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Superharmonic functions on Lipschitz domains *

by

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§ 1. Introduction. Examples due to E. Tolsted (see [5]) and A. Zygmund show there are positive superharmonic functions $u(z)$ in the unit disc D: $|z|<1$ which have nontangential limits almost nowhere on $|z|=1$. On the other hand, Littlewood [3] shows such u have radial limits almost everywhere. In [5], Tolsted extends Littlewood's result as follows. Let C be any continuous curve in D with one endpoint on $|z|=1$ at which C is tangent to a chord of D, and let $C(\theta)$ denote C rotated about the origin through an angle θ. Then $u(z)$ has a finite limit as z approaches $|z|=1$ along $C(\theta)$ for almost all θ.

Recently, L. Ziomek [6] proved the following result. Let $\Gamma_\delta(\theta) = \Gamma_\delta(\theta,\nu)$, $0<\nu<\pi$, denote the cone in D with height δ and aperture ν whose vertex is at $e^{i\theta}$ and whose axis is along the radius from 0 to $e^{i\theta}$. If $u(z)$ is positive and superharmonic in D there is a function $u(\theta)$ on the boundary such that

$$\lim_{\delta\to 0}\frac{1}{|\Gamma_\delta(\theta)|}\int\limits_{\Gamma_\delta(\theta)}|u(z)-u(\theta)|^p\,dz = 0$$

for almost all θ for every $0<\nu<\pi$ and $1\leqslant p<\infty$.

This result, and the others above, are actually true for superharmonic functions u which are not necessarily positive but whose negative part is L^1 bounded, that is,

$$\int\limits_0^{2\pi} u^-(re^{i\theta})\,d\theta \leqslant M<\infty \quad \text{for} \quad r<1,$$

where $u^- = \operatorname{Max}(-u, 0)$. This condition is sufficient to permit the Riesz decomposition of u into the sum of an L^1 bounded harmonic function and a (Green's) potential. Since the boundary behavior of L^1 bounded

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harmonic functions is known, it is enough to consider the potential part, which is positive and superharmonic.

Ziomek uses conformal mapping to extend the result to very general domains D of the plane. He then proves an analogous result when D is the unit ball in Euclidean n-space E^n. In this case, the p range is limited to $1 \leqslant p < \frac{n}{n-2}$, and he shows the result is false for $p \geqslant \frac{n}{n-2}$.

The purpose of this note is to extend Ziomek's result to more general domains $D \subset E^n$. Specifically, we will assume D is a *Lipschitz domain* — that is, a bounded open subset of E^n whose boundary ∂D is described locally by a function of class Lip 1 (see [2]). If $Q \in \partial D$, a truncated (open) cone Γ with vertex Q is called a *nontangential cone at* Q if there is another cone Γ' such that $\bar{\Gamma} - \{Q\} \subset \Gamma' \subset D$. If $\Gamma(Q)$ is any nontangential cone at Q, $\Gamma_\delta(Q)$ denotes $\Gamma(Q)$ truncated at the height δ. Following Ziomek, we say $u(P)$ has *nontangential limit* λ *in* L^p *at* Q if

$$\lim_{\delta\to 0} \frac{1}{|\Gamma_\delta(Q)|} \int_{\Gamma_\delta(Q)} |u(P)-\lambda|^p\, dP = 0$$

for every nontangential cone $\Gamma(Q)$ at Q.

Our main result is the following theorem.

THEOREM 1. *Let $u(P)$ be positive and superharmonic in a Lipschitz domain $D \subset E^n$. Then for $1 \leqslant p < n/(n-2)$, u has a nontangential limit in L^p at each $Q \in \partial D$ except for a set of harmonic measure zero.*

It will follow from the proof of the theorem that if $u = h+w$ is the Riesz decomposition of u into the sum of a positive harmonic function h and a potential w, then the nontangential boundary values in L^p of u coincide except in a set of harmonic measure zero with the nontangential boundary values (in L^∞) of h, that is, w has nontangential L^p limit equal to zero almost everywhere on ∂D.

We remark that just as when D is a ball, so also in this general case Theorem 1 is valid for superharmonic functions whose negative part is L^1 bounded — this being defined as for the ball except that averaging over interior spheres is replaced by integration with respect to harmonic measure over the boundaries of interior domains.

Our proof of Theorem 1 will depend heavily on results and terminology from [1] and [2], and to save space we shall often refer the reader to the necessary facts rather than recall them here. We will give two proofs of the theorem. The first and simplest of them, intended for the reader who is familiar with general potential theory, shows that the existence of such a limit is a corollary of the existence of a finite fine limit almost everywhere at the Martin boundary of D (see [4]), and the identification of the Martin boundary of D with its Euclidean boundary ∂D for D Lipschitz (see [2]). The second proof is the classical Lebesgue point proof, using estimates from [1] and [2]. The two methods are of course not independent.

Lemma 2 of § 2 contains the main inequality that is needed in proving Theorem 1. § 3 contains the proofs of Theorem 1 itself.

§ 2. An inequality for superharmonic functions.

LEMMA 1. *Let $g(M,P)$ be the Green's function of the unit ball B_1 of E^n. Then for $1 \leqslant p < n/(n-2)$,*

$$\sup_{M\in B_1} \int_{B_1} \{g(M,P)\}^p\, dP \leqslant c^p$$

where c depends only on p and n.

Proof. For $n > 2$,

$$\int_{B_1} \{g(M,P)\}^p\, dP \leqslant \int_{B_1} \frac{dP}{|M-P|}(n-2)p \leqslant c \int r^{n-1} r^{(2-n)p}\, dr,$$

which is finite provided $p < n/(n-2)$. When $n = 2$, we have in the same way

$$\int_{B_1} \{g(M,P)\}^p\, dP \leqslant c \int_0^1 r \left\{\log\frac{1}{r}\right\}^p dr,$$

which is finite for all $p < \infty$.

LEMMA 2. *Let $u(P)$ be positive and superharmonic in the unit ball B_1 of E^n. If for some $1 \leqslant p < n/(n-2)$,*

$$(2.1) \qquad \int_{B_\varrho} \{u(P)\}^p\, dP \geqslant A^p > 0,$$

then $u(P) \geqslant cA$ for all $P \in B_\varrho$, the ball with radius $\varrho < 1$, where c is a constant depending only on ϱ, p and n.

Proof. By the Riesz decomposition, $u(P) = h_1(P) + w_1(P)$, $P \in B_1$, where h_1 is positive and harmonic in B_1 and

$$w_1(P) = \int_{B_1} g(M,P)\, d\mu(M),$$

with $\int_{B_1} (1-|M|)\, d\mu(M) < \infty$. Splitting $B_1 = (B_1 - B_r) \cup B_r$, $r = (\varrho+1)/2$, we have $u(P) = h(P) + w(P)$ where h is positive and harmonic in B_r and

$$w(P) = \int_{B_r} g(M,P)\, d\mu(M).$$

It follows from (2.1) that either

$$\int_{B_\varrho} \{h(P)\}^p dP \geqslant \left(\frac{A}{2}\right)^p$$

or

$$\int_{B_\varrho} \{w(P)\}^p dP \geqslant \left(\frac{A}{2}\right)^p.$$

In the first case, there is a point $P_0 \in B_\varrho$ such that $h(P_0) \geqslant c\,\frac{A}{2}$, where c is the pth root of the volume of B_ϱ, and the lemma follows immediately from Harnack's inequality. In the second case, we have by Lemma 1

$$\frac{A}{2} \leqslant \Big(\int_{B_\varrho}\{w(P)\}^p dP\Big)^{1/p} \leqslant \int_{B_r}\Big(\int_{B_\varrho}\{g(M,P)\}^p dP\Big)^{1/p} d\mu(M) \leqslant c\int_{B_r} d\mu(M).$$

However, $g(M,P)$ is bounded below by a positive constant for $M \in B_r$ and $P \in B_\varrho$. Thus

$$\frac{A}{2} \leqslant c\int_{B_r} g(M,P)\,d\mu(M) = cw(P), \qquad P \in B_\varrho,$$

and again the lemma follows.

§ 3. Proof of Theorem 1. If $u(P)$ is positive and superharmonic in a Lipschitz domain D, we decompose $u = h+w$ where h is positive and harmonic in D and w is a Green's potential. Since h has an ordinary nontangential limit on ∂D except for a subset of harmonic measure zero (see [1]), it also has one in L^p for $p < \infty$. Hence it suffices to show that the potential part w has nontangential L^p limit zero except for a set of harmonic measure zero.

Given a sequence of points $P_k \in D$ which approach $Q_0 \in \partial D$ nontangentially, choose $r_0 > 0$ such that the closure of each ball $B'_k = B(P_k, r_0|P_k - Q_0|)$ with center P_k and radius $r_0|P_k-Q_0|$ lies in D. For $0 < r < r_0$, let $B_k = B(P_k, r|P_k - Q_0|)$. A simple covering argument shows that w has nontangential L^p limit zero at Q_0 if and only if

$$\frac{1}{|B_k|}\int_{B_k}\{w(P)\}^p dP \to 0 \tag{3.1}$$

as $k \to \infty$ for every such sequence of nontangential balls B_k at Q_0.

If we use the facts that w has fine limit zero almost everywhere on the Martin boundary of D (see [4], Theorem 21) and that the Martin boundary of D coincides with ∂D for Lipschitz domains (see [2]), (3.1) follows from Lemma 2. For if

$$\frac{1}{|B_k|}\int_{B_k}\{w(P)\}^p dP \geqslant \lambda > 0$$

for all k and some $1 < p < n/(n-2)$ then $w(P) \geqslant \lambda' > 0$ for $p \in \bigcup_k B_k$. By Lemma (5.3) of [2], $\bigcup_k B_k$ is not (semi) thin at Q_0 and so w cannot have fine limit zero at Q_0.

We can also give a Lebesgue point proof for Theorem 1. Restricting our attention to a small portion of ∂D, we may assume without loss of generality that D is starlike about a point P_0 (see Lemma 1 of § 5 of [1]), that w is finite at P_0 and that $P_0 = 0$. If $G(M,P)$ is the Green's function for D and D_r, $0 < r < 1$, is the contraction $\{rP\colon P \in D\}$ of D, we may write

$$w(P) = \int_D G(M,P)\,d\mu(M) = \int_{D_r} + \int_{D-D_r}$$

with

$$w(P_0) = \int_D G(M,P_0)\,d\mu(M) = \int_D d\nu(M) < \infty.$$

For the first integral on the right above we have

$$\lim_{p\to\partial D}\int_{D_r} G(M,P)\,d\mu(M) = 0.$$

This is true since $G(M,P) \to 0$ as $P \to \partial D$ (unrestrictedly) and $G(M,P)$ is uniformly bounded for $M \in D_r$ as $P \to \partial D$. Moreover, $\mu(D_r) < \infty$ since $G(M,P_0)$ has a positive lower bound for $M \in D_r$, and so the assertion follows from the Lebesgue dominated convergence theorem.

Since $\nu(D - D_r) \to 0$ as $r \to 1$, we need only consider the case

$$w(P) = \int_D \frac{G(M,P)}{G(M,P_0)}\,d\nu(M) \tag{3.2}$$

with $\nu(D)$ arbitrarily small.

Let (X, y), $X \in E^{n-1}$, y real, be the local coordinate system with center $Q_0 = (X_0, y_0) \in \partial D$ and positive y-axis along the ray Q_0P_0. Following [2], we will denote by $\psi(Q_0, r)$ the cylinder $\{(X,y)\colon |X - X_0| < r, |y - y_0| < rs\}$ with s sufficiently large to insure that the top of the cylinder is in D and the bottom is in the complement of D for all small $r > 0$. We write $\Delta(Q_0, r) = \partial D \cap \psi(Q_0, r)$ and call Δ a disc.

The measure ν on D induces a measure on ∂D also denoted by ν as follows. For $E \subset \partial D$, let E^* be the subset of D of all points rQ, $0 \leqslant r < 1$, $Q \in E$, and set $\nu(E) = \nu(E^*)$. If we form the Lebesgue decomposition of ν with respect to harmonic measure ω^{P_0} at P_0, $d\nu(Q) = f(Q)\,d\omega^{P_0}(Q) + d\sigma(Q)$ with f integrable and σ singular with respect to ω^{P_0}, then by Besicovitch's theorem on the differentiation of set functions (see [2]),

$$\frac{\nu(\Delta)}{\omega^{P_0}(\Delta)} \to f(Q)$$

as $\Delta(Q, r)$ shrinks to Q for almost all $(d\omega^{P_0})$ $Q \in \partial D$. Since $\nu(\partial D)$ can be chosen arbitrarily small in advance, the same is true of the L^1 norm of f with respect to $d\omega^{P_0}$. Chebyshev's inequality then implies that given $\varepsilon > 0$ there is, except for a set of points Q_0 with harmonic measure less than ε, a $\delta > 0$ depending on Q_0 such that $\nu(\Delta) \leqslant \varepsilon\omega^{P_0}(\Delta)$ for all $\Delta = \Delta(Q_0, r) \subset \Delta(Q_0, \delta)$.

Let $Q_0 \in \partial D$ be such a point and let P_k be a sequence of points of a standard nontangential cone at Q_0 which converge to Q_0. With B_k' and B_k as before,

$$\left(\frac{1}{|B_k|}\int_{B_k}\{w(P)\}^p\,dP\right)^{1/p} \leqslant \int_D \left(\frac{1}{|B_k|}\int_{B_k}\{G(M, P)\}^p\,dP\right)^{1/p} d\mu(M).$$

By Lemma 2 therefore,

$$\left(\frac{1}{|B_k|}\int_{B_k}\{w(P)\}^p\,dP\right)^{1/p} \leqslant c\int_D \{\inf_{P \in B_k} G(M, P)\}\,d\mu(M)$$

$$= c\int_D \left\{\inf_{P \in B_k}\frac{G(M, P)}{G(M, P_0)}\right\} d\nu(M).$$

We now estimate $\inf\limits_{P \in B_k}\dfrac{G(M, P)}{G(M, P_0)}$, considering several cases.

Case 1. $M \in B_k'$. For any fixed $M \in B_k'$, choose a point $\bar{P}_k \in B_k$ such that $|\bar{P}_k - M| > r_k$ where $r_k = r|Q_0 - P_k|$ is the radius of B_k. Since $G(M, P)$ is harmonic in each variable for $M \neq P$ and is positive, we may assume by Harnack's inequality that M is inside $\psi(Q_0, r_k)$ and $\bar{P}_k$ is the point in the center of the top of $\psi(Q_0, 2r_k)$. Applying Lemma (3.1) of [2] to the function $G(M, P)$ harmonic in $D - \psi(Q_0, r_k)$, taking $P = P_0 = 0$ in this inequality and using (3.8) of [1], we obtain

$$\frac{G(M, \bar{P}_k)}{G(M, P_0)} \leqslant \frac{c}{\omega^{P_0}(\Delta)},$$

and therefore

$$\inf_{P \in B_k}\frac{G(M, P)}{G(M, P_0)} \leqslant \frac{c}{\omega^{P_0}(\Delta)} \quad \text{for} \quad M \in B_k',$$

where $\Delta = \Delta(Q_0, |Q_0 - P_k|)$.

Case 2. $M \in \psi(Q_0, |Q_0 - P_k|) - B_k'$. If we choose $P = P_k$, the center of B_k, then the geometry of this situation is the same as in case 1, and we obtain the same estimate.

Case 3. M in the ring $\psi(Q_0, 2^j|Q_0 - P_k|) - \psi(Q_0, 2^{j-1}|Q_0 - P_k|)$, $j = 2, 3, \ldots, N$ where N is chosen so that $2^{N-1}|Q_0 - P_k| < \delta \leqslant 2^N|Q_0 - P_k|$. We again choose $P = P_k$, the center of B_k. Using Harnack's inequality, we may assume that M is in the part of the ring near ∂D of points whose distance from ∂D is at most a fixed constant multiple of $2^j|Q_0 - P_k|$. If we apply (2.4) of [2] to $G(M, P)$ as a function of P and put $P = P_k$ and then apply Lemma (3.1) of [2] to $G(M, P)$ and put $P = P_0$, it follows from the proof of Lemma 4 of [1] that

$$\frac{G(M, P_k)}{G(M, P_0)} \leqslant \frac{cc_j}{\omega^{P_0}(\Delta_j)},$$

where $\sum\limits_j c_j < \infty$ and $\Delta_j = \Delta(Q_0, 2^j|Q_0 - P_k|)$. Hence

$$\inf_{P \in B_k}\frac{G(M, P)}{G(M, P_0)} \leqslant \frac{cc_j}{\omega^{P_0}(\Delta_j)}$$

for $M \in \psi_j - \psi_{j-1}$, where $\psi_j = \psi(Q_0, 2^j|Q_0 - P_k|)$, $\Delta_j = \Delta(Q_0, 2^j|Q_0 - P_k|)$ and $\sum c_j < \infty$.

Case 4. $M \in D - \psi(Q_0, \delta)$. Using Harnack's inequality, applying (2.4) of [2] to $G(M, P)$ as a function of P and putting $P = P_k$, the center of B_k, we obtain

$$\inf_{P \in B_k}\frac{G(M, P)}{G(M, P_0)} \to 0$$

uniformly for $M \in D - \psi(Q_0, \delta)$ as $k \to \infty$.

Combining these estimates, we have

$$\int_D \left\{\inf_{P \in B_k}\frac{G(M, P)}{G(M, P_0)}\right\} d\nu(M) \leqslant \sum_{j=1}^{N}\int_{\psi_j - \psi_{j-1}} + \int_{D - \psi(Q_0, \delta)}$$

$$\leqslant c\sum_{j=1}^{N}\frac{c_j}{\omega^{P_0}(\Delta_j)}\nu(\psi_j) + o(1)\int_D d\nu(P)$$

$$\leqslant c\varepsilon\sum_{1}^{\infty} c_j + o(1).$$

It follows that

$$\limsup_{k\to\infty}\left(\frac{1}{|B_k|}\int_{B_k}\{w(P)\}^p\,dP\right)^{1/p}\leqslant c\,\varepsilon$$

except for a set of points Q_0 of harmonic measure at most ε. Since the constant c depends only on the nontangential cone at Q_0 and not on Q_0 itself, it is a simple matter to complete the proof of Theorem 1.

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A ring of analytic functions, II *

by

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This paper is devoted to a further study of the commutative locally convex algebra A of all complex-valued functions which are defined and holomorphic in the open unit disc U of the complex plane, where the ring multiplication is convolution (the Hadamard product), the other operations are the usual ones, and the topology is the compact-open. Specifically, we investigate the spectra of elements of this algebra and the operations of inversion and exponentiation.

In Section 2 we give simple proofs of the two main results of [2] whose proofs (in [2]) depended on an incorrect theorem (2.3 of [2]).

Section 3 consists of one theorem which gives a means for relating convergence in the compact convergence topology of A to certain properties of the corresponding sequences of Maclaurin coefficients.

Section 4 is concerned with the spectrum of an element of A. The algebra A may be identified algebraically with a certain subalgebra $\hat{A}$ of $C(N)$, since the space N of non-negative integers (usual topology) is in a natural way homeomorphic to the space of non-zero continuous homomorphisms of A onto C (with the usual Gelfand topology). For $x\in A$, we let $\hat{x}$ be the corresponding element of $C(N)$. Then $\hat{x}(n)=x_n$, the nth Maclaurin coefficient of the holomorphic function x. We show that the spectrum $\sigma(x)$ of x is between the range $R(\hat{x})$ of $\hat{x}$ and its closure and give examples to show that in general one cannot say any more about $\sigma(x)$. Since A is not locally m-convex, the functional calculus developed for such algebras by Michael is useless here and one is induced to look at a spectrum defined for general locally convex algebras by Allan (and others) for which there is an applicable functional calculus. We identity first the set of (Allan) bounded elements of A, and show that $\mathrm{Sp}(x) = R(\hat{x})^*$, where $\mathrm{Sp}(x)$ is the spectrum defined in terms of having or not having an (Allan) bounded inverse and "*" indicates the closure in the Riemann sphere. Thus, $\mathrm{Sp}(x)$ is easily computable, once $R(\hat{x})$ is known, whereas $\sigma(x)$ is not.

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