[3] K. Borsuk, *Concerning the notion of the shape of compacta*, Proc. Internat. Top. Symp. Hercegnovi 1968, Beograd 1969, pp. 98–104.
[4] S. T. Hu, *Theory of retracts*, Detroit 1965.
[5] S. Mardešić and J. Segal, *Shapes of compacta and* ANR-*systems*, Fund. Math. 72 (1971), pp. 41–59.

UNIVERSITY OF ZAGREB
UNIVERSITY OF WASHINGTON

Reçu par la Rédaction le 27. 1. 1970

Concerning the unions of absolute neighborhood retracts having brick decompositions*

by

Steve Armentrout (Iowa City, Ia.)

1. Introduction. In the study of retracts, one is interested in determining those properties of polyhedra that are also possessed by compact metric absolute neighborhood retracts. A basic property of polyhedra is that they can be decomposed into simplexes in such a way that if any number of them meet, their intersection is a face of each of them, and hence is a simplex. This property of polyhedra leads to the notion of a brick decomposition of a space.

If X is a topological space, then a *brick decomposition* of X is a finite collection $\{X_1, X_2, \ldots, X_n\}$ of compact metric absolute retracts in X such that (1) $X = X_1 \cup X_2 \cup \ldots \cup X_n$ and (2) if any number of the sets $X_1, X_2, \ldots,$ and X_n intersect, their intersection is an absolute retract.

Clearly, every polyhedron admits a brick decomposition. Further, any metric continuum admitting a brick decomposition is an absolute neighborhood retract [4, page 178]. However, not every compact metric absolute neighborhood retract has a brick decomposition [4, page 178]. The existence of compact metric absolute neighborhood retracts with no brick decomposition is related to the existence of such retracts with the singularity of Mazurkiewicz [4, page 152; 3].

In [4, page 179], Borsuk mentions the following open question: If X and Y are spaces such that X, Y, and $X \cap Y$ have brick decompositions, then does $X \cup Y$ have a brick decomposition? The purpose of this paper is to give a negative answer to this question.

The example that we describe here is obtained by an easy modification of the construction of [3]. A similar construction could be made using toroidal upper semicontinuous decompositions and the techniques of [2].

* Research supported in part by the National Science Foundation under Grant No. GP-9641, and in part by the Institute for Advanced Study.

If M is a manifold, then $\operatorname{Bd} M$ and $\operatorname{Int} M$ denote the boundary and interior, respectively, of M.

A collection C of sets in a metric space is a *null collection* if and only if for each positive number ε, there exist at most finitely many sets of C of diameter greater than ε.

2. Tapered Antoine's necklaces. Let D denote the disc $\{(x, y) = (x, y) \in E^2$ and $(x-2)^2+y^2 \leqslant 1\}$ in the plane E^2. Let T denote the solid torus obtained by rotating D about the Y-axis. Let E denote the disc which is the component of $T \cap E^2$ distinct from D. Let x and y denote the centers of D and E, respectively. Let A and B denote the two 3-cells that are the closures of the components of $T-(D \cup E)$.

In the construction of the spaces to be studied in this paper, we use sets related to the standard Antoine's necklaces in E^3, and which we shall call "tapered Antoine's necklaces". In Section 3, for each positive integer r, we shall construct two such sets, one in A and one in B. In this section we shall describe the construction for such sets in A, and give notation to be used later. Corresponding sets in B will be obtained by reflection through E^2.

By a *doubly infinite chain of solid tori* in A we shall mean a collection

$$\{\ldots, T_{-2}, T_{-1}, T_0, T_1, T_2, \ldots\}$$

of mutually disjoint unknotted polyhedral solid tori in $\operatorname{Int} A$ such that (1) if n and m are distinct integers, then T_n and T_m are linked if and only if $|n-m| = 1$, (2) for any neighborhood U of x, there is an integer λ such that if $i > \lambda$, $T_i \subset U$, and (3) for any neighborhood V of y, there is an integer μ such that if $j > \mu$, then $T_j \subset V$. See Figure 1. Note that the chain is a null collection. The *mesh* of the chain is $\max\{\operatorname{diam} T_i\colon i$ is an integer$\}$.

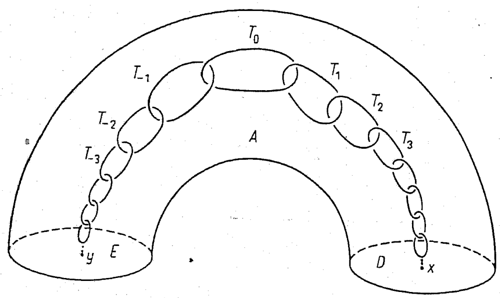

Fig. 1

Suppose that r is a positive integer (fixed in this section). Let

$$\{\ldots, T_{r(-2)}, T_{r(-1)}, T_{r0}, T_{r1}, T_{r2}, \ldots\}$$

be a doubly infinite chain of solid tori in A. We suppose that this chain has mesh at most one. For each integer i, let $\{T_{ri1}, T_{ri2}, \ldots, T_{rim_{ri}}\}$ be a chain of linked polyhedral unknotted solid tori in $\operatorname{Int} T_{ri}$ circling T_{ri} exactly twice; see Figure 2. We suppose that if $j = 1, 2, \ldots,$ or m_{ri},

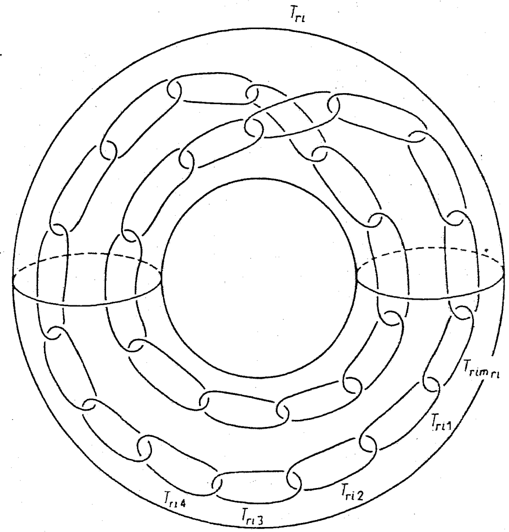

Fig. 2

$(\operatorname{diam} T_{rij}) < 1/2$. If i is an integer, $j = 1, 2, \ldots,$ or m_{ri}, let $\{T_{rij1}, T_{rij2}, \ldots$ $\ldots, T_{rijm_{rij}}\}$ be a chain of linked polyhedral solid tori in $\operatorname{Int} T_{rij}$, each of diameter less than $1/4$, circling T_{rij} exactly once; see Figure 3.

Let this process be continued, with subsequent chains circling exactly once, and let $M_{r1}, M_{r2}, M_{r3}, \ldots$ denote $\bigcup\limits_{i=-\infty}^{\infty} T_{ri}$, $\bigcup\limits_{i=-\infty}^{\infty} \bigcup\limits_{j=1}^{m_{ri}} T_{rij}$, $\bigcup\limits_{i=-\infty}^{\infty} \bigcup\limits_{j=1}^{m_{ri}} \bigcup\limits_{k=1}^{m_{rij}} T_{rijk}, \ldots$, respectively. Let N_r denote $\{x, y\} \cup (\bigcap\limits_{t=0}^{\infty} M_{rt})$; N_r is a *tapered Antoine's necklace* in A. Note that $N_r - \{x, y\} \subset \operatorname{Int} A$.

In the construction of N_r, the solid tori

$$\ldots, T_{r(-2)}, T_{r(-1)}, T_{r0}, T_{r1}, T_{r2}, \ldots$$

are the solid tori of the *first stage* of the construction of N_r, the solid tori T_{rij}, where i is an integer and $1 \leqslant j \leqslant m_i$, are the solid tori of the *second stage*, and so on.

If n is any positive integer, then a is a *stage n index* in the construction of N_r if and only if there exist integers $i_1, i_2, \ldots,$ and i_n such that $1 \leqslant i_2 \leqslant m_{ri_1}$, $1 \leqslant i_3 \leqslant m_{ri_1 i_2}, \ldots,$ and $1 \leqslant i_n \leqslant m_{ri_1 \ldots i_{n-1}}$, and $a = i_1 i_2 \ldots i_n$.

Clearly, if $z \in \operatorname{Int} A$, N_r may be constructed so that $z \in N_r$.

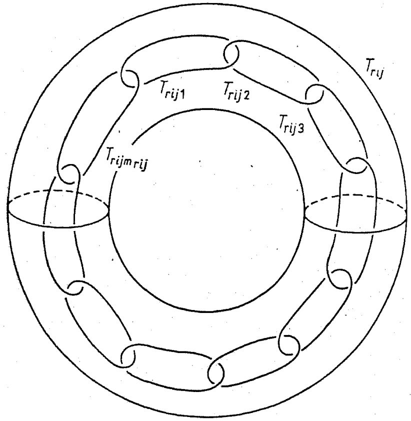

Fig. 3

For each integer i, we construct an arc A_{ri} lying in $\operatorname{Int} T_{ri}$ just as in section 2 of [3]. It is true that for each i, A_{ri} contains $N_r \cap T_{ri}$. The collection

$$\{\ldots, A_{r(-2)}, A_{r(-1)}, A_{r0}, A_{r1}, A_{r2}, \ldots\}$$

is a null collection of mutually disjoint arcs.

3. Construction of certain decompositions. Let $x_1, x_2, x_3, \ldots$ be a countable dense subset of $\operatorname{Int} A$.

Let N_1 be a tapered Antoine's necklace in A such that (1) $x_1 \in N_1$ and (2) each of the first stage solid tori used in describing N_1 has diameter less than 1.

For each positive integer i, there is a first stage solid torus T_{1i} used in describing N_1, and there is an arc A_{1i} in $\operatorname{Int} T_{1i}$, containing $T_{1i} \cap N_1$, and constructed as described above. Let $\mathcal{A}_1$ denote

$$\{\ldots, A_{1(-2)}, A_{1(-1)}, A_{10}, A_{11}, A, \ldots\},$$

and let A_1 denote $\{x, y\} \cup (\bigcup\{a\colon a \in \mathcal{A}_1\})$. Then $\mathcal{A}_1$ is a null collection of mutually disjoint arcs in $\operatorname{Int} A$, each of diameter less than one.

Let r_2 be the least positive integer q such that $x_q \notin A_1$. Let N_2 be a tapered Antoine's necklace in A, containing x_{r_2}, and, except for x and y, disjoint from A_1. Further, we required that each of the first stage solid tori used in describing N_2 be disjoint from A_1 and have diameter less than 1/2.

For each positive integer i, there is a first stage solid torus T_{2i} used in describing N_2, and there is an arc A_{2i} in $\operatorname{Int} T_{2i}$, containing $N_2 \cap T_{2i}$, and constructed as described above. Let $\mathcal{A}_2$ denote

$$\{\ldots, A_{2(-2)}, A_{2(-1)}, A_{20}, A_{21}, A_{22}, \ldots\},$$

and let A_2 denote $\{x, y\} \cup (\bigcup\{a\colon a \in \mathcal{A}_2\})$. Then $\mathcal{A}_2$ is a null collection of mutually disjoint arcs in $\operatorname{Int} A$, each of diameter less than 1/2.

Let this process be continued. There result a sequence $N_1, N_2, N_3, \ldots$ of tapered Antoine's necklaces in A, and a sequence $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \ldots$ of null collections of mutually disjoint arcs in $\operatorname{Int} A$ such that if for each positive integer r, $A_r = \{x, y\} \cup (\bigcup\{a\colon a \in \mathcal{A}_r\})$, the following hold: (1) $N_r \subset A_r$. (2) $x_r \in \bigcup_{i=1}^{r} A_i$. (3) For any integer i, $(\operatorname{diam} A_{ri}) < 1/2^{r-1}$.

Let $\mathcal{A}$ denote the collection $\bigcup\{\mathcal{A}_r\colon r = 1, 2, 3, \ldots\}$. It is clear that $\mathcal{A}$ is a null collection of mutually disjoint arcs in $\operatorname{Int} A$.

Let C be the disc $\{(x, y)\colon (x, y) \in E^2,\ |x| \leqslant 2,\ \text{and}\ |y| \leqslant 1\}$ in the plane E^2. Then $C \cup D \cup E$ is a disc in E^2. Let A^* denote $A \cup C$, and let B^* denote $B \cup C$.

Let H denote the collection consisting of all the arcs of the family $\mathcal{A}$, together with all singleton subsets of A^* not on arcs of $\mathcal{A}$. Since $\mathcal{A}$ is a null collection, it follows that H is an upper semicontinuous decomposition of A^*. Let X denote the associated decomposition space, and let h denote the projection map from A^* onto X.

B and B^* are the images under reflection in the plane E^2 of A and A^*, respectively. Let K denote the collection of images, under reflection in E^2, of sets of H. K is an upper semicontinuous decomposition of B^*. Let Y denote the associated decomposition space. Clearly Y is homeomorphic to X.

Let G denote $H \cup K$; it is easily verified that G is an upper semicontinuous decomposition of $A^* \cup B^*$. Let Z denote the associated decomposition space, and let g denote the projection map from $A^* \cup B^*$ onto Z.

It is clear that $Z = X \cup Y$. The remainder of the paper is devoted to proving the following two facts:

(1) X, Y, and $X \cap Y$ have brick decompositions.

(2) Z has no brick decomposition.

4. Preliminary lemmas. We establish some preliminary lemmas in this section. The first two are adaptations, to the situation described in this paper, of Lemmas 2 and 4 of [3].

LEMMA 1. *Suppose that r is a positive integer, i is an integer, and U is an open subset of $A^* \cup B^*$ such that (1) U is a union of elements of G and (2) U contains a singular disc Δ such that $\mathrm{Bd}\Delta \subset T_{ri}$ and $\mathrm{Bd}\Delta \nsim 0$ in T_{ri}. Then U contains loops γ and λ such that (1) $\gamma \subset T_{r(i-1)}$ and $\gamma \nsim 0$ in $T_{r(i-1)}$, and (2) $\lambda \subset T_{r(i+1)}$ and $\lambda \nsim 0$ in $T_{r(i+1)}$.*

LEMMA 2. *Suppose that $U_0, U_1, U_2, \ldots$ is a sequence of open subsets of $A^* \cup B^*$ such that for each j, $U_{j+1} \subset U_j$ and each loop in U_{j+1} is homotopic to 0 in U_j. Suppose V is open in $A^* \cup B^*$, $V \subset \bigcap_{j=0}^{\infty} U_j$, and for some integers k and l, $A_{kl} \subset V$. Then there is a loop γ in $U_0 \cap T_{kl}$ such that $\gamma \nsim 0$ in T_{kl}.*

LEMMA 3. *Suppose the hypothesis of Lemma 2, and in addition, each of $V, U_0, U_1, U_2, \ldots$ is a union of elements of G. Suppose U_x and U_y are neighborhoods (in $A^* \cup B^*$) of x and y, respectively. Then there exist integers s and t such that (1) $t < s$, (2) $T_{ks} \subset U_x$, (3) $T_{kt} \subset U_y$, and (4) if i is an integer such that $t \leqslant i \leqslant s$, there is a loop γ_i in $U_1 \cap T_{ki}$ such that $\gamma_i \nsim 0$ in T_{ki}.*

Proof. From the construction, there exist integers t and s such that $t < s$, $T_{ks} \subset U_x$, and $T_{kt} \subset U_y$. We may suppose that $s-l = l-t$; let m denote $s-l$.

The sequence $U_{m+1}, U_{m+2}, U_{m+3}, \ldots$ and the set V satisfy the hypothesis of Lemma 2. Hence there is a loop γ_l in $U_{m+1} \cap T_{kl}$ such that $\gamma_l \nsim 0$ in T_{kl}. Since each loop in U_{m+1} is trivial in U_m, γ_l bounds a singular disc Δ_l in U_m. Then by Lemma 1, there exist (a) loop γ_{l+1} in $U_m \cap T_{k(l+1)}$ such that $\gamma_{l+1} \nsim 0$ in $T_{k(l+1)}$ and (b) a loop γ_{l-1} in $U_m \cap T_{k(l-1)}$ such that $\gamma_{l-1} \nsim 0$ in $T_{k(l-1)}$.

After finitely many repetitions of this procedure, we obtain loops γ_s in $U_1 \cap T_{ks}$ and γ_t in $U_1 \cap T_{kt}$ such that $\gamma_s \nsim 0$ in T_{ks} and $\gamma_t \nsim 0$ in T_{kt}.

Since for each j, $U_j \subset U_1$, then each of the loops $\gamma_s, \gamma_{s-1}, \ldots, \gamma_l, \ldots, \gamma_{t+1}$, and γ_t lies in U_1.

LEMMA 4. *Suppose Q is a 3-cell in E^3, γ and λ are disjoint linked loops in $\mathrm{Int}Q$, and λ bounds a singular disc Δ in E^3 such that in a neighborhood of $\mathrm{Bd}Q$, Δ is polyhedral. Then there is an arc β in $\Delta \cap \mathrm{Int}Q$ joining a point of γ to a point of λ.*

Proof. There is a 3-cell Q_0 such that $Q_0 \subset \mathrm{Int}Q$, $\gamma \cup \lambda \subset \mathrm{Int}Q_0$, Δ is polyhedral in a neighborhood of $\mathrm{Bd}Q_0$ and in general position with $\mathrm{Bd}Q_0$. Let δ be a map from a 2-simplex Δ_0 onto Δ such that $\delta[\mathrm{Bd}\Delta_0] = \lambda$ and each component of $\delta^{-1}[\Delta \cap \mathrm{Bd}Q_0]$ is a simple closed curve in $\mathrm{Int}\Delta_0$. Let D_0 be the component of $\Delta_0 - \delta^{-1}[\Delta \cap \mathrm{Bd}Q_0]$ containing $\mathrm{Bd}\Delta_0$. Then $\delta[D_0]$ lies in Q_0, and hence in $\mathrm{Int}Q$.

We shall prove now that γ intersects $\delta[D_0]$. For each boundary curve μ of D_0 distinct from $\mathrm{Bd}\Delta_0$, $\delta[\mu] \subset \mathrm{Bd}Q_0$. Hence there is an extension δ^* of $\delta|D_0$ such that $\delta^*[\Delta_0 - D_0] \subset \mathrm{Bd}Q_0$. Since γ and λ are linked, γ intersects $\delta^*[\Delta_0]$. Since $\gamma \subset \mathrm{Int}Q_0$ and $\delta^*[\Delta_0] \cap \mathrm{Int}Q_0 \subset \delta[D_0]$, γ intersects $\delta[D_0]$.

Hence $\delta[D_0]$ contains an arc β joining a point of γ and a point of λ. Clearly $\beta \subset \Delta \cap \mathrm{Int}Q$.

5. Properties of X, Y, and Z.

LEMMA 5. *Each of X, Y, and Z is a compact absolute neighborhood retract.*

Proof. This follows immediately from Corollary 12.14 of Chapter V of [4] provided each of X, Y, and Z has finite dimension. We shall prove that Z has finite dimension.

Z is the space of a certain decomposition G of the set $A^* \cup B^*$, and $A^* \cup B^* \subset E^3$. Further, G has only a countable number of nondegenerate elements, and each is an arc. Enlarge G to a decomposition G^* or E^4 by adding to the sets of G all singletons in $E^4-(A^* \cup B^*)$; here we regard E^3 as a subset of E^4. Now each arc of G lies in E^3 and hence by the Corollary on page 337 of [5], each arc of G is cellular in E^4. Hence G^* is a cellular (or pointlike) decomposition of E^4. Further, if φ is the associated projection map sending E^4 onto the decomposition space E^4/G^* associated with G^*, then $\varphi[A^* \cup B^*]$ is homeomorphic to Z.

By Corollary 2 of [1], E^4/G^* can be embedded in E^5. Hence Z is finite dimensional.

The following lemma is the main result of this section. Recall that g is the projection map from $A^* \cup B^*$ onto Z.

LEMMA 6. *Suppose M is a compact absolute retract in Z such that for some open subset Λ of $g[\mathrm{Int}A]$ and some integers k and l, $g[A_{kl}] \subset \Lambda \subset M$. Suppose W is any open set in Z containing M, and U_x and U_y are open sets in $A^* \cup B^*$ containing x and y, respectively. Then there is an arc θ in $g^{-1}[W] \cap \mathrm{Int}A$ from a point of U_x to a point of U_y.*

Proof. By Lemma 5, Z is a compact absolute neighborhood retract. Hence we may apply Lemma 7 of [3]. Let W_0 denote W. Hence there exists a sequence $W_0, W_1, W_2, \ldots$ of open sets in Z such that for each i, $M \subset W_i$, $W_{i+1} \subset W_i$, and each loop in W_{i+1} is homotopic to 0 in W_i.

For each i, let U_i denote $g^{-1}[W_i]$. Then for each i, U_i is open in $A^* \cup B^*$ and a union of elements of G. Further, by Lemma 9 of [3], for each i, each loop in U_{i+1} is homotopic to 0 in U_i.

Let V denote $g^{-1}[\Lambda]$. Since $g[\mathrm{Int}A]$ is open in Z, V is open in $A^* \cup B^*$. Since $M \subset \bigcap_{i=0}^{\infty} W_i$, then $V \subset \bigcap_{i=0}^{\infty} U_i$. Since there exist integers k and l such that $g[A_{kl}] \subset \Lambda$ it follows that $A_{kl} \subset V$.

By Lemma 3, there exist integers s and t with $t < s$ such that $T_{ks} \subset U_x$, $T_{kt} \subset U_y$, and if i is an integer such that $t \leqslant i \leqslant s$, there is a loop γ_i in $U_i \cap T_{ki}$ such that $\gamma_i \nsim 0$ in T_{ki}.

Now each loop in U_1 is trivial in U_0. Hence for each i such that $t \leqslant i \leqslant s$, γ_i bounds a singular disc Δ_i in U_0.

Let A_0 be a 3-cell in $\mathrm{Int}A$ such that $\bigcup_{i=t}^{s}\gamma_i \subset \mathrm{Int}A_0$. Since $\mathrm{Bd}A_0 \subset \mathrm{Int}A$, we may, for each i such that $t \leqslant i \leqslant s$, adjust Δ_i in a neighborhood of $\mathrm{Bd}A_0$ so that the adjusted Δ_i is polyhedral and lies in U_0. We assume then that each Δ_i has these properties.

Now by construction and the fact that for each i such that $t \leqslant i \leqslant s$, γ_i lies in T_{ki} and $\gamma_i \sim 0$ in T_{ki}, it follows that if $t \leqslant i < s$, γ_i and γ_{i+1} are linked. Therefore, by Lemma 4, if $t \leqslant i < s$, there is an arc β_i in $\Delta_i \cap \mathrm{Int}A$ joining a point of γ_i to a point of γ_{i+1}. Then $(\bigcup_{i=t}^{s}\gamma_i) \cup (\bigcup_{i=t}^{s-1}\beta_i)$ is an arcwise connected continuum lying in $(\bigcup_{i=t}^{s}\Delta_i) \cap \mathrm{Int}A$, and since $(\bigcup_{i=t}^{s}\Delta_i) \subset U_0$, this continuum lies in $U_0 \cap \mathrm{Int}A$. Since $\gamma_s \subset T_{ks} \subset U_x$ and $\gamma_t \subset T_{kt} \subset U_y$, there is an arc θ lying in $U_0 \cap \mathrm{Int}A$ joining a point of U_x and a point of U_y.

In the construction of X, we made use of a countable dense subset $x_1, x_2, \ldots$ of $\mathrm{Int}A$. The construction was done so that if r is any positive integer, there exist integers k and l such that $x_r \in A_{kl}$. Note that $\{g(x_r)\colon r = 1, 2, 3, \ldots\}$ is dense in $g[\mathrm{Int}A]$, and since $\mathrm{Int}A$ is a union of elements of G, $g[\mathrm{Int}A]$ is open in Z.

For each positive integer r, let y_r be the image, under reflection in E^2, of x_r, and for integers k and l, let B_{kl} denote the image, under reflection in E^2, of A_{kl}.

Lemma 7. *Suppose M is a compact absolute retract in Z such that for some open subset Ω of $g[\mathrm{Int}B]$ and some integers m and n, $g[B_{mn}] \subset \Omega \subset M$. Suppose U_x and U_y are open sets in $A^* \cup B^*$ containing x and y, respectively, and W is any open set in Z containing M. Then there is an arc φ in $g^{-1}[W] \cap \cap \mathrm{Int}B$ from a point of U_x to a point of U_y.*

Lemma 8. *Suppose M is a compact absolute retract in Z such that for some open subset Λ of $g[\mathrm{Int}A]$ and some integers k and l, $g[A_{kl}] \subset \Lambda \subset M$. Then both x and y belong to $g^{-1}[M]$.*

Proof. Since M is compact, so is $g^{-1}[M]$. We shall prove that if U is any open set in $A^* \cup B^*$ containing $g^{-1}[M]$, both x and y belong to the closure of U. It then follows easily that x and y belong to $g^{-1}[M]$.

Let U be any open set in $A^* \cup B^*$ containing $g^{-1}[M]$. Since G is upper semicontinuous, there is an open set V in $A^* \cup B^*$ such that $g^{-1}[M] \subset V \subset U$ and V is a union of elements of G. Hence $g[V]$ is open in Z. Let U_x and U_y be open sets in $A^* \cup B^*$ containing x and y, respectively.

By Lemma 6, there is an arc θ in $V \cap \mathrm{Int}A$ intersecting both U_x and U_y. Since $V \subset U$, U intersects U_x and U_y. Hence x and y belong to the closure of U.

Lemma 9. *Suppose M is a compact absolute retract in Z such that for some open subset Ω of $g[\mathrm{Int}B]$ and some integers m and n, $g[B_{mn}] \subset \Omega \subset M$. Then both x and y belong to $g^{-1}[M]$.*

6. Existence of brick decompositions.

Lemma 10. *Each of X, Y, and $X \cap Y$ has a brick decomposition.*

Proof. It is easy to see that the disc $C \cup D \cup E$ has a brick decomposition $\mathfrak{C}$ into discs, having D and E as elements, and such that no disc of $\mathfrak{C}$ intersects both D and E. Since $X \cap Y$ is homeomorphic to $C \cup D \cup E$, then $X \cap Y$ has a brick decomposition.

Recall that X is the space associated with the decomposition H of A^*, and that h is the projection map from A^* onto X. Now A is a 3-cell and is a union of elements of H. By Corollary 12.14 of Chapter V of [4], $h[A]$ is a compact absolute retract. The fact that $h[A]$ is finite-dimensional is a corollary of facts established in the proof of Lemma 5, and of the fact that $h[A]$ and $g[A]$ are homeomorphic.

Let $\mathfrak{B}$ consist of $h[A]$ and each set $h[\sigma]$ where σ is a disc of $\mathfrak{C}$ distinct from D and E. Recall that no disc of $\mathfrak{C}$ intersects both D and E. It follows that $\mathfrak{B}$ is a brick decomposition of X. Since Y is homeomorphic to X, Y has a brick decomposition.

Lemma 11. *Z has no brick decomposition.*

Proof. Suppose Z has a brick decomposition $\{Z_1, Z_2, \ldots, Z_p\}$. Now $g[\mathrm{Int}A]$ is an open subset of Z, and hence there exists an integer q such that Z_q contains an open subset Λ of $g[\mathrm{Int}A]$. Then since $\{g(x_i) = i = 1, 2, \ldots\}$ is dense in $g[\mathrm{Int}A]$, there exists an integer i such that $g(x_i) \in \Lambda$.

By a similar argument, there exist integers r and j such that $g(y_j)$ lies in an open set Ω in $g[\mathrm{Int}B]$ and $\Omega \subset Z_r$.

Now it follows from Lemmas 8 and 9 that both Z_q and Z_r contain $g(x)$ and $g(y)$, and hence Z_q and Z_r intersect. Since $\{Z_1, Z_2, \ldots, Z_p\}$ is a brick decomposition, $Z_q \cap Z_r$ is a compact absolute retract, and therefore, by 2.9, Chapter V of [4], $Z_q \cup Z_r$ is a compact absolute retract. Let M denote $Z_q \cup Z_r$. We have shown that (1) there exist an open subset Λ of $g[\mathrm{Int}A] \cap M$ and an integer i such that $g(x_i) \in \Lambda$ and (2) there exist an open subset Ω of $g[\mathrm{Int}B] \cap M$ and an integer j such that $g(x_j) \in \Omega$.

It follows that there exist integers k and l such that $g[A_{kl}] = g(x_i)$, and integers m and n such that $g[B_{mn}] = g(y_i)$. Hence $g[A_{kl}] \subset \Lambda$ and $g[B_{mn}] \subset \Omega$.

Since M is a compact absolute retract in Z, then by Lemma 6 of [3], there is an open set W in Z such that $M \subset W \subset Z$ and each loop in W is homotopic to 0 in Z. By Lemma 9 of [3], each loop in $g^{-1}[W]$ is homotopic to 0 in $g^{-1}[Z]$, or in $A^* \cup B^*$.

Since both x and y belong to $g^{-1}[M]$ and $g^{-1}[M] \subset g^{-1}[W]$, there exist disjoint open 3-cells U_x and U_y containing x and y, respectively, and contained in $g^{-1}[W]$.

By Lemma 6, there is an arc θ in $(\operatorname{Int} A) \cap g^{-1}[W]$ from a point of U_x to a point of U_y. By Lemma 7, there is an arc φ in $(\operatorname{Int} B) \cap g^{-1}[W]$ from a point of U_x to a point of U_y. Let γ be a simple closed curve formed by joining θ and φ by an arc in U_x and by an arc in U_y. Then γ is a loop in $g^{-1}[W]$ and it is clear that $\gamma \nsim 0$ in $A^* \cup B^*$. However, W has the property that each loop in $g^{-1}[W]$ is homotopic to 0 in $A^* \cup B^*$. This is a contradiction, and thus Z has no brick decomposition.

The following summarizes our results.

THEOREM. *There exist compact metric spaces X and Y such that* (1) *X, Y, and $X \cap Y$ have brick decompositions, but* (2) *$X \cup Y$ has no brick decomposition.*

References

[1] S. Armentrout, *On embedding decomposition spaces of E^n in E^{n+1}*, Fund. Math. 61 (1967), pp. 1–21.
[2] — *On the strong local simple connectivity of the decomposition spaces of toroidal decompositions*, Fund. Math. 69 (1970), pp. 15–37.
[3] — *On the singularity of Mazurkiewicz in absolute neighborhood retracts*, Fund. Math. 69 (1970), pp. 131–145.
[4] K. Borsuk, *Theory of retracts*, Warsaw 1967.
[5] D. R. McMillan, Jr., *A criterion for cellularity in a manifold*, Ann. of Math. 79 (1964), pp. 327–337.

UNIVERSITY OF IOWA
THE INSTITUTE FOR ADVANCED STUDY

Reçu par la Rédaction le 3. 2. 1970

Determinateness in the low projective hierarchy *

by

Harvey M. Friedman (Madison, Wisc.)

Introduction. This paper contains results on Σ_1^1 determinateness, determinateness for certain fragments of the Boolean algebra generated by the $\boldsymbol{\Sigma}_1^1$ sets, and $\boldsymbol{\Delta}_2^1$ determinateness. We use L for the class of constructible sets. In the first section, we use L_α for the set of all sets of level $< \alpha$ in the constructible hierarchy, and $L_\alpha(x)$ for the set of all sets of level $< \alpha$ in the constructible hierarchy starting from x. In Sections 2, 3, 4, however, we found it convenient to use, respectively, $L(\alpha)$ and $L^x(\alpha)$, and to use L^x for the class of sets constructible from x. (No confusion will arise as to which of the possible notions of relative constructibility is used.) By K determinateness, where $K \subset (N^N \times N^N)$, we mean that $(\forall A \in K)$ (A is determined). By "A is determined" we mean that the game G_A has a winning strategy for either player I or player II, where G is played as follows: players I, II play alternately, starting with I. Each move is an integer. The result of the game is an element x of $N^N \times N^N$, and I is deemed the winner if $x \in A$; II is deemed the winner otherwise.

In Section 1 we consider Σ_1^1 determinateness (a relativized version, Theorem 1′, is stated in Section 4). Previously, there were two main results about Σ_1^1 determinateness. The first is the result of D. Martin [4] that $\boldsymbol{\Sigma}_1^1$ determinateness follows from (*) of Section 2, which in turn follows from measurable cardinals. The second is that $\Sigma_1^1(\boldsymbol{\Sigma}_1^1)$ determinateness implies that every uncountable Σ_1^1 ($\boldsymbol{\Sigma}_1^1$) subset of N^N contains a perfect subset (Davis [2], Theorems 4.1, 4.2). The former result suggested that it would be worthwhile to find a consistency proof for Σ_1^1 ($\boldsymbol{\Sigma}_1^1$) determinataness using the consistency of some currently formulated axioms about sets. The latter result showed that, by Solovay [10], Σ_1^1 ($\boldsymbol{\Sigma}_1^1$) determinateness implies that $N^N \cap L$ is countable ($N^N \cap L(x)$ is countable for all $x \subset \omega$), and so Σ_1^1 determinateness cannot be proved from the currently formulated axioms about sets, since they are all compatible with $(\forall x)(x \in L)$. But it was still possible that Σ_1^1 determinateness was

* This research was partially supported by NSF GP-13335. This is a revised version.