeq \underline{1}_Y ,$$

which proves that $\underline{X} \simeq \underline{Y}$.

COROLLARY 1. *Let $\underline{X}$ and $\underline{X}'$ be two* ANR*-systems associated with the same compact Hausdorff space X. Then $\underline{X} \simeq \underline{X}'$.*

We see from Theorem 7 and Corollary 1 that every compact Hausdorff space X determines a homotopy type of ANR-systems, namely the type which contains all systems $\underline{X}$ associated with X. This type is called the *shape* of X and is denoted by $[X]$ or following K. Borsuk by $\mathrm{Sh}(X)$. Theorem 12 implies that the shape of X is a homotopy invariant, i.e. $X \simeq Y$ implies $[X] = [Y]$. However, one can have spaces X and Y of different homotopy type but of the same shape. The simplest example is the circle S^1 and the "Warsaw circle" (the graph of the $\sin\frac{1}{x}$ curve closed by an arc). Also note that each class $[\underline{X}]$ is determined by some space X, namely by $\mathrm{Inv\,lim}\, \underline{X}$.

THEOREM 13. *Let $\underline{X}$ and $\underline{Y}$ be* ANR*-systems associated with X and Y respectively, and let $\underline{f}$: $\underline{X} \to \underline{Y}$ be a map of systems. If Y is an* ANR, *then there exists a map f: $X \to Y$ such that $\underline{f}$ is associated with f.*

Proof. Since Y is an ANR we can consider a special system associated with Y. Its index set $A = \{1\}$ and its only coordinate space $Y_1 = Y$; we denote this system by $\{Y\}$. By Theorem 10, there are maps of systems $\underline{g}$: $\underline{Y} \to \{Y\}$ and $\underline{h}$: $\{Y\} \to \underline{Y}$ associated with 1_Y: $Y \to Y$. Since $\underline{g}\underline{f}$: $\underline{X} \to \{Y\}$ is a map of systems there is a map $g_1 f_{g(1)}$: $X_{fg(1)} \to Y$. We now define f: $X \to Y$ by

$$f = g_1 f_{g(1)} p_{fg(1)} .$$

By definition, $\underline{g}\underline{f}$ is associated with f. On the other hand, $\underline{h}$ is associated with 1_Y: $Y \to Y$, so that (by Theorem 8) $\underline{h}\underline{g}\underline{f}$: $\underline{X} \to \underline{Y}$ is associated with f: $X \to Y$. By Theorem 11, $\underline{h}\underline{g} \simeq \underline{1}_Y$ and so Theorem 3 and 9 imply that $\underline{f}$ is associated with f.

Remark 3. Let X be an ANR and $\{X\}$ the special ANR-system associated with X. A morphism $\underline{f}$: $\{X\} \to \underline{Y}$ consists of a collection of maps f_β: $X \to Y_\beta$, $\beta \in B$, such that $q_{\beta\beta'} f_{\beta'} \simeq f_\beta$ for all $\beta \leqslant \beta'$. In general one cannot find a map f: $X \to Y$ such that $\underline{f}$ is associated with f.

THEOREM 14. *Let X and Y be* ANR*'s. If X and Y are of the same shape, then they are of the same homotopy type.*

Proof. By Corollary 1, we can take as representatives of $[X]$ and $[Y]$ the special ANR-systems $\{X\}$ and $\{Y\}$. By assumption, there are maps of systems $\underline{f}$: $\{X\} \to \{Y\}$ and $\underline{g}$: $\{Y\} \to \{X\}$ such that $\underline{g}\underline{f} \simeq \underline{1}_X$ and $\underline{f}\underline{g} \simeq \underline{1}_Y$. However, in this case they reduce to maps f: $X \to Y$, g: $Y \to X$ such that $gf \simeq 1_X$ and $fg \simeq 1_Y$.

All the proofs in this section carry over to the relative case after the obvious modifications needed for pairs are made.

9. Cohomology groups. In this section we shall show that Čech cohomology groups are shape invariants.

THEOREM 15. *Let $\underline{X}$ and $\underline{Y}$ be ANR-systems associated with X and Y respectively, and let $\underline{f}$: $\underline{X}\to\underline{Y}$ be a map of systems. Then f induces a homomorphism $\underline{f}^*$: $H^q(Y;\,G)\to H^q(X;\,G)$ for every integer q and Abelian group G. Furthermore, $(\underline{g}\underline{f})^*=\underline{f}^*\underline{g}^*$, $\underline{1}^*=1$, and $\underline{f}\simeq\underline{g}$ implies $\underline{f}^*=\underline{g}^*$.*

Proof. The cohomology functor H^q transforms the inverse systems $\underline{X}=\{X_\alpha,p_{\alpha\alpha'},A\}$ and $\underline{Y}=\{Y_\beta,q_{\beta\beta'},B\}$ into direct systems of groups

$$H^q(\underline{X};\,G)=\{H^q(X_\alpha;\,G),\,p^*_{\alpha\alpha'},\,A\}\,,$$
$$H^q(\underline{Y};\,G)=\{H^q(Y_\beta;\,G),\,q^*_{\beta\beta'},\,B\}\,.$$

Note that the continuity of Čech cohomology implies that the direct limits of these systems are the groups $H^q(X;\,G)$ and $H^q(Y;\,G)$ respectively. The homotopy

$$f_\beta p_{f(\beta)f(\beta')}\simeq q_{\beta\beta'}f_{\beta'}$$

induces the equality

$$p^*_{f(\beta)f(\beta')}f^*_\beta=f^*_{\beta'}q^*_{\beta\beta'}\,.$$

Therefore, the mapping of systems $\underline{f}=\{f_\beta\}$: $\underline{X}\to\underline{Y}$ induces a homomorphism $\{f^*_\beta\}$: $H^q(\underline{Y};\,G)\to H^q(\underline{X};\,G)$ of direct systems of groups (see [5], VIII) and this homomorphism induces in the usual way ([5], VIII) a homomorphism of groups $\underline{f}^*$: $H^q(Y;\,G)\to H^q(X;\,G)$. It follows from the definition that the composite map of ANR-systems $\underline{g}\underline{f}$ induces a composite of homomorphisms of direct systems of cohomology groups and this induces the composite homomorphism $\underline{f}^*\underline{g}^*$ of limit groups, so that $(\underline{g}\underline{f})^*=\underline{f}^*\underline{g}^*$.

Finally, let $\underline{f}\simeq\underline{g}$. Then for every $\beta\in B$ there is an $\alpha\geqslant f(\beta),\,g(\beta)$ such that $f_\beta p_{f(\beta)\alpha}\simeq g_\beta p_{g(\beta)\alpha}$ and consequently

$$p^*_{f(\beta)\alpha}f^*_\beta=p^*_{g(\beta)\alpha}g^*_\beta\,. \tag{1}$$

Composing with p^*_α on the left, we conclude from (1) that

$$p^*_{f(\beta)}f^*_\beta=p^*_{g(\beta)}g^*_\beta\,.$$

On the other hand, by the definition of f^* and g^*, we have

$$\underline{f}^*q^*_\beta=p^*_{f(\beta)}f^*_\beta\,,$$
$$\underline{g}^*q^*_\beta=p^*_{g(\beta)}g^*_\beta\,.$$

Consequently, for every $\beta\in B$,

$$\underline{f}^*q^*_\beta=\underline{g}^*q^*_\beta\,,$$

which implies $\underline{f}^*=\underline{g}^*$.

THEOREM 16. *If X and Y are compact Hausdorff spaces of the same shape, then $H^q(X;\,G)\approx H^q(Y;\,G)$ for each integer q and Abelian group G.*

Proof. Let $\underline{X}$ and $\underline{Y}$ be ANR-systems associated with X and Y respectively. Then there are maps of systems $\underline{f}$: $\underline{X}\to\underline{Y}$, $\underline{g}$: $\underline{Y}\to\underline{X}$ such that $\underline{g}\underline{f}\simeq\underline{1}_X$, $\underline{f}\underline{g}\simeq\underline{1}_Y$. It follows from Theorem 15 that $\underline{f}^*$: $H^q(Y;\,G)\to$ $\to H^q(X;\,G)$ is an isomorphism.

In the relative case a map of systems $\underline{f}$: $(\underline{X},\underline{X}_0)\to(\underline{Y},\underline{Y}_0)$ induces a homomorphism of the relative cohomology groups $\underline{f}^*$: $H^q(Y,Y_0;\,G)\to$ $\to H^q(X,X_0;\,G)$. The analogues of Theorems 15 and 16 remain valid.

Remark 4. Note that there exist compact Hausdorff spaces X which are not of the shape of any compact metric space. For example, consider the Cartesian product X of an uncountable collection of 0-spheres (discrete 2-point spaces). Since $H^0(X;\,Z)$ is the group of all continuous mappings f: $X\to Z$ (see e.g. [5], p. 254), it is clear that $H^0(X;\,Z)$ is an uncountable group. On the other hand, every compact metric space Y is the inverse limit of a sequence of polyhedra (Theorem 7) and therefore $H^q(Y;\,Z)$ is a countable group for each q. Thus we may conclude from Theorem 16 that X and Y are not of the same shape.

10. Shape classification of sphere-like continua. As an example of the ANR-system approach to shapes we will classify S^m-like continua. We first classify P-adic solenoids S_P where $P=(p_1,p_2,\ldots)$ is a sequence of primes. S_P is defined as the limit of the inverse sequence $\underline{S}_P=\{X_n,\pi_{n,n+1},N\}$ where $X_n=\{z|\,|z|=1\}$ is the unit circle in the complex plane and the map $\pi_{n,n+1}$: $X_{n+1}\to X_n$ is given by $\pi_{n,n+1}(z)=z^{p_n}$. Two sequences of primes $P=(p_1,p_2,\ldots)$ and $Q=(q_1,q_2,\ldots)$ are said to be equivalent, written as $P\sim Q$, provided it is possible to delete a finite number of terms from each so that every prime occurs the same number of times in each of the deleted sequences (see [1], p. 210).

THEOREM 17. *Let S_P and S_Q be two solenoids. Then the following three statements are equivalent:*

(i) *S_P and S_Q are of the same shape;*

(ii) *$P\sim Q$;*

(iii) *S_P and S_Q are homeomorphic.*

Proof. (i)$\Rightarrow$(ii). Let $[S_P]=[S_Q]$. Then by Theorem 16, $H^1(S_P;\,Z)$ $\approx H^1(S_Q;\,Z)$. From the continuity of Čech cohomology it follows that $H^1(S_P;\,Z)$ is isomorphic to the group F_P of P-adic rationals, i.e. rationals of the form $m/p_1p_2\ldots p_s$, where $m\in Z$. However, $F_P\approx F_Q$ implies $P\sim Q$ (see [13], p. 198).

(ii) $\Rightarrow$(iii) has been noted by R. H. Bing ([1], p. 210).

(iii) $\Rightarrow$(i) follows from Theorem 12.

Remark 5. S. Godlewski has also obtained the results of Theorem 17 [6].

For every sequence of primes $P = (p_1, p_2, \ldots)$ we now consider the inverse sequence $\underline{S}_P^m = \{X_n, \pi_{n,n+1}, N\}$, where each X_n is the m-sphere S^m and $\pi_{n,n+1}\colon S^m \to S^m$ is any map of degree p_n. We denote the inverse limit of $\underline{S}_P^m$ by S_P^m. By Theorem 5, the shape of S_P^m is completely determined.

THEOREM 18. *Two spaces S_P^m and S_Q^m are of the same shape if and only if $P \sim Q$.*

Proof. Consider the solenoid S_P and its expansion $\underline{S}_P$. Applying the $(m-1)$-fold suspension S^{m-1} we obtain the inverse sequence

$$S^{m-1}(\underline{S}_P) = \{S^{m-1}(S^1),\ S^{m-1}(\pi_{n,n+1}),\ N\}$$

whose limit is $S^{m-1}(S_P)$. Let $f_n\colon X_n = S^m \to S^{m-1}(S^1) = S^m$ be any mapping of degree 1. Then the maps f_n form a map of systems $\underline{f}\colon \underline{S}_P^m \to S^{m-1}(\underline{S}_P)$ because of the Hopf classification theorem for maps of spheres. In fact $\underline{f}$ is a homotopy equivalence and thus S_P^m is of the same shape as $S^{m-1}(S_P)$. In this way the problem reduces to showing that $S^{m-1}(S_P)$ and $S^{m-1}(S_Q)$ are of the same shape if and only if $P \sim Q$.

If $S^{m-1}(S_P)$ and $S^{m-1}(S_Q)$ are of the same shape, then by Theorem 16,

$$H^m\big(S^{m-1}(S_P);\ Z\big) \approx H^m\big(S^{m-1}(S_Q);\ Z\big). \tag{1}$$

However, the first group is isomorphic to $H^1(S_P; Z)$ and the second group to $H^1(S_Q; Z)$ so that (1) implies $H^1(S_P; Z) \approx H^1(S_Q; Z)$, which again implies $P \sim Q$. Conversely, if $P \sim Q$, then S_P and S_Q are homeomorphic and therefore so are $S^{m-1}(S_P)$ and $S^{m-1}(S_Q)$. This completes the proof.

A metric continuum X is said to be S^m-like provided for each $\varepsilon > 0$ there is a mapping $f_\varepsilon\colon X \to S^m$ onto S^m such that $\operatorname{diam} f^{-1}(y) < \varepsilon$ for any $y \in S^m$.

THEOREM 19. *Every S^m-like continuum X has the shape of a point, S^m or S_P^m.*

Proof. X admits an inverse sequence expansion $\{X_n, \pi_{n,n+1}, N\}$ where all X_n are m-spheres (see [10] or [8]). Let k_n be the degree of $\pi_{n,n+1}$. We can always assume that all $k_n \geqslant 0$ (this can be achieved by omitting a finite number of initial terms and by taking compositions of consecutive bonding maps with an even number of negative degrees). If there are infinitely many zeros among the degrees, then X is of the shape of a point because the maps of degree 0 can be replaced by constant maps without affecting the shape (see Theorem 5). Thus, if X is not of the shape of a point, we can assume that $k_n \geqslant 1$. If there is an n_0 such that $k_n = 1$ for $n \geqslant n_0$, then by replacing the bonding maps by identities, we conclude that X is of the shape of S^m. Otherwise, we can assume that all $k_n \geqslant 2$ (this can be achieved by taking suitable compositions of consecutive bonding maps). We now decompose the bonding map $\pi_{n,n+1}$

into a product of maps from S^m into S^m each of prime degree. This yields a limit space S_P^m of the same shape as X and the proof is completed.

Theorems 18 and 19 classify all S^m-like continua with respect to their shape.

11. Shapes of 0-dimensional compacta.

THEOREM 20. *Two 0-dimensional compact Hausdorff spaces are of the same shape if and only if they are homeomorphic.*

Proof. First notice that every 0-dimensional compact Hausdorff space X is the inverse limit of an ANR-system $\underline{X} = \{X_\alpha, p_{\alpha\alpha'}, A\}$, where all X_α are finite sets. Indeed, $\underline{X}$ can be obtained by considering the nerves X_α of finite coverings α of X formed by disjoint open sets and the natural projections $p_{\alpha\alpha'}$ uniquely determined by inclusion. Note that for a given $\alpha' \in A$ there are only finitely many $\alpha \in A$ refined by α' so that $(A, \leqslant)$ is closure-finite and $\underline{X}$ is an ANR-system associated with X (see [16], Theorem 5, p. 459).

Now assume that X and Y are compact Hausdorff spaces of the same shape and that $\underline{X}$ and $\underline{Y}$ are associated ANR-systems with X_α and Y_β finite sets. Then there exist maps of systems $\underline{f}\colon \underline{X} \to \underline{Y}$ and $\underline{g}\colon \underline{Y} \to \underline{X}$ such that $\underline{g}\underline{f} \simeq \underline{1}_X$ and $\underline{f}\underline{g} \simeq \underline{1}_Y$. Since the components of Y_β are single points, the homotopy

$$f_\beta p_{f(\beta)f(\beta')} \simeq q_{\beta\beta'} f_{\beta'}, \qquad \beta \leqslant \beta',$$

becomes an equality

$$f_\beta p_{f(\beta)f(\beta')} = q_{\beta\beta'} f_{\beta'}.$$

Therefore, $\{f_\beta\}$ is actually a map of inverse systems (in the sense of [5], VIII) and so induces a map $f\colon X \to Y$ such that for every $\beta \in B$

$$f_\beta p_{f(\beta)} = q_\beta f. \tag{1}$$

Similarly, we have a map $g\colon Y \to X$ such that for every $\alpha \in A$

$$g_\alpha q_{g(\alpha)} = p_\alpha g. \tag{2}$$

Furthermore, the homotopy $\underline{g}\underline{f} \simeq \underline{1}_X$ implies that for every $\alpha \in A$ there is an $\alpha' \in A$, $\alpha' \geqslant \alpha$, $fg(\alpha)$, such that

$$g_\alpha f_{g(\alpha)} p_{fg(\alpha)\alpha'} = p_{\alpha\alpha'}. \tag{3}$$

Consequently, for every $\alpha \in A$,

$$g_\alpha f_{g(\alpha)} p_{fg(\alpha)} = p_\alpha. \tag{4}$$

By (1), (4) becomes

$$g_\alpha q_{g(\alpha)} f = p_\alpha, \tag{5}$$

which, by (2), gives

$$p_a g f = p_a . \tag{6}$$

Since (6) holds for every $a \in A$, we conclude that

$$gf = 1_X . \tag{7}$$

Similarly, we obtain

$$fg = 1_Y . \tag{8}$$

(7) and (8) show that $f\colon X \to Y$ is a homeomorphism which completes the proof.

COROLLARY 2. *Among countable metric compacta there are $\aleph_1$ different shapes.*

Proof. Countable metric compacta are 0-dimensional and therefore their shape classification coincides with their topological classification. However, S. Mazurkiewicz and W. Sierpiński have shown ([12], Theorem 2, p. 22) that there are $\aleph_1$ different topological types among the countable metric compacta.

Remark 6. K. Borsuk has shown that there are $2^{\aleph_0}$ different shapes of plane compacta ([3], Theorem 10.7, p. 238). Corollary 2 shows that already on the line there are uncountably many different shapes.

References

[1] R. H. Bing, *A simple closed curve is the only homogeneous bounded plane continuum that contains an arc*, Canad. J. Math. 12 (1960), pp. 209–213.
[2] K. Borsuk, *Theory of retracts*, Warszawa 1967.
[3] — *Concerning homotopy properties of compacta*, Fund. Math. 62 (1968), pp. 223–254.
[4] — *Concerning the notion of the shape of compacta*, Proc. Internat. Top. Symp. Hercegnovi 1968, Beograd 1969, pp. 98–104.
[5] S. Eilenberg and N. Steenrod, *Foundations of algebraic topology*, Princeton 1952.
[6] S. Godlewski, *On the shape of solenoids*, Bull. Acad. Polon. Sci., Ser. Sci. Math., Astr. et Phys. 17 (1969), pp. 623–627.
[7] S. T. Hu, *Theory of retracts*, Detroit 1965.
[8] I. Lončar and S. Mardešić, *A note on inverse sequences of* ANR's, Glasnik Mat. 3 (1968), pp. 41–48.
[9] S. Mardešić, *On inverse limits of compact spaces*, Glasnik Mat.-Fiz. Astron. 13 (1958), pp. 249–255.
[10] — J. Segal, *ε-mappings onto polyhedra*, Trans. Amer. Math. Soc. 109 (1963), pp. 146–164.
[11] — — *Equivalence of the Borsuk and the* ANR-*system approach to shapes*, Fund. Math. 72 (1971), pp. 61–68.

[12] S. Mazurkiewicz and W. Sierpiński, *Contribution à la topologie des ensembles dénombrables*, Fund. Math. 1 (1920), pp. 17–27.
[13] M. C. McCord, *Inverse limit sequences with covering maps*, Trans. Amer. Math. Soc. 114 (1965), pp. 197–209.
[14] J. Nagata, *Modern general topology*, Amsterdam 1968.
[15] B. Pasynkov, *O polièdral'nyh spektrah i razmernosti bikompaktov, v častnosti bikompaktnyh grupp*, Dokl. Akad. Nauk SSSR 121 (1958), pp. 45–48.
[16] — *O spektrah i razmernosti topologičeskih prostranstv*, Mat. Sbornik 57 (99) (1962) pp. 449–476.

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