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Hilbertian, Besselian, and semi-shrinking bases *

by

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1. Introduction. Let (x_i, f_i) be a Schauder basis for a Banach space X such that $0 < \inf \|x_i\| \leqslant \sup \|x_i\| < +\infty$. Then (f_i) is a basic sequence in X^* with coefficient functionals in $[f_i]^*$ which are similar to (x_i) in X by the "norm determining" property of $[f_i]$ (cf. [15]). We write (f_i, x_i) to designate this fact.

The basis (x_i, f_i) is said to be

(i) *p-Hilbertian* $(1 < p < +\infty)$ if whenever $\sum_i |a_i|^p < +\infty$, then $\sum_i a_i x_i \in X$;

(ii) *p-Besselian* $(1 < p < +\infty)$ if whenever $\sum_i a_i x_i \in X$, then $\sum_i |a_i|^p < +\infty$;

(iii) *semi-shrinking* [10], [11] (or of type wc_0 [3]) if the sequence $\{x_i\}$ converges weakly to zero in X (in symbols, $(x_i) \xrightarrow{w} 0$).

In the case $p = 2$ of (i) and (ii), the basis is called *Hilbertian* (resp. *Besselian*) [8], [9].

The purpose of this paper is to continue the study of these types of bases begun in [9], [10], and [11] and to demonstrate the use of tensor product methods in this investigation.

Section 3 is devoted to the study of general properties of p-Hilbertian and p-Besselian bases and contains a characterization of Hilbertian bases in a space X in terms of a certain basic sequence in $X \otimes_\varepsilon X$. The permanence properties of the tensor product of Hilbertian and Besselian bases in $X \otimes_\varepsilon Y$ and $X \otimes_\pi Y$ are also discussed.

In section 4 we use the results of section 3 and tensor product methods to construct examples of semi-shrinking bases which are not shrinking (and which have certain additional properties). At one time it was an

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open question whether every semi-shrinking basis is shrinking [10]. Since that time Pełczyński and Szlenk have given an example of a semi-shrinking, non-shrinking basis [10] and Retherford has shown that the space (d) of Davis and Dean [2] also has such a basis [11]. In each case the basis constructed is unconditional and boundedly complete. We exhibit a continuum of mutually non-similar bases each of which is semi-shrinking, non-shrinking, conditional, and non-boundedly complete.

2. Preliminary results. We recall the following definitions and results.

Let X and Y be Banach spaces. We will denote by $X\otimes_\varepsilon Y$ the completion of the algebraic tensor product of X and Y in the norm

$$\Big\|\sum_{i=1}^{n} x_i\otimes y_i\Big\| = \sup_{\substack{\|f\|\leqslant 1, f\epsilon X^*\\ \|g\|\leqslant 1, g\epsilon Y^*}} \Big|\sum_{i=1}^{n} f(x_i)g(y_i)\Big|,$$

while we denote by $X\otimes_\pi Y$ the completion of $X\otimes Y$ in the norm

$$\Big\|\sum_{i=1}^{n} x_i\otimes y_i\Big\| = \inf\Big\{\sum_{j=1}^{k}\|x_j'\|\ \|y_j\|:\ \sum_{j=1}^{k} x_j'\otimes y_j' = \sum_{i=1}^{n} x_i\otimes y_i\Big\}.$$

If M is a closed subspace of X and N is a closed subspace of Y, then $M\otimes_\varepsilon N$ is a closed subspace of $X\otimes_\varepsilon Y$([13], p. 35). We write $M\otimes_\varepsilon N \subset X\otimes_\varepsilon Y$. Also, $X^*\otimes_\varepsilon Y^*$ is a closed subspace of $(X\otimes_\pi Y)^*$([13], p. 43).

The space $(X\otimes_\varepsilon Y)^*$ consists exactly of those bounded bilinear forms v on $X\times Y$ that can be represented in the form

$$v(w) = \int_{S\times T} w_0(x', y')\,d\mu(x', y'),$$

where S and T are suitable closed equicontinuous subsets of X_σ^* and Y_σ^* and w_0 is the restriction of the bilinear form w on $X^*\times Y^*$ to $S\times T$([12], p. 168).

If (x_i) is a basis for X and (y_i) a basis for Y, then $(x_i\otimes y_j)$ ordered as

$$\begin{array}{c|c|c|c}
x_1\otimes y_1 & x_1\otimes y_2 & x_1\otimes y_3 & \cdots\\
x_2\otimes y_1 & x_2\otimes y_2 & x_2\otimes y_3 & \cdots\\
x_3\otimes y_1 & x_3\otimes y_2 & x_3\otimes y_3 & \cdots\\
\vdots & \vdots & \vdots &
\end{array}$$

is a basis for $X\otimes_\varepsilon Y$ and $X\otimes_\pi Y$ called the *tensor product basis* [4]. The subsequence $(x_i\otimes y_i)$ of $(x_i\otimes y_j)$ is called the *tensor diagonal* of the bases (x_i) and (y_i) [5].

3. Properties of p-Hilbertian and p-Besselian bases. We begin with some simple lemmas. The first demonstrates the duality existing between Hilbertian and Besselian bases.

LEMMA 3.1. *The basis (x_i, f_i) for X is p-Hilbertian (p-Besselian) $(1<p<+\infty)$ if and only if the basic sequence (f_i, x_i) in X^* is q-Besselian (q-Hilbertian) $(p^{-1}+q^{-1}=1)$.*

Proof. If (x_i) is p-Hilbertian, then by the Banach-Steinhaus theorem the linear map $T\colon l^p\to X$ defined by $T((a_i)) = \sum_i a_i x_i$ is continuous. Therefore the adjoint map $T^*\colon X^*\to l^q$ $(p^{-1}+q^{-1}=1)$ is continuous and $T^*(\sum_i b_i f_i) = (b_i)\,\epsilon\, l^q$ for all $\sum_i b_i f_i\,\epsilon\,[f_i]$. By definition (f_i) is q-Besselian.

The other implication is proved in the same way.

LEMMA 3.2. *If (x_i) is a p-Hilbertian basis for X $(1<p<+\infty)$, then (x_i) is semi-shrinking.*

Proof. As the proof of Lemma 3.1 shows, there is a continuous one-to-one linear map $T\colon l^p\to X$ such that $Te_i = x_i$ (where e_i denotes the i-th unit vector in l^p). For any f in X^*, $T^*(f)$ is in l^p $(p^{-1}+q^{-1}=1)$ and $T^*(f)(e_i)\xrightarrow{i} 0$ since (e_i) is semi-shrinking in l^p [6]. But $T^*(f)e_i = f(Te_i) = f(x_i)$. Therefore (x_i) is semi-shrinking.

THEOREM 3.3. *Let (x_i, f_i) be a p-Hilbertian basis for X and (y_i, g_i) a q-Hilbertian basis for Y $(p^{-1}+q^{-1}=1)$. Then the tensor diagonal $(x_i\otimes y_i)$ in $X\otimes_\varepsilon Y$ is similar to the unit vector basis of c_0.*

Proof. Let $(a_i)\,\epsilon\, c_0$. Then

$$\Big\|\sum_{i=m}^{n} a_i x_i\otimes y_i\Big\| = \sup_{\substack{\|x^*\|\leqslant 1\\ \|y^*\|\leqslant 1}}\Big|\sum_{i=m}^{n} a_i x^*(x_i)y^*(y_i)\Big|.$$

Since the spaces $[f_i]$ and $[g_i]$ are norm-determining over X and Y respectively [15] there is an $M\geqslant 1$ such that this last is less than or equal to

$$M\sup_{m\leqslant i\leqslant n}|a_i|\sup_{\substack{\|x^*\|\leqslant 1, x^*\epsilon[f_i]\\ \|y^*\|\leqslant 1, y^*\epsilon[g_i]}}\sum_{i=m}^{n}|x^*(x_i)|\ |y^*(y_i)|.$$

By Lemma 3.1, (f_i, x_i) is q-Besselian and (g_i, y_i) is p-Besselian. Therefore by the continuity of the mappings T^* described in Lemma 3.1,

$$\sup_{\substack{\|x^*\|\leqslant 1\\ x^*\epsilon[f_i]}}\Big[\sum_{i=m}^{n}|x^*(x_i)|^q\Big]^{1/q}\leqslant \|T_1^*\|,$$

and

$$\sup_{\substack{\|y^*\|\leqslant 1\\ y^*\epsilon[g_i]}}\Big[\sum_{i=m}^{n}|y^*(y_i)|^p\Big]^{1/p}\leqslant \|T_2^*\|$$

(where T_1 is the mapping corresponding to (x_i) in 3.1 and T_2 corresponds to (y_i)).

Applying Hölder's inequality we then have

$$\sup_{\substack{\|x^*\|\leqslant 1, x^*\epsilon[f_i]\\ \|y^*\|\leqslant 1, y^*\epsilon[g_i]}} \sum_{i=m}^{n} |x^*(x_i)|\ |y^*(y_i)| \leqslant \|T_1^*\|\ \|T_2^*\|$$

for all m and n. By our previous inequalities it follows that $\|\sum_{i=m}^{n} a_i x_i \otimes y_i\| \overset{m,n}{\to} 0$ and $(x_i \otimes y_i)$ is similar to the unit vector basis of c_0.

In the case $p = q = 2$ the following proposition (which is a partial converse of Theorem 3.3) is true.

PROPOSITION 3.4. *Let (x_i) be a basis for X such that the tensor diagonal $(x_i \otimes x_i)$, in $X \otimes_\varepsilon X$ is of type P [14]. Then (x_i) is Hilbertian.*

Proof. If (x_i) is of type P in $X \otimes_\varepsilon X$, then by definition

$$\sup_n \Big\| \sum_{i=1}^{n} x_i \otimes x_i \Big\| = \sup_n \sup_{\substack{\|x^*\|\leqslant 1\\ \|y^*\|\leqslant 1}} \Big| \sum_{i=1}^{n} x^*(x_i) y^*(x_i) \Big| < +\infty.$$

In particular, $\sup_n \sup_{\|x^*\|\leqslant 1} \big| \sum_{i=1}^{n} [x^*(x_i)]^2 \big| < +\infty$.

Now if $(a_i) \epsilon l^2$, then

$$\Big\| \sum_{i=m}^{n} a_i x_i \Big\| = \sup_{\|x^*\|\leqslant 1} \Big| \sum_{i=m}^{n} a_i x^*(x_i) \Big|$$
$$\leqslant \Big[\sup_{\|x^*\|\leqslant 1} \sum_{i=m}^{n} |x^*(x_i)|^2 \Big]^{1/2} \Big[\sum_{i=m}^{n} |a_i|^2 \Big]^{1/2} \overset{m,n}{\to} 0.$$

Therefore (x_i) is Hilbertian.

Combining Theorem 3.3 and Proposition 3.4 we obtain the following classification of Hilbertian bases in terms of a tensor diagonal.

COROLLARY 3.5. *A basis (x_i) for X is Hilbertian if and only if the tensor diagonal $(x_i \otimes x_i)$ in $X \otimes_\varepsilon X$ is similar to (e_i) in c_0.*

An interesting corollary of Theorem 3.3 is

COROLLARY 3.6. *If X contains a p-Hilbertian basic sequence (x_i) and Y contains a q-Hilbertian basic sequence (y_i) $(p^{-1}+q^{-1} = 1)$, then $X \otimes_\varepsilon Y$ is not reflexive.*

Proof. The corollary follows immediately from Theorem 3.3, the fact that $[x_i] \otimes_\varepsilon [y_i]$ is a closed subspace of $X \otimes_\varepsilon Y$, and the fact that every closed subspace of a reflexive Banach space is reflexive.

We will need the following result in order to demonstrate that the duality existing between the ε and π topologies on $X \otimes Y$ corresponds to the duality between the notions of Hilbertian and Besselian bases.

LEMMA 3.7. *Let (x_i, f_i) be a basis for X having a subsequence (x_{n_i}) which is similar to (e_i) in c_0. Then the corresponding subsequence (f_{n_i}) of (f_i) in X^* is similar to (e_i) in l^1.*

Proof. If (x_{n_i}) is similar to (e_i) in c_0, then the sequence of coefficient functionals (g_{n_i}) in $[x_{n_i}]^*$ is similar to (e_i) in l^1. But clearly $g_{n_i} = f_{n_i}|_{[x_{n_i}]}$ for each i. Hence if $\sum_i a_{n_i} f_{n_i}$ converges, then $\sum_i a_{n_i} f_{n_i}|_{[x_{n_i}]} = \sum a_{n_i} g_{n_i}$ converges. Thus $\sum_i |a_{n_i}| < +\infty$ and (f_{n_i}) is similar to (e_i) in l^1.

Remark. Since $[f_i]$ is norm determining over X, it follows from Lemma 3.7 that if (x_i, f_i) is a basis for X such that a subsequence (f_{n_i}) is similar to (e_i) in c_0, then (x_{n_i}) is similar to (e_i) in l^1.

THEOREM 3.8. *Let (x_i, f_i) be a p-Besselian basis for X and (y_i, g_i) a q-Besselian basis for Y $(p^{-1}+q^{-1} = 1)$. Then the tensor diagonal $(x_i \otimes y_i)$ in $X \otimes_\pi Y$ is similar to (e_i) in l^1.*

Proof. By Lemma 3.1, (f_i, x_i) is q-Hilbertian and (g_i, y_i) is p-Hilbertian. Hence by Theorem 3.3 $(f_i \otimes g_i)$ in $[f_i] \otimes_\varepsilon [g_i] \subset (X \otimes_\pi Y)^*$ is similar to (e_i) in c_0. It then follows from the previous remark that $(x_i \otimes y_i)$ is similar to (e_i) in l^1.

We note that the duality expressed in Theorems 3.3 and 3.8 cannot be extended further. That is, there is no analogue of Proposition 3.5 to be obtained by replacing "Hilbertian" by "Besselian", "ε" by "π", and "c_0" by "l^1" in that proposition.

EXAMPLE 3.9. *Let (x_i) be the (conditional) basis for l^1 defined by $x_1 = e_1$, $x_n = e_{n-1} - e_n$ for $n > 1$. Then $(x_i \otimes x_i)$ in $l^1 \otimes_\pi l^1$ is similar to (e_i) in l^1, but (x_i) is not Besselian.*

Proof. It is well known that in $l^1 \otimes_\pi Z$, $\|\sum_{i=1}^{n} e_i \otimes z_i\| = \sum_{i=1}^{n} \|z_i\|$ [1]. Using this fact one easily computes that for $n \geqslant 3$,

$$\Big\| \sum_{i=1}^{n} a_i x_i \otimes x_i \Big\| = \sum_{i=1}^{n-1} \big| a_i + a_{i+1} \big| + 2 \sum_{i=2}^{n} |a_i| + 3|a_n|.$$

Hence if $\sum_i a_i x_i \otimes x_i$ converges then $\sum_i |a_i| < +\infty$, implying $(x_i \otimes x_i)$ in $l^1 \otimes_\pi l^1$ is similar to (e_i) in l^1.

To show that (x_i) is not Besselian, define $a_1 = a_2 = a_3 = a_4 = \frac{1}{2}$, $a_5 = a_6 = \ldots = a_{20} = \frac{1}{4}$, and in general continue this pattern with every "block" of 2^{2n} numbers (a_j) each equal to $1/2^n$ $(n = 1, 2, \ldots)$. Then by definition of the basis (x_i) and the sequence (a_i) it follows that $\sum_{i=1}^{\infty} a_i x_i$ converges in l^1. However, it is clear from the definition of the set (a_i) that $\sum_{i=1}^{\infty} |a_i|^2 = +\infty$. Therefore (x_i) is not Besselian.

We will conclude this section with some results and examples concerning the tensor product $(x_i \otimes y_j)$ of Hilbertian and Besselian bases (x_i) and (y_i).

PROPOSITION 3.10. *Let (x_i) and (y_i) be bases for Banach spaces X and Y respectively. Then (x_i) and (y_i) are Hilbertian if and only if $(x_i \otimes y_j)$ is a Hilbertian basis for $X \otimes_\varepsilon Y$.*

Proof. If (x_i) and (y_i) are Hilbertian, then as in Lemma 3.1 there exist continuous linear mappings $T_1\colon l^2 \to X$ and $T_2\colon l^2 \to Y$ such that $T_1(e_i) = x_i$ and $T_2(e_i) = y_i$. It is well known that the tensor product mapping $T_1 \otimes T_2\colon l^2 \otimes_\varepsilon l^2 \to X \otimes_\varepsilon Y$ is continuous [13], and $T_1 \otimes T_2 (e_i \otimes e_j) = x_i \otimes y_j$.

Now $(e_i \otimes e_j)$ is an orthonormal basis for the Hilbert space $l^2 \otimes_\sigma l^2$ (where σ denotes the Hilbert-Schmidt crossnorm) and $\varepsilon \leqslant \sigma$ [13]. Hence if $\sum |a_{ij}|^2 < +\infty$, then $\sum a_{ij} e_i \otimes e_j \in l^2 \otimes_\sigma l^2$, implying $\sum a_{ij} e_i \otimes e_j \in l^2 \otimes_\varepsilon l^2$. By the continuity of $T_1 \otimes T_2$ it follows that $\sum a_{ij} x_i \otimes y_j$ is in $X \otimes_\varepsilon Y$ and $(x_i \otimes y_j)$ is Hilbertian.

The converse is trivial since any subsequence of a Hilbertian basis is Hilbertian.

COROLLARY 3.11. *Let (x_i) and (y_i) be bases for Banach spaces X and Y. Then (x_i) and (y_i) are Besselian if and only if the tensor product basis $(x_i \otimes y_j)$ for $X \otimes_\pi Y$ is Besselian.*

Proof. Let (f_i) and (g_i) denote the coefficient functionals associated with the bases (x_i) and (y_i) respectively. Then if (x_i) and (y_i) are Besselian, (f_i) and (g_i) are Hilbertian by Lemma 3.1. It follows from Proposition 3.10 that the basis $(f_i \otimes g_j)$ for $[f_i] \otimes_\varepsilon [g_i] \subset (X \otimes_\pi Y)^*$ is Hilbertian. Therefore by Lemma 3.1, $(x_i \otimes y_j, f_i \otimes g_j)$ is Besselian in $X \otimes_\pi Y$.

In general the tensor product $(x_i \otimes y_j)$ of Hilbertian bases (x_i) and (y_i) is not Hilbertian in $X \otimes_\pi Y$, even in the case where $x_i = y_i$ for all i. In fact, the unit vector basis (e_i) in l^2 is Hilbertian but $(e_i \otimes e_j)$ in $l^2 \otimes_\pi l^2$ is similar to (e_i) in l^1, a non-Hilbertian basis [5]. Since every subsequence of a Hilbertian basis is Hilbertian it follows that $(e_i \otimes e_j)$ in $l^2 \otimes_\pi l^2$ is not Hilbertian. Similarly, (e_i) in l^p is Besselian for $1 \leqslant p \leqslant 2$ but there exists $1 < p < 2$ such that $(e_i \otimes e_j)$ in $l^p \otimes_\varepsilon l^p$ is similar to (e_i) in l^4, a non-Besselian basis [5]. It follows that $(e_i \otimes e_j)$ in $l^p \otimes_\varepsilon l^p$ is not Besselian.

In contrast to these observations we have the following results.

PROPOSITION 3.12. *Let (e_i) denote the unit vector basis for l^1 (a Besselian basis). Then $(e_i \otimes e_j)$ in $l^1 \otimes_\varepsilon l^1$ is Besselian.*

Proof. Suppose $x = \sum a_{ij} e_i \otimes e_j$ converges in $l^1 \otimes_\varepsilon l^1$. Then $\left\| \sum_{i,j=1}^{N} a_{ij} e_i \otimes e_j \right\|_\varepsilon \leqslant K \|x\|$ for some $K \geqslant 1$ and all N. By definition then,

$$\sup_{\substack{\|f\| \leqslant 1 \\ \|g\| \leqslant 1 \\ f, g \in l^\infty}} \left| \sum_{i,j=1}^{N} a_{ij} f(e_i) g(e_j) \right| \leqslant K \|x\| \qquad (N = 1, 2, \ldots).$$

Now if $(s_i)_{i=1}^N$ and $(t_j)_{j=1}^N$ are sequences of real numbers for which $|s_i| \leqslant 1$ for all $1 \leqslant i \leqslant N$ and $|t_j| \leqslant 1$ for all $1 \leqslant j \leqslant N$, then set $f_0 = (s_i')_{i=1}^\infty$, $g_0 = (t_j')_{j=1}^\infty$, where

$$s_i' = \begin{Bmatrix} s_i & \text{if } 1 \leqslant i \leqslant N \\ 0 & \text{if } i > N \end{Bmatrix} \quad \text{and} \quad t_j' = \begin{Bmatrix} t_j & \text{if } 1 \leqslant j \leqslant N \\ 0 & \text{if } j > N \end{Bmatrix}.$$

Clearly f_0 and g_0 are elements of l^∞ and $\|f_0\| \leqslant 1$, $\|g_0\| \leqslant 1$. Hence by the above

$$\left| \sum_{i,j=1}^{N} a_{ij} f_0(e_i) g_0(e_j) \right| = \left| \sum_{i,j=1}^{N} a_{ij} s_i t_j \right| \leqslant K \|x\|.$$

It follows from Littlewood's inequality [7] that $\sum_j \left(\sum_i a_{ij}^2 \right)^{1/2} < +\infty$, and so certainly $\sum_j \left(\sum_i a_{ij}^2 \right) < +\infty$. By definition, then, $(e_i \otimes e_j)$ is Besselian in $l^1 \otimes_\varepsilon l^1$.

COROLLARY 3.13. *Let (e_i) denote the unit vector basis for c_0 (a Hilbertian basis). Then $(e_i \otimes e_j)$ in $c_0 \otimes_\pi c_0$ is Hilbertian.*

Proof. The coefficient functionals in l^1 associated with (e_i) in c_0 are (e_i). Since $l^1 \otimes_\varepsilon l^1$ is a closed subspace of $(c_0 \otimes_\pi c_0)^*$ and since by the above $(e_i \otimes e_j)$ in $l^1 \otimes_\varepsilon l^1$ is Besselian, it follows from Lemma 3.1 that $(e_i \otimes e_j)$ in $c_0 \otimes_\pi c_0$ is Hilbertian.

4. Semi-shrinking bases. In this section we will use the results of section 3 to construct a continuum of semi-shrinking bases, each of which is non-shrinking, conditional, and non-boundedly complete. The method illustrates the usefulness of tensor products for constructing examples and counter-examples in basis theory.

We will need the following result.

PROPOSITION 4.1. *Let (x_i) and (y_i) be semi-shrinking bases for X and Y respectively. Then the basis $(x_i \otimes y_j)$ for $X \otimes_\varepsilon Y$ is semi-shrinking.*

Proof. Let $g \in (X \otimes_\varepsilon Y)^*$. Then (as we mentioned in section 2) $g(w) = \int_{S \times T} w_0(x', y') \, d\mu(x', y')$, where S and T are equicontinuous subsets of X_σ^* and Y_σ^* respectively and w_0 is the restriction of the bilinear form w on $X^* \times Y^*$ to $S \times T$ (recall that every w in $X \otimes Y$ can be viewed as a bilinear form on $X^* \times Y^*$). Now $S \times T$ is a compact metric space and $x_i \otimes y_j(x', y') = x_i(x') y_j(y')$ for all (x', y') in $S \times T$. Since by assumption $x_i(x') \xrightarrow{i} 0$ and $y_j(y') \xrightarrow{j} 0$, we have $(x_i \otimes y_j)$ (as a sequence in $C(S \times T)$) converges pointwise to zero on $S \times T$ and hence converges weakly to zero in $C(S \times T)$. It follows that $g(x_i \otimes y_j) = \int_{S \times T} x_i \otimes y_j(x', y') \, d\mu(x', y') = \mu(x_i \otimes y_j) \to 0$, and $(x_i \otimes y_j)$ is semi-shrinking.

PROPOSITION 4.2. *Let (x_i) be a p-Hilbertian basis for X and (y_i) a q-Hilbertian basis for Y $(p^{-1}+q^{-1}=1)$. If either (x_i) or (y_i) is non-shrinking, then the basis $(x_i \otimes y_j)$ for $X \otimes_\varepsilon Y$ is semi-shrinking but non-shrinking and non-boundedly complete.*

Proof. By Lemma 3.2 each of (x_i) and (y_j) is semi-shrinking. Therefore by Proposition 4.1 the basis $(x_i \otimes y_j)$ for $X \otimes_\varepsilon Y$ is semi-shrinking. However, since one of (x_i) and (y_i) is not shrinking and every subsequence of a shrinking basis is shrinking, it easily follows that $(x_i \otimes y_j)$ cannot be shrinking. Finally, the diagonal $(x_i \otimes y_i)$ is similar to (e_i) in c_0 (a non-boundedly complete basis) by Theorem 3.3 and it follows that $(x_i \otimes y_j)$ is not boundedly complete.

PROPOSITION 4.3. *Let (y_i) denote the basis for the space (d) of Davis and Dean* [2]. *Then (y_i) is q-Hilbertian for every $1 < q < +\infty$.*

Proof. Suppose $\sum |a_i|^q < +\infty$. By definition of the norm in (d),

$$\Big\|\sum_{i=m}^{n} a_i y_i\Big\| = \sup_{p\in\mathscr{P}} \sum_{i=m}^{n} \frac{|a_{p(i)}|}{i} \leqslant \sup_{p\in\mathscr{P}} \Big[\sum_{i=m}^{n} |a_{p(i)}|^q\Big]^{1/q} \Big[\sum_{i=m}^{n} \frac{1}{i^p}\Big]^{1/p}.$$

This last tends to zero as $m, n \to \infty$, implying $\sum_i a_i y_i \in (d)$ and (y_i) is q-Hilbertian.

We can now state and prove the main result of this section.

THEOREM 4.4. *There exists a continuum of mutually non-similar bases each of which is semi-shrinking, non-shrinking, conditional, and non-boundedly complete.*

Proof. Let (e_i) denote the unit vector basis for l^p $(1 < p < +\infty)$ and (y_i) the basis for the space (d) mentioned in Proposition 4.3. Then the basis $(e_i \otimes y_j)$ for $l^p \otimes_\varepsilon (d)$ has the property that it is semi-shrinking, non-shrinking, and non-boundedly complete (Propositions 4.2 and 4.3). Clearly if $p_1 \neq p_2$ the bases $(e_i \otimes y_j)$ in $l^{p_1} \otimes_\varepsilon (d)$ and in $l^{p_2} \otimes_\varepsilon (d)$ are non-similar (if not, $(e_i \otimes y_1)$ in $l^{p_1} \otimes_\varepsilon (d)$ would be similar to $(e_i \otimes y_1)$ in $l^{p_2} \otimes_\varepsilon (d)$, implying l^{p_1} and l^{p_2} are isomorphic, a contradiction for $p_1 \neq p_2$).

For convenience, let $X_p = l^p \otimes_\varepsilon (d)$ and let $(z_i^{(p)})$ denote the basis $(e_i \otimes y_j)$ for X_p. It is well known that for each $1 < p < +\infty$ the space l^p has a conditional basis $(w_i^{(p)})$ [9] which is shrinking [6]. Hence by our results the basis $(w_i^{(p)} \otimes z_j^{(p)})$ in $l^p \otimes_\varepsilon X_p$ is semi-shrinking, non-shrinking, non-boundedly complete, and conditional $(1 < p < +\infty)$. By the same argument as the above it is clear that if $p_1 \neq p_2$, the bases $(w_i^{(p_1)} \otimes z_j^{(p_1)})$ and $(w_i^{(p_2)} \otimes z_j^{(p_2)})$ are not similar.

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