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On separable Banach spaces containing all separable reflexive Banach spaces

by

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1. Introduction. The main result of this paper is the following

THEOREM 1. *If a separable Banach space X has the property that every separable reflexive Banach space is isomorphic to a subspace of X, then x is not isomorphic to the dual space of any Banach space.*

This theorem completes the result (due to Szlenk [5]) that if a Banach space satisfies the hypothesis of Theorem 1, then it has a non-separable dual and improves in the separable case a result of Lindenstrauss ([3], Theorem 3.1), where the isometric isomorphisms are considered. Our Corollary 1 and results of [5] and [3] give the complete answer on the problem raised by Banach and Mazur (cf. [6], Problem 49). From Theorem 1 we deduce the non-existence of a boundedly complete basis which is universal for all boundedly complete normalized bases. This confirmed a conjecture of Pełczyński [4], p. 267. This result was also obtained recently by Zippin [7] who used a different method.

To prove Theorem 1 we used a modification of Szlenk's method [5].

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2. Definitions and notations. The capital letters $X, Y, Z, \ldots$, will denote Banach spaces and the letters $f, x, y, z, \ldots$, will denote the elements of Banach spaces. The term "subspace" means "closed linear subspace". The symbol K_X will denote the closed unit ball of the space X. By X^* and X^{**} we shall denote the first and the second dual of a Banach space X. We shall always identify the Banach space X with its canonical image in X^{**}. The letters α, β, γ will be reserved for denoting ordinal numbers. By ω_1 we shall denote the first uncountable ordinal number. By the (X, X^*)*-topology* we mean the X^*-topology of X called also the *weak topology of* X. By the (X^*, X)*-topology* we mean the X topology of X^*

called also the *weak star topology of* X^* (cf. [2], V, § 3, for definitions). The convergent in (X, X^*)-topology (resp. in (X^*, X)-topology) will be denoted by $x_n \xrightarrow[(X,X^*)]{} x_0$ (resp. $x_n^* \xrightarrow[(X^*,X)]{} x_0^*$). We shall preserve the terminology of [1].

We shall need the following

Definition 1. Let $\mathscr{A}$ be a class of Banach spaces. The Banach space $X \in \mathscr{A}$ is said to be *isomorphically universal for the class* $\mathscr{A}$ if each member of $\mathscr{A}$ is isomorphic (i.e. linearly homeomorphic) to a subspace of X.

Next we recall the definition and the basic properties of the *Szlenk index* (cf. [5]).

Definition 2. Let G and Γ be bounded sets in Banach spaces X and X^* respectively. Let us assume that Γ is (X^*, X)-compact. For every $\varepsilon > 0$ and to each ordinal number a we assign (by transfinite induction) a set $P_a(\varepsilon; G, \Gamma)$ as follows:

1. $P_0(\varepsilon; G, \Gamma) = \Gamma$;

2. $P_{a+1}(\varepsilon; G, \Gamma) = \{f \in X^*$: there exist $x_m \in G$ and $f_m \in P_a(\varepsilon; G, \Gamma)$ for $m = 1, 2, \ldots$, such that $f_m \xrightarrow[(X^*,X)]{} f$, $x_m \xrightarrow[(X,X^*)]{} 0$ and $\limsup\limits_m |f_m(x_m)| \geqslant \varepsilon\}$;

3. If a is a limit number, then $P_a(\varepsilon; G, \Gamma) = \bigcap\limits_{\gamma<a} P_\gamma(\varepsilon; G, \Gamma)$.

Let us set

$$\eta(\varepsilon; G, \Gamma) = \sup\{a < \omega_1 \colon P_a(\varepsilon; G, \Gamma) \neq \emptyset\}.$$

Definition 3. The *Szlenk index* of a Banach space X with separable dual is the ordinal number

$$\eta(X) = \sup_{\varepsilon>0} \eta(\varepsilon; K_X, K_{X^*}).$$

The following basic properties of the index $\eta(\cdot)$ were established by Szlenk [5]:

S.1. *If X^* is separable, then $\eta(X) < \omega_1$.*

S.2. *If Banach spaces X and Y are isomorphic, then $\eta(X) = \eta(Y)$.*

S.3. *For any countable ordinal number a there exists a separable, reflexive Banach space X_a with a basis such that $\eta(X_a) \geqslant a$* (cf. [5] and [4], Theorem 4). *The family X_a can be constructed as follows: $X_0 = l_2$, $X_{a+1} = (X_a \oplus l_2)_1$ and if a is a limit number $X_a = (\prod\limits_{\gamma<a} X_\gamma)_2$.*

3. The main result. In this section we consider only Banach spaces with separable duals. Let us introduce the following concept.

Definition 4. Let Γ and Q be bounded sets in X^* and X^{**} respectively, and let Γ be (X^*, X)-compact. For each ordinal number a and real number $\varepsilon > 0$ we define the sets $B_a(\varepsilon; \Gamma, Q)$ as follows:

1. $B_0(\varepsilon; \Gamma, Q) = \Gamma$.

2. $B_{a+1}(\varepsilon; \Gamma, Q) = \{f \in X^*$: there exist $f_n \in B_a(\varepsilon; \Gamma, Q)$ and $x_n^{**} \in Q$ for $n = 1, 2, \ldots$, such that $x_n^{**} \xrightarrow[(X^{**},X^*)]{} 0$, $f_n \xrightarrow[(X^*,X)]{} f$ and $\liminf\limits_n |x_n^{**}(f_n)| \geqslant \varepsilon\}$.

3. If a is a limit number, then $B_a(\varepsilon; \Gamma, Q) = \bigcap\limits_{\gamma<a} B_\gamma(\varepsilon; \Gamma, Q)$.

Lemma 1. *Let G, Γ be bounded sets in X and X^* respectively and let Γ be (X^*, X)-compact. Denote by $\tilde{G}$ the (X^{**}, X^*)-closure of G regarded as a subset of X^{**}. Then for each a and each $\varepsilon > 0$ we have*

$$P_a(\varepsilon; G, \Gamma) = B_a(\varepsilon; \Gamma, \tilde{G}).$$

Proof. Obviously $P_0(\varepsilon; G, \Gamma) = \Gamma = B_0(\varepsilon; \Gamma, \tilde{G})$.

Next we show inductively that $P_\beta(\varepsilon; G, \Gamma) \subset B_\beta(\varepsilon; \Gamma, \tilde{G})$ for each $\beta < \omega_1$.

Suppose, that for a countable ordinal number $a \geqslant 0$ we have $P_a(\varepsilon; G, \Gamma) \subset B_a(\varepsilon; \Gamma, \tilde{G})$. If $f \in P_{a+1}(\varepsilon; G, \Gamma)$, then there exist $x_n \in G$ and $f_n \in P_a(\varepsilon; G, \Gamma)$ for $n = 1, 2, \ldots$ such that $f_n \xrightarrow[(X^*,X)]{} f$, $x_n \xrightarrow[(X,X^*)]{} 0$ and $\limsup\limits_k |f_n(x_n)| \geqslant \varepsilon$. Let us extract an increasing sequence of indices (n_k) such that $\liminf\limits_k |f_{n_k}(x_{n_k})| \geqslant \varepsilon$; by inductive hypothesis $f_{n_k} \in B_a(\varepsilon; \Gamma, \tilde{G})$. Since x_{n_k} (considered as elements of X^{**}) belong to $\tilde{G}$ for $k = 1, 2, \ldots$, and $x_{n_k} \xrightarrow[(X^{**},X^*)]{} 0$, we obtain $f \in B_{a+1}(\varepsilon; \Gamma, \tilde{G})$. Clearly if a is a limit ordinal number and for each $a' < a$ we have $P_{a'}(\varepsilon; G, \Gamma) \subset B_{a'}(\varepsilon; \Gamma, \tilde{G})$, then $P_a(\varepsilon; G, \Gamma) \subset B_a(\varepsilon; \Gamma, \tilde{G})$.

Now we show inductively that $B_a(\varepsilon; \Gamma, \tilde{G}) \subset P_a(\varepsilon; G, \Gamma)$. If $f \in B_{a+1}(\varepsilon; \Gamma, \tilde{G})$, then there exist $x_n^{**} \in \tilde{G}$ and $f_n \in B_a(\varepsilon; \Gamma, \tilde{G})$ for $n = 1, 2, \ldots$, such that $x_n^{**} \xrightarrow[(X^{**},X^*)]{} 0$, $f_n \xrightarrow[(X^*,X)]{} f$ and $\liminf\limits_n |x_n^{**}(f_n)| \geqslant \varepsilon$. Since the (X^{**}, X^*)-topology in $\tilde{G}$ is metrisable for separable X^* (cf. [2], p. 426), we can define the sets

$$U_n = \{x^{**} \in \tilde{G} \colon |x^{**}(f_n) - x_n^{**}(f_n)| < 1/n\} \cap \{x^{**} \in \tilde{G} \colon \varrho(x_n^{**}, x^{**}) < 1/n\} \quad \text{for } n = 1, 2, \ldots,$$

where ϱ is a metric for $\tilde{G}$ equipped with (X^{**}, X^*)-topology. Clearly the sets $U_n \subset \tilde{G}$ are non-empty (because $x_n^{**} \in U_n$) and open in (X^{**}, X^*)-topology. Since G is dense in $\tilde{G}$ in this topology there exists an $x_n \in G \cap U_n$. We have

$$\limsup_n |f_n(x_n)| \geqslant \liminf_n |f_n(x_n)| \geqslant \liminf_n (|x_n^{**}(f_n)| - 1/n) \geqslant \varepsilon$$

and $x_n \xrightarrow[(X^{**},X^*)]{} 0$ in X^{**} or, equivalently, $x_n \xrightarrow[(X,X^*)]{} 0$ in X. So we get $f \in P_{a+1}(\varepsilon; G, \Gamma)$. Thus $B_{a+1}(\varepsilon; \Gamma, \tilde{G}) \subset P_{a+1}(\varepsilon; G, \Gamma)$. Clearly if a is a limit ordinal number and for each $a' < a$ we have $B_{a'}(\varepsilon; \Gamma, \tilde{G}) \subset P_{a'}(\varepsilon; G, \Gamma)$, then $B_a(\varepsilon; \Gamma, \tilde{G}) \subset P_a(\varepsilon; G, \Gamma)$. Thus for each countable ordinal number a we obtain $P_a(\varepsilon; G, \Gamma) = B_a(\varepsilon; \Gamma, \tilde{G})$ and the proof is complete.

LEMMA 2. *Let Z, Y be Banach spaces such that Y^* is separable, Z is reflexive and Z^* is a subspace of Y^*. Then for each $0 \leqslant \alpha < \omega_1$ and for real $\varepsilon > 0$*

$$B_\alpha(\varepsilon; K_{Z^*}, K_{Z^{**}}) \subset B_\alpha(\tfrac{1}{2}\varepsilon; K_{Y^*}, K_{Y^{**}}). \tag{1}$$

Proof. We use the transfinite induction. Obviously

$$B_0(\varepsilon; K_{Z^*}, K_{Z^{**}}) = K_{Z^*} \subset K_{Y^*} = B_0(\tfrac{1}{2}\varepsilon; K_{Y^*}, K_{Y^{**}}).$$

Suppose that (1) is true for $\beta < \alpha < \omega_1$. We shall show then that (1) holds for α. This is trivial for the limit ordinals. Assume therefore that $\alpha = \beta+1$. Let $f \in B_\alpha(\varepsilon; K_{Z^*}, K_{Z^{**}})$. (If $B_\alpha(\varepsilon; K_{Z^*}, K_{Z^{**}})$ is empty it is nothing to prove.) Then there exist $f_n \in B_\beta(\varepsilon; K_{Z^*}, K_{Z^{**}})$ and $z_n^{**} \in K_{Z^{**}}$ for $n = 1, 2, \ldots$, such that $z_n^{**} \xrightarrow[(Z^{**},Z^*)]{} 0$ and $f_n \xrightarrow[(Z^*,Z)]{} f$ and $\liminf\limits_n |z_n^{**}(f_n)| \geqslant \varepsilon$. Since z_n^{**} is a functional defined on the subspace Z^* of Y^*, we can extend it to an $y_n^{**} \in K_{Y^{**}}$ (by the Hahn-Banach theorem). Since $K_{Y^{**}}$ is compact in (Y^{**}, Y^*)-topology, we can choose a subsequence $(y_{n_k}^{**})$ such that $y_{n_k}^{**} \xrightarrow[(Y^{**},Y^*)]{} y_0^{**}$ for some $y_0^{**} \in K_{Y^{**}}$. Let us put

$$\tilde{y}_{n_k}^{**} = \tfrac{1}{2}(y_{n_k}^{**} - y_0^{**}).$$

Obviously, we have $\tilde{y}_{n_k}^{**} \xrightarrow[(Y^{**},Y^*)]{} 0$. Now let us consider the sequence (f_{n_k}). It is clear that $f_{n_k} \xrightarrow[(Z^*,Z)]{} f$. Thus by reflexivity of Z $f_{n_k} \xrightarrow[(Z^*,Z^{**})]{} f$. This implies that $f_{n_k} \xrightarrow[(Y^*,Y^{**})]{} f$. Hence $f_{n_k} \xrightarrow[(Y^*,Y)]{} f$. The inductive hypothesis implies that $f_{n_k} \in B_\beta(\frac{1}{2}\varepsilon; K_{Y^*}, K_{Y^{**}})$. Moreover, we have

$$\begin{aligned}\liminf_k |\tilde{y}_{n_k}^{**}(f_{n_k})| &= \tfrac{1}{2}\liminf_k |(y_{n_k}^{**} - y_0^{**}) f_{n_k}| \\ &= \tfrac{1}{2}\liminf_k |y_{n_k}^{**}(f_{n_k}) - y_0^{**}(f_{n_k})| = \tfrac{1}{2}\liminf_k |y_{n_k}^{**}(f_{n_k})| \geqslant \tfrac{1}{2}\varepsilon\end{aligned}$$

because for $z^* \in Z^*$ we have

$$y_0^{**}(z^*) = \lim_k y_{n_k}^{**}(z^*) = \lim_k z_{n_k}^{**}(z^*) = 0.$$

Thus we get $f \in B_\alpha(\frac{1}{2}\varepsilon; K_{Y^*}, K_{Y^{**}})$ what means nothing else but

$$B_\alpha(\varepsilon; K_{Z^*}, K_{Z^{**}}) \subset B_\alpha(\tfrac{1}{2}\varepsilon; K_{Y^*}, K_{Y^{**}}).$$

This completes the inductive proof.

LEMMA 3. *Let Y be a Banach space with the separable dual and let Z be reflexive and let Z^* be a subspace of Y^*. Then for $\varepsilon > 0$*

$$\eta(\varepsilon; K_Z, K_{Z^*}) \leqslant \eta(\tfrac{1}{2}\varepsilon; K_Y, K_{Y^*})$$

and

$$\eta(Z) \leqslant \eta(Y).$$

Proof. Use Lemmas 1 and 2 and the fact that the unite ball of Y^{**} is the (Y^{**}, Y^*)-closure of the unite ball of Y ([1], p. 41).

PROPOSITION 1. *Let $(X_\alpha)_{\alpha<\omega_1}$ be a family of separable reflexive spaces with the property that $\eta(X_\alpha) \geqslant \alpha$ for $0 \leqslant \alpha < \omega_1$. Let Y be a separable Banach space which contains subspaces isomorphic to each X_α^*, $0 \leqslant \alpha < \omega_1$. Then Y is not isomorphic to a dual space of any Banach space.*

Proof. Assume to the contrary that Y is isomorphic to the separable dual X^* of a Banach space X. Let V_α be a subspace of X^* isomorphic to X_α^*. Since X_α is reflexive, V_α is isometrically isomorphic to the dual Z_α^* of a Banach space Z_α isomorphic to X_α. By S.2, we have $\eta(Z_\alpha) = \eta(X_\alpha)$. On the other hand, by Lemma 3, we infer that $\eta(X) \geqslant \eta(Z_\alpha) = \eta(X_\alpha) \geqslant \alpha$ for each $\alpha < \omega_1$. But this contradicts S.1. This completes the proof.

Proof of Theorem 1. Combine Proposition 1 with S.3.

From Theorem 1 we deduce some corollaries.

COROLLARY 1. *There is no isomorphically universal Banach space in the class of separable dual spaces.*

This is an obvious consequence of Theorem 1 and the fact that every reflexive Banach space is a dual one.

The next application concerns the universal bases (see [5] and [7]).

Two bases (x_n) and (y_n) in Banach spaces X and Y respectively are said to be *equivalent* if a series $\sum\limits_n c_n x_n$ converges if and only if the series $\sum\limits_n c_n y_n$ converges. If (n_k) is an increasing sequence of indices, then the sequence (x_{n_k}) is called a *subbasis* of a basis (x_n). Clearly the subbasis is a basis in the subspace which it spans. A basis (x_n) is boundedly complete if the series $\sum\limits_n t_n x_n$ is convergent whenever $\sup\limits_n \|\sum\limits_{k=1}^n t_k x_k\| < +\infty$.

DEFINITION 5. Let $\mathscr{B}$ be a family of bases. A basis (x_n) is said to be *universal for $\mathscr{B}$* if every basis in $\mathscr{B}$ is equivalent to a subbasis of (x_n).

COROLLARY 2. *There is no boundedly complete basis universal for the family of all normalized boundedly complete bases.*

Proof. Let $(X_\alpha)_{\alpha<\omega_1}$ be a family of reflexive spaces described in S.3. Since X_α^* are reflexive and have bases, because X_α have this property, the bases are boundedly complete ([1], p. 71). Thus if Y is a Banach space with a basis which is universal for the class of all normalized boundedly complete bases, then it contains subspaces isomorphic to each X_α^* for $0 \leqslant \alpha < \omega_1$. Hence, by Proposition 1, Y is not isomorphic to the dual of a separable Banach space. Hence Y does not have a boundedly complete basis, because a Banach space with a boundedly complete basis is isomorphic to a dual of a Banach space ([1], p. 70). This completes the proof.

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