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Reçu par la Rédaction le 1. 7. 1970

STUDIA MATHEMATICA, T. XXXIX. (1971)

Differentiation in locally convex spaces

by

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In this paper we shall present an approach to a theory of differentiation in a few important classes of locally convex spaces. Differentiation in Banach spaces is one of the most useful tools of non-linear analysis (cf. [7], [8] and [10]). Our main goal is a natural generalization of this theory to a greater class of locally convex spaces. We give one of the possible realizations of this program. The other approaches can be found in reviews [1], [2], [15] and papers [3], [4], [9], [11], [14], [18] and [20]. The main idea of the differentiation is an approximation of a given map by a linear map. This approximation can be defined in many ways. The definition which is used in this paper can be found also in [14] and [18]. It is valid for an arbitrary locally convex space. However, it is not possible in the general case to obtain the mean value theorem and some results connected with it. Therefore, we consider only two classes of locally convex spaces: metrizable, quasi-normable spaces and *DF-S* spaces (*DF* spaces which are also Schwartz spaces).

We obtain the mean value theorem, the Taylor formula with the estimation of the remainder, and theorems on partial differentiability. The class of spaces which we consider in this paper includes normable spaces, Fréchet–Schwartz spaces and their duals (e. g. $\mathscr{S}$, $\mathscr{S}'$, $\mathscr{E}$, $\mathscr{E}'$).

The results of this paper are very close to those of paper [16], which dealt with a different model of the theory for *F-S* spaces.

1. Notation. We shall consider topological locally convex spaces over the field of real or complex numbers, called in this paper *locally convex spaces*. We shall assume the Hausdorf axiom. Let E be a locally convex space; $\mathscr{N}(E)$ denotes the set of all closed, absolutely convex neighbourhoods of zero in E. If $U\in\mathscr{N}(E)$, then $\|\cdot\|_U$ is the seminorm generated by U and $E_U := E/N(U)$, where $N(U) = \{e\in E;\ \|e\|_U = 0\}$. The symbol $\hat{E}$ will stand for the completion of E. $\mathscr{B}(E)$ will denote the set of all closed, absolutely convex bounded sets of E.

If E is a locally convex space, then E'_s (E'_b) is the dual space to E endowed with the weak (strong) topology. Let F be a locally convex space; then $L(E,F)$ ($L(E,E;F)$, $L^n(E,F)$) denotes the space of linear (respectively: bilinear, n-linear) continuous maps from E (resp. $E\times E$, $E\times\ldots\times E$) to F. These spaces, endowed with the simple convergence topology (weak) or the bounded convergence topology (strong), will be denoted by: $L_s(E,F)$, $L_s(E,E;F)$, $L_s^n(E,F)$, $L_b(E,F)$, $L_b(E,E;F)$, $L_b^n(E,F)$, respectively.

2. Theory of differentiation in locally convex spaces.

Definition. We say that a map T from an open set Ω of a locally convex space E to a locally convex space F is *Gâteaux-differentiable at a point* $e_0\in\Omega$ if there exists a $\nabla T(e_0)\in L(E,F)$ such that the map

$$E\ni h\to r(e_0,h):=T(e_0+h)-T(e_0)-\nabla T(e_0)h\in F$$

has the following property:

$$\lim_{t\to 0}\left(\frac{1}{t}r(e_0,\mathrm{th})\right)=0,\qquad t\in\boldsymbol{R}\text{ or }t\in\boldsymbol{C}.$$

This definition is the same as in the case of normed spaces (cf [13]). We take the following definition of Fréchet differentiability (cf [2], [14], [18]):

Definition. Let T be a mapping from an open set $\Omega\subset E$ to F. We say that T is *Fréchet-differentiable at a point* $e_0\in\Omega$ if there exists a mapping $L\in L(E,F)$ such that the map

$$E\ni h\to r(e_0,h):=T(e_0+h)-T(e_0)-Lh\in F$$

has the following property: for every $V\in\mathcal{N}(F)$ there exists a $U\in\mathcal{N}(E)$ such that for every Moore–Smith sequence $\{h_\lambda\}_{\lambda\in\Lambda}$ convergent to zero in E we have

$$\lim_{\lambda\in\Lambda}\frac{\|r(e_0,h_\lambda)\|_V}{\|h\|_U}=0.$$

The mapping L is called the *Fréchet derivative of T at e_0* and is denoted by $T'(e_0)$ or $DT(e_0)$. The derivative of T — if it exists — is determinated uniquely.

PROPOSITION 1.

1° *Let T be a Fréchet-differentiable mapping at e_0; then T is also Gâteaux-differentiable at e_0 and $DT(e_0)=\nabla T(e_0)$.*

2° *If T is Fréchet-differentiable at e_0, then T is continuous at e_0.*

3° *A linear combination of mappings differentiable at e_0 is differentiable at e_0 and its derivative is a linear combination of derivatives.*

4° *The superposition rule holds:* $(T_2\circ T_1)'(e_0)=T_2'(T_1(e_0))\circ T'_1(e_0)$.

PROPOSITION 2. *Let E be a metrizable locally convex space and F an arbitrary locally convex space. Let $\Omega\subset E$ be an open set and let f be a mapping from $\Omega\times E$ to F satisfying the following conditions:*

1° *for every $e\in\Omega$ $f(e,\cdot)\in L(E,F)$,*

2° *the mapping $\Omega\ni e\to f(e,\cdot)\in L_b(E,F)$ is continuous at $e_0\in\Omega$.*

Then f is continuous at every point (e_0,h), $h\in E$.

Proof. We shall prove that f is continuous at the point $(e_0,0)$. Let $(e_n,h_n)\xrightarrow[n\to\infty]{}(e_0,0)$. If $f(e_n,h_n)\nrightarrow 0$, then there exists a $V\in\mathcal{N}(E)$ such that $f(e_{n_k},h_{n_k})\notin V$ for a certain subsequence $(n_k)_{k=1}^\infty$ (*). It follows from 2° that $f(e_n,\cdot)\to f(e_0,\cdot)$ in $L_b(E,F)$; hence, the set $\{f(e_n,\cdot)\}_{n=1}^\infty$ is bounded in $L_b(E,F)$. Since E is a quasi-barrelled space, this set is equicontinuous. Thus there exists a $U\in\mathcal{N}(E)$ such that $f(e_n,u)\in V$ for $u\in U$, $n\in N$. But this contradicts (*). q. e. d.

Definition (cf. [12]). A locally convex space E is called *quasi-normable* if for every equicontinuous set $A\subset E'$ there exists a $V\in\mathcal{N}(E)$ such that the topology induced in A by the topology of E_b' is equivalent to the topology of uniform convergence on V.

LEMMA 1 (cf. [12]). *A locally convex space E is quasi-normable if and only if for every $U\in\mathcal{N}(E)$ there exists a $V\in\mathcal{N}(E)$ such that for every $\lambda>0$ there exists a bounded set B_λ such that $V\subset\lambda U+B_\lambda$.*

The class of quasi-normable spaces includes all normable and all Schwartz space (cf. [12]).

Lemma 1 implies

LEMMA 2. *Let E be a quasi-normable space, F a normable space and A an equicontinuous set in $L(E,F)$. Then there exists a $V\in\mathcal{N}(E)$ such that the topology in A induced by the topology of $L_b(E,F)$ is equivalent to the topology of uniform convergence on V.*

Definition. Let T be a mapping from an open set $\Omega\subset E$ to F. We say that T is *continuously differentiable at* $e_0\in\Omega$ if T is diferentiable in some neighbourhood $\mathcal{O}$ of e_0 and the mapping $E\supset\mathcal{O}\ni e\to T'(e)\in L_b(E,F)$ is continuous at e_0.

PROPOSITION 3. *Let F be a metrizable locally convex space and let E, G be arbitrary locally convex spaces. If T_1 is a mapping from $\Omega_1\subset E$ ($e_0\in\Omega_1$) to F continuously differentiable at e_0, and T_2 is a mapping from $\Omega_2\subset F$ ($T_1(e_0)\in\Omega_2$) to G continuously differentiable at $T_1(e_0)$, then the mapping $T_2\circ T_1$ is continuously differentiable at e_0.*

The proof follows from Proposition 2.

THEOREM 1. *Let E be a quasi-normable metrizable space and F a locally convex space. Let T be a mapping from an open set $\Omega\subset E$ to F continuously differentiable at some neighbourhood of $e_0\in\Omega$. Then for every $V\in\mathcal{N}(F)$ there* *exist* $U, W \in \mathcal{N}(E)$ *such that for every* $e_1 \in e_0 + U$

$$\sup_{h \in W} \|T'(e)h - T'(e_1)h\|_V \xrightarrow[e \to e_1]{} 0.$$

Proof. Let $V \in \mathcal{N}(F)$. Applying Proposition 2 to the mapping $(e, h) \to T'(e)h \in F$, we find, that there exists a $U_1 \in \mathcal{N}(E)$ such that the set $A := \{T'(e_0 + u_1)\}_{u_1 \in U_1} \subset L(E, F_V)$ is a set of equicontinuous mappings. Let $U \in \mathcal{N}(E)$ so that $U + U \subset U_1$ and let $e \to e_1 \in e_0 + U$; then $T'(e) \to T'(e_1)$ in $L_b(E, F_V)$. We infer from Lemma 2 that there exists $W \in \mathcal{N}(E)$ such that for A the $L_b(E, F_V)$-convergence is equivalent to the uniform convergence on W. q. e. d.

We can express this theorem in a weaker form.

THEOREM 1'. *Let E be a metrizable quasi-normable space and F a locally convex space. Let T be a mapping from $\Omega \subset E$ to F continuously differentiable at $e_0 \in \Omega$. Then for every $V \in \mathcal{N}(F)$ there exists a $W \in \mathcal{N}(E)$ such that*

$$\sup_{h \in W} \|T'(e)h - T'(e_0)h\|_V \xrightarrow[e \to e_0]{} 0.$$

THEOREM 2 (Mean Value Theorem). *If the hypothesis of Theorem 1 is fulfilled, then for every $V \in \mathcal{N}(F)$ there exist $U, W \in \mathcal{N}(E)$ such that, for every $e_1 \in e_0 + U$ and $h \in W$,*

$$\|T(e_1 + h) - T(e_1)\|_V \leqslant C\|h\|_W,$$

where

$$C := \sup_{k, s \in W} \|T'(e_1 + k)s\|_V < \infty.$$

Proof. Applying the mean value theorem (cf. [8]) to the mapping $[0, 1] \ni t \to T(e_1 + th) \in F_V$, we have

$$\|T(e_1 + h) - T(e_1)\|_V \leqslant \sup_{0 \leqslant \theta \leqslant 1} \|T'(e_1 + \theta h)h\|_V \leqslant \sup_{k, s \in W} \|T'(e_1 + k)s\|_V \|h\|_W$$

for every $W \in \mathcal{N}(E)$ and every $h \in W$.

By Proposition 2 there exist $M < \infty$, $U_1, W_1 \in \mathcal{N}(E)$ such that, for every $e \in e_0 + U_1$, $\|T'(e)s\|_V \leqslant M\|s\|_{W_1}$. Let $U, W \in \mathcal{N}(E)$ that $U + U \subset U_1$, $W \subset W_1 \cap U$; then, for every $e_1 \in e_0 + U$, $\sup_{K, s \in W} \|T'(e_1 + k)s\|_V \leqslant M$. q. e. d.

Remark. In this proof we make no use of the quasi-normability of E.

THEOREM 3. *Let the hypothesis of Theorem 1 be fulfilled; then for every $V \in \mathcal{N}(F)$ there exist $U, W \in \mathcal{N}(E)$ such that, for every $e_1 \in e_0 + U$, $Y \in \mathcal{N}(E)$ and $h \in Y$, we have $\|r(e_1, h)\|_V \leqslant C(e_1, Y)\|h\|_W$, where*

$$(*) \quad C(e_1, Y) := \sup_{k \in Y, s \in W} \|T'(e_1 + k)s - T'(e_1)s\|_V \quad \textit{and} \quad \lim_{Y \in N(E} C(e_1, Y) = 0$$

Proof. The derivative of the mapping $E \ni e \to g(e) := T(e) - T'(e_1)e$ is $g'(e) = T'(e) - T'(e_1)$. Applying Theorem 2 to g, we have

$$\|r(e_1, h)\|_V = \|g(e_1 + h) - g(e_1)\|_V \leqslant \sup_{k \in Y, s \in W} \|T'(e_1 + k)s - T'(e_1)s\|_V \|h\|_W.$$

Formula $(*)$ folows immediately from Theorem 1. q. e. d.

THEOREM 3'. *Let E be a metrizable quasi-normable space, F a locally convex space, and $T: E \supset \Omega \to F$ a mapping continuously differentiable at $e_0 \in \Omega$. Then for every $V \in \mathcal{N}(F)$ there exists a $W \in \mathcal{N}(E)$ such that, for every $Y \in \mathcal{N}(E)$ and $h \in Y$, we have $\|r(e_0, h)\|_V \leqslant C(Y)\|h\|_W$, where*

$$C(Y) := \sup_{k \in Y, s \in W} \|T'(e_0 + k)s - T'(e_0)s\|_V \quad \textit{and} \quad \lim_{Y \in N(E)} C(Y) = 0.$$

This theorem follows from Theorem 1' and Lemma 2. Let us notice that Theorem 3' is valid if we replace the Fréchet differentiability by the Gâteaux differentiability. Thus we have

THEOREM 4. *Let E be a metrizable quasi-normable space, F a locally convex space and $T: E \supset \Omega \to F$ a mapping continuously Gâteaux-differentiable at $e_0 \in \Omega$. Then T is Fréchet-differentiable at e_0 and $\nabla T(e_0) = T'(e_0)$.*

From Proposition 2 we obtain

LEMMA 3. *If E is a metrizable locally convex space and F a locally convex space, then the spaces $L_b(E, L_b(E, F))$ and $L_b(E, E; F)$ are canonically isomorphic.*

Similarly, $L_b^n(E; F) \cong L_b(E, L_b(\ldots L_b(E, F)))$.

PROPOSITION 4. *Let E be a metrizable locally convex space and F a locally convex space. Let f be a mapping from $\Omega \times \underbrace{E \times \ldots \times E}_{n}$ to F (Ω an open subset of E) which satisfies the following conditions:*

1° *for every $e \in \Omega$, $f(e, \ldots, \cdot) \in L^n(E, F)$,*

2° *the mapping $\Omega \ni e \to f(e, \ldots, \cdot) \in L_b^n(E, F)$ is continuous at $e_0 \in \Omega$.*

Then the mapping $\Omega \times \underbrace{E \times \ldots \times E}_{n} \ni (e, h_1, \ldots, h_n) \to f(e, h_1, \ldots, h_n) \in F$ is continuous at the point $(e_0, h_1, \ldots, h_n)$ for every $h_j \in E$, $j = 1, \ldots, n$.

The proof follows by induction from Proposition 2 and Lemma 3.

Definition. Let E be a metrizable locally convex space. Let T be a mapping from a subset of E to F, differentiable at a neighbourhood Ω of $e_0 \in E$. We say that T is *twice differentiable at* e_0 if the mapping $E \supset \Omega \ni e \to T'(e) \in L_b(E, F)$ is differentiable at e_0. The derivative of this mapping is called the *second derivative of T at e_0* and is denoted by $T''(e_0)$ or $D^2T(e_0)$.

Of course, $T''(e_0) \in L(E, E; F)$ (cf. Lemma 3).

PROPOSITION 5. *Let E be a metrizable locally convex space and F a locally convex space. If a mapping $T: E \supset \Omega \to F$ is twice differentiable at $e_0 \in \Omega$, then $T''(e_0)$ is a symmetrical bilinear mapping.*

Proof. For every $V \in \mathcal{N}(F)$ and $h, s \in E$ the mapping

$$\boldsymbol{R}^2 \ni (a, b) \to g(a, b) := T(e_0 + ah + bs) \in F_V$$

is twice differentiable. Since

$$g''(0,0)(\alpha, \beta)(\gamma, \delta) = T''(e_0)(\alpha h + \beta s)(\gamma h + \delta s), \quad \alpha, \beta, \gamma, \delta, \in \boldsymbol{R},$$

we infer from the symmetry of second derivatives in normed spaces that

$$\|T''(e_0)(h, s) - T''(e_0)(s, h)\|_V = 0.$$

q. e. d.

The higher order derivatives are defined by induction.

Definition. Let E be a metrizable locally convex space and F a locally convex space. Let T be a mapping from an open set $\Omega \subset E$ to F. We say that T is *n-times differentiable at* $e_0 \in \Omega$ if T is $(n-1)$-times differentiable at some neighbourhood $\mathcal{O}$ of e_0 and the mapping $E \supset \mathcal{O} \ni e \to T^{(n-1)}(e) \in L_b^{n-1}(E, F)$ is differentiable at e_0.

The derivative of this mapping is called the *n-th order derivative of T at e_0* and is denoted by $T^{(n)}(e_0)$. By Lemma 3, $T^{(n)}(e_0) \in L^n(E, F)$.

PROPOSITION 6. *An n-th order derivative is an n-linear symmetrical mapping.*

If $E = \boldsymbol{R}$ (or $\boldsymbol{C}$) and T is a mapping from an open set $I \subset \boldsymbol{R}$ ($\boldsymbol{C}$) n-times differentiable at $e_0 \in I$, then — with regard to $L_b^n(\boldsymbol{R}, F) \cong F$ — we have

$$T^{(n)}(e_0) = \lim_{h \to 0} \frac{1}{h}\left(T^{(n-1)}(e_0 + h) - T^{(n-1)}(e_0)\right), \quad h \in \boldsymbol{R} \ (\boldsymbol{C}),$$

where $T^{(0)} := T$.

LEMMA 4. *Let E be a quasi-normable space and F a normable space. Let A be an equicontinuous set of n-linear mappings from $\underbrace{E \times \ldots \times E}_{n}$ to F. Then there exists a $V \in \mathcal{N}(E)$ such that the topology in A induced by the topology of $L_b^n(E, F)$ is equivalent to the topology of uniform convergence on $\underbrace{V \times \ldots \times V}_{n}$.*

Proof. Let $U \in \mathcal{N}(E)$ and $V \in \mathcal{N}(E)$ be such that for every $\lambda > 0$ there exists a bounded set B_λ such that $V \subset \lambda(U + B_\lambda)$ (Lemma 1). Let $\tilde{V} := \{f \in L^n(E, F): \|f(v_1, \ldots, v_n)\| \leqslant 1, v_j \in V, j = 1, \ldots, n\}$. Then $\frac{1}{2}(\tilde{U} \cap \tilde{B}_\lambda) \subset \lambda^n \tilde{V} \cap \tilde{U}$. q. e. d.

LEMMA 5. *If E is a metrizable locally convex space and F a complete locally convex space, then the space $L_b^n(E, F)$ is complete.*

Proof. It is known (cf. [5]) that under this hypothesis the space $L_b(E, F)$ is complete. Applying Lemma 1, we finish the proof.

If F is a complete locally convex space and f a continuous mapping from $[a, b] \subset \boldsymbol{R}$ to F, then we can define a Riemann integral of f (see [1]). (If F is Banach space, this construction is effected in [4].) The following rezult is valid:

LEMMA 6. *Let F be a complete locally convex space and f a mapping $(p+1)$-times differentiable, from $]a, b[$ to F. Then for every $t_0, t \in]a, b[$ we have:*

$$f(t) = \sum_{k=0}^{p} f^{(k)}(t_0) \frac{(t - t_0)^k}{k!} + \int_{t_0}^{t} f^{(p+1)}(s) \frac{(t-s)^p}{p!}\, ds.$$

THEOREM 5 (Taylor formula). *Let E be a metrizable locally convex space, F a complete locally convex space and T a mapping from an open set $\Omega \subset E$ to F, $(p+1)$-times continuously differentiable on Ω. Then for every $e_0 \in \Omega$ there exists a $U \in \mathcal{N}(E)$ such that for every $h \in U$*

$$T(e_0 + h) - T(e_0) = \sum_{k=1}^{p} \frac{1}{k!} T^{(k)}(e_0)(h, \ldots, h) +$$
$$+ \left(\int_0^1 \frac{(1-s)^p}{p!} T^{(p+1)}(e_0 + sh)\, ds\right)(h, \ldots, h).$$

Proof. Let $U \in \mathcal{N}(E)$ be such that $e_0 + 2U \subset \Omega$ and let $h \in U$. The mapping $]-\delta, 1+\delta[\ni t \to f(t) = T(e_0 + th) \in F$ is $(p+1)$-times differentiable and $f^{(k)}(t) = T^{(k)}(e_0 + th)(h, \ldots, h)$. Applying Lemma 6, we obtain the theorem. q. e. d.

We shall use the following notation:

$$r^{(p)}(e, h) := T(e + h) - \sum_{k=0}^{p} \frac{1}{k!} T^{(k)}(e)(h, \ldots, h).$$

THEOREM 6. *Let E be a metrizable quasi-normable space and F a complete locally convex space. If a mapping $T: E \supset \Omega \to F$ is p-times continuously differentiable on Ω, then for every $e_0 \in \Omega$ and every $V \in \mathcal{N}(F)$ there exist $U, W \in \mathcal{N}(E)$ such that for every $Y \in \mathcal{N}(E)$ and every $e \in e_0 + U$, $h \in Y$,*

$$\|r^{(p)}(e, h)\|_V \leqslant C(e, Y)\|h\|_W^p,$$

where

$$C(e, Y) := \frac{1}{p!} \sup_{k \in Y, q_i \in W} \|T^{(p)}(e + k)(q_1, \ldots, q_p) - T^{(p)}(e)(q_1, \ldots, q_p)\|_V.$$

Besides $\lim_{Y \in \mathcal{N}(E)} C(e, Y) = 0$.

Proof. For every $V \in \mathcal{N}(F)$, $U, W, Y \in \mathcal{N}(E)$, $e \in e_0+U$, $h \in Y$, we have by the Taylor formula:

$$\|r^{(p)}(e,h)\|_V = \Big\| \int_0^1 \frac{(1-s)^{p-1}}{(p-1)!}\big(T^{(p)}(e+sh)(h,\dots,h)-T^{(p)}(e)(h,\dots,h)\big)\,ds\Big\|_V$$
$$\leqslant C(e,Y)(\|h\|_W)^p.$$

It follows from Proposition 3 and Lemma 4 that, for every $V \in \mathcal{N}(F)$, there exist $U, W \in \mathcal{N}(E)$ such that, for every $e \in e_0+U$, $\lim\limits_{Y \in \mathcal{N}(E)} C(e,Y) = 0$. q. e. d.

Definition. Let E, G, F be locally convex spaces and T a mapping from $\Omega \subset E\times G$ to F. We say that T is *partially differentiable at* $(e_0, g_0) \in \Omega$ in the direction of the space E if the mapping $e \to T(e, g_0) \in F$ is differentiable at e_0. The derivative of this mapping is called the *partial derivative of T at the point* (e_0, g_0) in the direction of the space E and is denoted by $T'_E(e_0, g_0)$ or $D_1T(e_0,g_0)$. Similarly, we define the partial derivative in the direction of space G. The mapping T, which is differentiable at (e_0, g_0), is also partially differentiable at (e_0, g_0) in the direction of both spaces and: $T'(e_0,g_0) = T'_E(e_0,g_0)\circ\pi_E + T'_G(e_0,g_0)\circ\pi_G$, where π_E, π_G are projections from $E\times G$ on E and G, respectively. Similarly, $T'_E(e_0,g_0) = T'(e_0,g_0)\circ i_E$, $T'_G(e_0,g_0) = T'(e_0,g_0)\circ i_G$, where i_E, i_G are injections E, G into $E\times G$.

In the following we shall use the fact that the finite product of metrizable quasi-normable spaces is also a space of this type.

Theorem 7. *Let E, G be a metrizable quasi-normable spaces and F a locally convex space. A mapping $T: E\times G \supset \Omega \to F$ is continuously differentiable on Ω if and only if it is partially continuously differentiable (in both variables) on Ω.*

Proof. We shall prove that the continuously partially differentiable mapping is also differentiable. Let $(e_0, g_0)\in\Omega$, $h \in E$, $s \in G$,

$$T(e_0+h, g_0+s) - T(e_0,g_0) - D_1T(e_0,g_0)h - D_2T(e_0,g_0)s$$
$$= D_1T(e_0, g_0+s)h - D_1T(e_0,g_0)h + \overset{1}{r}\big((e_0,g_0+s),h\big) + \overset{2}{r}\big((e_0,g_0),s\big).$$

From Proposition 2, Theorem 3, Lemma 2 and the continuity of the partial derivative we infer that for every $V\in\mathcal{N}(F)$ there exist $W\in\mathcal{N}(G)$ $Q\in\mathcal{N}(E)$ such that $\|D_1T(e_0,g_0+s)h - D_1T(e_0,g_0)h\|_V \leqslant a(s)\|h\|_Q$, where $a(s)\to 0$ for $s\to 0$,

$$\|\overset{1}{r}\big((e_0,g_0+s),h\big)\|_V \leqslant \|D_1T(e_0+h, g_0+s)h - D_1T(e_0,g_0)h\|_V$$
$$+\|D_1T(e_0,g_0+s)h - D_1T(e_0,g_0)h\|_V \leqslant \beta(h,s)\|h\|_Q + a(s)\|h\|_Q,$$

where $\beta(h,s)\to 0$ for $h\to 0$, $s\to 0$,

$$\|\overset{2}{r}\big((e_0,g_0),s\big)\|_V \leqslant \gamma(s)\|s\|_W \quad \text{where} \quad \gamma(s)\to 0 \text{ for } s\to 0.$$

Hence T is differentiable at (e_0, g_0) and

$$T'(e_0,g_0) = D_1T(e_0,g_0)\circ\pi_1 + D_2T(e_0,g_0)\circ\pi_2.$$

The continuity of the derivative follows from this formula. q. e. d. Simultaneously, we have proved

Theorem 7′. *Let E, G, F be spaces which satisfy the hypothesis of Theorem 7. If a mapping $T: E\times G\supset\Omega\to F$ is partially differentiable at (e_0,g_0) in the first variable and partially continuously differentiable in the second variable on some neighbourhood of (e_0,g_0), then T is differentiable at (e_0,g_0).*

The above rezults can be extended to higher order derivatives and several variables. In this case partial derivatives do not depend on the order of iteration, e. g. $D^2_{12}T(e,g) = D^2_{21}T(e,g)$. This fact follows from the symmetry of the n-th order derivative (Proposition 6).

3. Some properties of DF spaces.

Definition. A locally convex space E is called a *DF space* if it satisfies the following conditions:

1° E has a countable base of bounded sets,

2° every strongly bounded set in E' which is a countable union of equicontinuous sets is also an equicontinuous set.

It can be shown that the strong dual to a metrizable locally convex space is a DF space and that the strong dual to an F-S space (Fréchet and Schwartz space) is a DF-S space (i. e. a DF space which is also a Schwartz space). In addition, it is reflexive and therefore a barrelled space; cf. [5], [12].

Lemma 7 (cf. [12]). *Let E be a DF space, and $(V_j)_{j=1}^{\infty}$ a sequence of neighbourhoods of zero in E. Then there exist $V\in\mathcal{N}(E)$ and a sequence $\lambda_j>0$, $j=1,2,\dots$, such that $\lambda_jV\subset V_j$, $j=1,2,\dots$*

Lemma 8 (cf. [12]). *Let E be a DF space. A convex set $U\subset E$ is a neighbourhood of zero in E if and only if, for every absolutely convex bounded set A in E, $U\cap A$ is a neighbourhood of zero in A equipped with the induced topology.*

Lemma 9. *Let E be a DF space and G a locally convex space. Let $A\subset L_b(E,G)$ be a precompact set; then A is a set of equicontinuous mappings.*

Proof. We must show that for every $W\in\mathcal{N}(G)$ the set $V := \bigcap\limits_{f\in A} f^{-1}(W)$ is a neighbourhood of zero in E. In the light of Lemma 8 it is sufficient to show that for every absolutely convex bounded set $B\subset E$ there exists a $U\in\mathcal{N}(E)$ such that $B\cap U\subset V$. Let $B^0_W = \{f\in L(E,G): f(B)\subset W\}$. B^0_W

is a neighbourhood of zero in $L_b(E, G)$. By the precompactness of A there exist $f_1, \ldots, f_n \in A$ such that $A \subset \bigcup_{j=1}^{n} (f_j + \frac{1}{2}B_W^0)$. Let $U \in \mathcal{N}(E)$ be such that $\|f_j(x)\|_W \leqslant \frac{1}{2}$ for $j = 1, \ldots, n$ and $x \in U$. Then for every $y \in B \cap U$ and $f \in A$ we have $\|f(y)\|_W \leqslant 1$; i. e. $B \cap U \subset V$. q. e. d.

PROPOSITION 7. *Let E be a DF-S space and G a locally convex space. Let f be a mapping from $\Omega \times E$ ($\Omega \subset E$) to G which satisfies the following conditions:*

1° *for every $e \in \Omega$, $f(e, \cdot) \in L(E, G)$,*

2° *for every $W \in \mathcal{N}(G)$ the mapping $\Omega \ni e \to f(e, \cdot) \in L_b(E, G_W)$ is uniformly continuous on an open neighbourhood $\mathcal{O}_W$ of $e_0 \in \Omega$* (i. e., for every $\varepsilon > 0$ and $A \in \mathcal{B}(E)$ there exists a $U \in \mathcal{N}(E)$ such that, for every $e_1, e_2 \in \mathcal{O}_W$ such that $e_1 - e_2 \in U$, we have $\sup_{h \in A} \|f(e_1, h) - f(e_2, h)\|_W < \varepsilon$).

Then for every $W \in \mathcal{N}(G)$ there exist $U, V \in \mathcal{N}(E)$ such that, for every $e_1, e_2 \in e_0 + V$, $\sup_{h \in U} \|f(e_1, h) - f(e_2, h)\|_W \leqslant 1$.

Proof. Let $(A_i)_{i=1}^{\infty}$ form a base of bounded sets in E. Let $W \in \mathcal{N}(G)$. For every $i \in N$ there exists a $Y_i \in \mathcal{N}(E)$ such that, for every $e_1, e_2 \in \mathcal{O}_W$, $e_1 - e_2 \in Y_i$, $h \in A_i$, we have $\|f(e_1, h) - f(e_2, h)\|_W \leqslant 1$.

By Lemma 7 there exist $Y \in \mathcal{N}(E)$ and a sequence $\lambda_i > 0$, $i = 1, \ldots$, such that $\lambda_i Y \subset Y_i$, $i = 1, 2, \ldots$ We see that the mapping $E_Y \supset \mathcal{O}_W \ni e \to f(e, \cdot) \in L_b(E, G_W)$ is uniformly continuous on $\mathcal{O}_W$ in the topology of the space E_Y. Let $V \in \mathcal{N}(E)$, $V \subset Y$, $e_0 + V \subset \mathcal{O}_W$, and let V be precompact in the topology of E_Y (the property of Schwartz spaces); then $f(e_0 + V, \cdot) \subset L_b(E, G_W)$ is a precompact set, as the image of a precompact set under the uniformly continuous mapping. From Lemma 9 the set $f(e_0 + V, \cdot)$ is a set of equicontinuous mappings. Hence there exists a $U \in \mathcal{N}(E)$ such that $\|f(e_0 + v, h)\|_W \leqslant \frac{1}{2}$ for $h \in U$, $v \in V$. Thus $\|f(e_1, h) - f(e_2, h)\|_W \leqslant 1$ for $h \in U$, $e_1, e_2 \in e_0 + V$. q. e. d.

Definition. Let E_i, $i = 1, \ldots, n$, and let G be locally convex spaces and h an n-linear mapping from $\prod_{i=1}^{n} E_i$ to G. We say that h is *hypocontinuous* if, for every $W \in \mathcal{N}(G)$ and for an arbitrary system of n bounded sets $A_1, \ldots, A_n$, $A_i \subset E_i$, there exist $U_1, \ldots, U_n$, $U_i \in \mathcal{N}(E_i)$, such that, for every $1 \leqslant j \leqslant n$, $\|h(e_1, \ldots, e_n)\|_W \leqslant 1$ for $e_i \in A_i$, $i \neq j$, $e_j \in U_j$.

We say that the family H of n-linear mappings from $\prod_{i=1}^{n} E_i$ to G is *equihypocontinuous* if for every $W \in \mathcal{N}(G)$ and for an arbitrary system of bounded sets $(A_i)_{i=1}^{n}$ ($A_i \subset E_i$) there exist $(U_i)_{i=1}^{n}$, $U_i \in \mathcal{N}(E_i)$ such that, for every $h \in H$, $1 \leqslant j \leqslant n$, $e_j \in U_j$ and $e_i \in A_i$ for $i \neq j$, we have $\|h(e_1, \ldots \ldots, e_n)\|_W \leqslant 1$.

LEMMA 10 (cf. [12]). *Let E_i, $i = 1, \ldots, n$, be DF spaces and let G be a locally convex space. An n-linear mapping $h: \prod_{i=1}^{n} E_i \to G$ is continuous if and only if it is hypocontinuous. A family H of n-linear mappings from $\prod_{i=1}^{n} E_i$ to G is an equicontinuous family if and only if it is an equihypocontinuous family.*

LEMMA 11. *Let E_i, $i = 1, \ldots, n$, be DF-S spaces and let G be a locally convex space; then*

$$L_b\big(E_1, L_b(E_2, \ldots, L_b(E_n, G))\big) \cong L_b(E_1, \ldots, E_n; G).$$

Proof. $n = 2$. It is obvious that $L(E_1, E_2; G) \subset L\big(E_1, L_b(E_2, G)\big)$. On the other hand, the mapping $h \in L\big(E_1, L_b(E_2, G)\big)$ determines the bilinear mapping $\tilde{h}: E_1 \times E_2 \to G$. We shall show that $\tilde{h}$ is hypocontinuous. We infer from the continuity of h that, for every $W \in \mathcal{N}(G)$, $A_1 \in \mathcal{B}(E_1)$, $A_2 \in \mathcal{B}(E_2)$, there exists a $U_1 \in \mathcal{N}(E_1)$ such that $h(e_1)e_2 \in W$ for $e_1 \in U_1$, $e_2 \in A_2$. But A_1 is a precompact set (E_1 is a Schwartz space); hence $h(A_1)$ is a precompact set in $L_b(E_2, G)$. By Lemma 9, $h(A_1)$ is a equicontinuous family of linear mappings. Thus there exists a $U_2 \in \mathcal{N}(E_2)$ such that $h(A_1)U \subset W$. Applying Lemma 10, we have the continuity of $\tilde{h}$. We have proved the algebraic isomorphism. The topological isomorphism is evident. For $n > 2$ we prove the lemma by induction. q. e. d.

LEMMA 12. *Let E be a DF-S space and G a locally convex complete space. Then the space $L_b^n(E, G)$ is complete.*

Proof. The completeness of $L_b(E, G)$ follows from [12]. For multilinear mappings we make use of Lemma 11. q. e. d.

Lemma 9 can be generalized to n-linear mappings.

LEMMA 13. *Let E be a DF space and G a locally convex space. Let $A \subset L_b^n(E, G)$ be a precompact set. Then A is a set of equicontinuous mappings.*

Proof. $n = 2$. It is sufficient to show that A is a set of equihypocontinuous mappings (Lemma 10), i. e., that for every $W \in \mathcal{N}(G)$ and every $B \in \mathcal{B}(E)$ there exist $U_1, U_2 \in \mathcal{N}(E)$ such that, for every $f \in A$,

$$f(x, y) \in W, \quad \text{for } x \in B,\ y \in U_1, \tag{1}$$

$$f(x, y) \in W, \quad \text{for } x \in U_2,\ y \in B. \tag{2}$$

We shall prove (1). Let $x \in B$, $f_x(y) := f(x, y)$. The set $\{f_x\}_{x \in B} \subset L(E, G)$. Let $W \in \mathcal{N}(G)$ and $V := \bigcap_{\substack{x \in B \\ f \in A}} f_x^{-1}(W)$. A is a precompact set, and therefore there exist $f_1, \ldots, f_k \in A$ such that $A \subset \bigcup_{j=1}^{k} (f_j + \frac{1}{2}B_W^{0,0})$, where $B_W^{0,0} =$

$\{f\in L(E,E;G): f(B,B)\subset W\}$. Let $U\in\mathcal{N}(E)$ be such that $\|f_j(u_1,u_2)\|_W\leqslant\frac{1}{2}$ for $u_1,u_2\in U$, $j=1,\ldots,k$. Let $B\subset nU$, $n\geqslant 1$. Then, for every $f\in A$, every $x\in B$ and $y\in\left(\frac{1}{n}U\right)\cap B$, we have $f(x,y)\subset W$. Hence $\left(\frac{1}{n}U\right)\cap B\subset V$. It follows from Lemma 8 that V is a neighbourhood of zero in E. (1) is proved. In the same way we can prove (2). For $n>2$ the proof is similar. q. e. d.

From Lemma 13 we have

PROPOSITION 8. *Let E be a DF-S space and G a locally convex space. Let f be a mapping from $\Omega\times\underbrace{E\times\ \times E}_{n}$ to G ($\Omega\subset E$) which satisfies the following conditions:*

1° *for every $e\in\Omega$, $f(e,\cdot,\ldots,\cdot)\in L^n(E,G)$,*

2° *for every $W\in\mathcal{N}(G)$ the mapping $\Omega\ni e\to f(e)\in L^n_b(E,G)$ is uniformly continuous on some neighbourhood $\mathcal{O}_W$ of a point e_0.*

Then, for every $W\in\mathcal{N}(G)$, there exist $U,V\in\mathcal{N}(E)$ such that, for every $e_1,e_2\in e_0+V$,

$$\sup_{h_i\in U}\|f(e_1,h_1,\ldots,h_n)-f(e_2,h_1,\ldots,h_n)\|_W\leqslant 1.$$

4. Differentiability in *DF-S* spaces. Within this section we shall use the following notion.

Definition. Let E,F be locally convex spaces and T a mapping from an open set $\Omega\subset E$ to F, differentiable on Ω. We say that T is *locally uniformly continuously differentiable in a neighbourhood of $e_0\in\Omega$* if for every $W\in\mathcal{N}(F)$ there exists a neighbourhood $\mathcal{O}_W$ of e_0 such that the mapping $\mathcal{O}_W\ni e\to T'(e)\in L_b(E,F_W)$ is uniformly continuous on $\mathcal{O}_W$.

Obviously the local uniform continuous differentiability in a neighbourhood of e_0 implies the continuity of the derivative at this point.

THEOREM 8. *Let E be an DF-S space and F a locally convex space. Let T be a mapping from an open set $\Omega\subset E$ to F, locally uniformly continuously differentiable in a neighbourhood of $e_0\in\Omega$. Then, for every $W\in\mathcal{N}(F)$, there exist $U,V,Q\in\mathcal{N}(E)$ such that:*

1° *for every $e\in e_0+V$, $T'(e)\in L(E_U,F_W)$,*

2° *the mapping $e_0+V\ni e\to T'(e)\in L(E,F_W)$ is uniformly continuous in the topology of uniform convergence on Q, i. e., for every $\varepsilon>0$ there exists a $P\in\mathcal{N}(E)$ such that for every $e_1,e_2\in e_0+V$, $e_1-e_2\in P$,*

$$\sup_{h\in Q}\|T'(e_1)h-T'(e_2)h\|_W<\varepsilon.$$

Proof. 1° follows immediately from Proposition 7.

2° Let U,V be such as in Proposition 7. Let $Q\subset U$ and let Q be precompact in E_U. Then we get 2° from the fact that for an equicontinuous family of linear mappings the simple convergence topology is equivalent to the precompact convergence topology [5].

THEOREM 9 (Mean-Value Theorem). *Let the hypothesis of Theorem 8 be satisfied. Then for every $W\in\mathcal{N}(F)$ there exist $U,V\in\mathcal{N}(E)$ such that, for every $e\in e_0+V$, $Y\in\mathcal{N}(E)$ and $h\in Y$, we have $\|r(e,h)\|_W\leqslant C(e,Y)\|h\|_U$, where $C(e,Y)=\sup\limits_{\substack{k\in Y\\ s\in W}}\|T'(e+k)s-T'(e)s\|_W$.*

Besides $\lim\limits_{Y\in\mathcal{N}(E)}C(e,Y)=0$ and, for every $u\in U$,

$$\|T(e+u)-T(e)\|_W\leqslant A\|u\|_U,\quad \textit{where } A=\sup_{k,s\in U}\|T'(e+k)s\|_W.$$

This theorem is equivalent to the following fact: a mapping from a DF-S space to a locally convex space which is locally uniformly continuously differentiable in the sense of Gâteaux in the neighbourhood of e_0 is also Fréchet-differentiable at e_0.

PROPOSITION 9. *Let E, F be DF-S spaces and G a locally convex space. Let T_1 be a mapping from an open set $\Omega_1\subset E$, $e_0\in\Omega_1$, to F, and let T_2 be a mapping from an open set $\Omega_2\subset F$, $T_1(e_0)\in\Omega_2$, to G. Let T_1 be locally uniformly continuously differentiable in a neighbourhood of e_0, let T_1 be uniformly continuous in a neighbourhood $\mathcal{O}$ of e_0 and let T_2 be locally uniformly continuously differentiable in a neighbourhood of $f_0=T_1(e_0)$. Then $T_2\circ T_1$ is locally uniformly continuously differentiable in a neighbourhood of e_0.*

Proof. Let $W\in\mathcal{N}(G)$. Let us take $V,Q\in\mathcal{N}(F)$ as in Theorem 8. Then we have $\|T_2'(f)s\|_W\leqslant 1$ for $f\in f_0+V$, $s\in Q$, and the mapping $e_0+V\ni f\to T_2'(f)\in L(F_Q,G_W)$ is uniformly continuous in the norm topology in $L(F_Q,G_W)$. Let Z, P be such that $\|T_1'(e)h\|_Q\leqslant 1$ for $e\in e_0+P$, $h\in Z$, and the mapping $e_0+P\ni e\to T_1'(e)\in L(E_Z,F_Q)$ is uniformly continuous in the norm topology in $L(E_Z,F_Q)$. We infer from the uniform continuity of T_1 on $\mathcal{O}$ that there exists a $Y\in\mathcal{N}(E)$ such that: 1° $T_1(e)\in f_0+V$ for $e\in e_0+Y$, 2° the mapping $e_0+Y\ni e\to T_1(e)\in F$ is uniformly continuous on e_0+Y. Let $R=Y\cap P$; then from the inequality

$$\|T_2'(T_1(e_1))\circ T_1'(e_1)h-T_2'(T_1(e_2))\circ T_1'(e_2)h\|_W$$
$$\leqslant\big\|\big(T_2'(T_1(e_1))-T_2'(T_1(e_2))\big)T_1'(e_1)h\big\|_W+\big\|T_2'(T_1(e_2))\big(T_1'(e_1)-T_1'(e_2)\big)h\big\|_W$$

it follows that the mapping $e_0+R\ni e\to T_2'(T_1(e))\circ T_1'(e)\in L(E_Z,G_W)$ is uniformly continuous. Hence, the mapping $e_0+R\ni e\to(T_2\circ T_1)'\in L_b(E,G_W)$ is uniformly continuous on e_0+R. q. e. d.

If E is a DF-S space, F a locally convex space and T a differentiable mapping from an open set $\Omega\subset E$ to F, then we can introduce the notation of the second derivative of T in the same way as we did in Section 2 for a mapping from a metrizable space. We can also prove that the second derivative is a bilinear continuous symmetric mapping from $E\times E$ to F. A similar result is valid for the n-th order derivative (see Lemma 11).

Definition. Let T be a mapping n-times differentiable, from an open set $\Omega\times E$ to F. We say that the n-th derivative is *locally uniformly continuous in a neighbourhood of* $e_0\in\Omega$ if for every $W\in\mathcal{N}(F)$ there exists a neighbourhood $\mathcal{O}_W$ of e_0 such that the mapping $\mathcal{O}_W\ni e\to T^{(n)}(e)\in L_b^n(E, F_W)$ is uniformly continuous on $\mathcal{O}_W$. If E is DF-S space, then from the local uniform continuity of $T^{(n)}(\cdot)$ in a neighbourhood of e_0 we infer the local uniform continuity of $T^{(0)}, T^{(1)}, \ldots, T^{(n-1)}$ in a neighbourhood of e_0. Hence we can introduce the notion of a mapping n-times locally uniformly continuously differentiable in a neighbourhood of e_0.

THEOREM 10. *Let E be a DF-S space, F a locally convex space and T a mapping from an open set $\Omega\subset E$ to F, n-times locally uniformly continuously differentiable in a neighbourhood of $e_0\in\Omega$. Then for every $W\in\mathcal{N}(F)$ there exist $V, U, Q\in\mathcal{N}(E)$ such that:*

1° *for every $e\in e_0+V$, $T^{(n)}(e)\in L^n(E_U, F_W)$,*

2° *the mapping $e_0+V\ni e\to T^{(n)}(e)\in L^n(E_U, F_W)$ is uniformly continuous in the topology of the uniform convergence on $\underbrace{Q\times\ldots\times Q}_{n}$ in $L^n(E_U, F_W)$.*

The proof follows from Proposition 8 and Lemma 4.

THEOREM 11. *Let E be a DF-S space, F a complete locally convex space and T a mapping from an open set $\Omega\subset E$ to F, $(p+1)$-times locally uniformly continuously differentiable in a neighbourhood of $e_0\in\Omega$. Then there exists a $U\in\mathcal{N}(E)$ such that for every $h\in U$ we have*

$$T(e_0+h) = \sum_{k=0}^{p}\frac{1}{k!}T^{(k)}(e_0)(h,\ldots,h)+\left(\int_0^1\frac{(1-s)^p}{p!}T^{(p+1)}(e_0+sh)\,ds\right)(h,\ldots,h).$$

Besides, for every $W\in\mathcal{N}(E)$ there exist $U, V\in\mathcal{N}(E)$ such that, for every $Q\in\mathcal{N}(E)$, $e\in e_0+V$, $h\in Q$, we have

$$\|r^{(p)}(e,h)\|_W\leqslant C(e,Q)\|h\|_W^p \quad\text{and}\quad \lim_{Q\in\mathcal{N}(E)} C(e,Q) = 0$$

(cf. Theorem 6).

This theorem follows from Theorem 10 and Lemma 12.

Using the results of this section, we can prove that the mapping $T: E\times G\to F$ (E and G being DF-S spaces) which is partially locally uniformly continuously differentiable in a neighbourhood of (e,g) in both variables, is also differentiable at (e,g) (cf. Theorems 7 and 7′).

5. Differentiability on dense subspaces of DF-S spaces.

LEMMA 14. *Let E be a barrelled space, and $(B_\alpha)_{\alpha\in\mathcal{A}}$ a base of bounded sets in E. Let F be a linear subspace of E such that*

$$(*)\qquad \bigcup_{\alpha\in\mathcal{A}}\overline{B_\alpha\cap F} = E.$$

Then F is a quasi-barrelled space.

Proof. Let $A\subset F$ be an absolutely convex closed set which absorbs all bounded sets in F, i. e., for every $\alpha\in\mathcal{A}$ there exists $\lambda_\alpha>0$ such that $\lambda_\alpha(B_\alpha\cap F)\subset A$. Since $\lambda_\alpha(\overline{B_\alpha\cap F})\subset\bar{A}$, we infer from $(*)$ that $\bar{A}$ is an absorbing set in E. Hence $\bar{A}\in\mathcal{N}(E)$. But $A=\bar{A}\cap F$ (A is closed in F). Thus A is a neighbourhood of zero in F. q. e. d.

Remark. It follows from condition $(*)$ that F is dense in E. We also infer from $(*)$ that every element of E is the limit of a bounded Moore–Smith sequence whose elements belong to F.

If E is a DF-S barrelled space (e. g. the strong dual to an F-S space) and F its subspace which satisfies condition $(*)$ of Lemma 14, then F is a DF-S space. Indeed: 1° F has a countable base of bounded sets and is quasi-barrelled, and thus it is an DF-S space; 2° every subspace of a Schwartz space is also a Schwartz space.

To the space F we can apply the theory developed in the previous section. We shall need the following

LEMMA 15 (cf. [12]). *If E is a locally convex space and F its subspace of type DF, then, for every bounded set A in $\bar{F}$, there exists a bounded set $B\subset F$ such that $A\subset\bar{B}$.*

COROLLARY 1. *If E is a locally convex space, F its dense subspace of type DF and $(B_\alpha)_{\alpha\in\mathcal{A}}$ a base of bounded sets in E, then every bounded set $A\subset E$ is contained in one of sets $\overline{B_\alpha\cap F}$. Obviously, we have $\bigcup_{\alpha\in\mathcal{A}}\overline{B_\alpha\cap F}=E$.*

From Lemma 14 and Corollary 1 we get

COROLLARY 2. *Let E be a barrelled DF space and F its dense subspace. F is a DF space exactly if $\bigcup_{\alpha\in\mathcal{A}}\overline{B_\alpha\cap F}=E$. Moreover, F is quasi-barrelled.*

LEMMA 16. *If E is a locally convex space, F its dense subspace of type DF and G a complete locally convex space, then $L_b(E,G)\cong L_b(F,G)$.*

Proof. $L(E,G)\subset L(F,G)$, but every mapping $f\in L(F,G)$ can be uniqely extended to $\tilde{f}\in L(E,G)$. We have proved the algebraic isomorphism. The topology of $L_b(E,G)$ is not weaker than the topology in $L_b(F,G)$, but from Lemma 15 we see that these topologies are equivalent. q. e. d.

A similar result can be obtained for n-linear mappings.

EXAMPLES.

1° $E=\mathcal{E}'(\boldsymbol{R}^n)$ (the space of distributions with compact supports endowed with the strong topology), $F=C_0^\infty(\boldsymbol{R}^n)$ with the topology induced from E.

These spaces satisfy the conditions of Corollary 2, because every distribution with a compact support can be obtained as the limit of a sequence of functions belonging to $C_0(\boldsymbol{R}^n)$ (cf. [19]).

2° $E = \mathscr{S}'(\boldsymbol{R}^n)$ (the space of tempered distributions endowed with the strong topology), $F = C_0^\infty(\boldsymbol{R}^n)$ with the topology induced by E.

6. Supplementary results. The theory developed in the previous sections can be extended to all quasi-barrelled DF spaces if we make stronger assumptions.

PROPOSITION 7'. *Let E be a quasi-barrelled DF space and G a locally convex space. Let f be mapping from $\Omega \times E$ ($\Omega \subset E$) to G which satisfies the following conditions:*

1° *for every $e \in \Omega$, $f(e, \cdot) \in L(E, G)$,*

2° *for every $W \in \mathscr{N}(G)$ the mapping $\mathcal{O}_W \ni e \to f(e, \cdot) \in L_b(E, G_W)$ is uniformly continuous on some neighbourhood $\mathcal{O}_W$ of $e_0 \in \Omega$,*

3° *for every bounded set $B \subset \mathcal{O}_W$, $f(B, \cdot)$ is bounded in $L_b(E, G_W)$.*

Then for every $W \in \mathscr{N}(G)$ there exist U, $V \in \mathscr{N}(E)$ such that, for every $e_1, e_2 \in e_0 + V$, $\sup_{h \in U} \|f(e_1, h) - f(e_2, h)\|_W \leqslant 1$.

LEMMA 17. *If E is a quasi-normable space, Ω a neighbourhood of zero in E, $Y \in \mathscr{N}(E)$ and $Y + Y \subset \Omega$, then there exists a $V \in \mathscr{N}(E)$ such that, for every $\lambda > 0$, there exists a bounded set $B \subset \Omega$ such that $V \subset \lambda Y + B$.*

Proof. From the quasi-normability of E we infer that there exists a $V \subset Y$ such that for every $\lambda > 0$ there exists a bounded set A such that $V \subset \lambda Y + A$. Of course, we can take $0 \leqslant \lambda \leqslant 1$. Then $V \subset \lambda Y + A \cap \Omega$ and $B = A \cap \Omega$. q. e. d.

Proof of Proposition 7'. It is known that a quasi-barrelled DF space is quasi-normable. Let $W \in \mathscr{N}(G)$. As in the proof of Proposition 7, there exists a $Y \in \mathscr{N}(E)$ such that 1° $Y + Y \in \mathcal{O}_W - e_0$, 2° the mapping $E_Y \supset \mathcal{O}_W \ni e \to f(e, \cdot) \in L_b(E, G_W)$ is uniformly continuous on $\mathcal{O}_W$ in the topology of E_Y. Let V be such a neighbourhood of zero in E that Lemma 17 is satisfied. Then, for $e \in e_0 + V$, we have $e = e_0 + y + b$, where $y \in \lambda_i Y \subset Y_i$, $b \in B_i \subset \mathcal{O}_W - e_0$ and $Y_i \in \mathscr{N}(E)$ are such that

$$\sup_{b \in B_i} \sup_{a \in A_i} \|(f(b + y + e_0) - f(b + e_0))a\|_W \leqslant 1$$

(cf. the proof of Proposition 7). Hence, by 3° we have

$$\sup_{\substack{e \in e_0 + V \\ a \in A_i}} \|f(e, a)\|_W \leqslant 1 + \sup_{\substack{e \in A_i \\ b \in B_i}} \|f(e_0 + b)a\|_W < \infty.$$

Thus the set $\{f(e, \cdot)\}_{e \in e_0 + V} \subset L_b(E, G_W)$ is bounded and hence equicontinuous. q. e. d.

This proposition can be extended to the case of n-spaces. It can be shown from Lemma 10 that, for a quasi-barrelled DF space E and a locally convex space G, $L_b(E, L_b(E, G)) \cong L_b(E, E; G)$. A similar result is valid for n-linear mappings.

In this way we can obtain by stronger assumptions the Mean-Value Theorem and the other results which follow from it.

The class of quasi-barrelled DF spaces includes, for example, all spaces which are the inductive limits of sequences of normed spaces (e. g. $\mathscr{D}_K(\boldsymbol{R}_n)$, $\mathscr{D}'_{l^2}(\boldsymbol{R}^n)$, cf. [19]) and all spaces which are the inductive limits of sequences of locally convex spaces where the transition maps are weakly compact (cf. [17]).

In the next paper it will be shown that in the case of complex spaces one can obtain interesting results concerning the connection between differentiability and analicity.

I would like to express my thanks to Professor K. Maurin for his inspiring interest in this work.

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KATEDRA METOD MATEMATYCZNYCH FIZYKI UNIWERSYTETU WARSZAWSKIEGO
DEPARTMENT OF MATHEMATICAL METHODS IN PHYSICS, WARSAW UNIWERSITY

Reçu par la Rédaction le 1. 7. 1970

'UDIA MATHEMATICA, T. XXXIX. (1971)

Continuous tensor products of Hilbert spaces and product operators

by

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Contents

INTRODUCTION

The paper consists of two parts. In the first part the problem of the construction of a continuous tensor product of Hilbert spaces is considered. There are two natural ways to approach this problem.

The first way is by defining in a given Hilbert space $\mathfrak{H}$ the so-called *tensor structure*, i. e., by assigning to every partition of a certain Boolean algebra a unitary mapping from $\mathfrak{H}$ onto an infinite (incomplete) tensor product of Hilbert spaces [8]. The notion of tensor structure appears in a different form in a paper by Araki and Woods [1]. The authors have found the general model for tensor structures. They show that in the most interesting case of a non-atomic Boolean algebra (the "continuous" case) the Hilbert space is in a natural way isomorphic to an exponential