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On the extension of Lipschitz-Hölder maps on L^p spaces

by

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1. Introduction. Let (M_1, d_1) and (M_2, d_2) be metric spaces and, for each subset D of M_1 and positive number α , define $\text{Lip}(D, M_2; \alpha)$ to be the set of all maps $f: D \rightarrow M_2$ which satisfy a Lipschitz-Hölder continuity condition of order α , that is,

$$d_2[f(x_1), f(x_2)] \leq [d_1(x_1, x_2)]^\alpha \quad \text{for all } x_1, x_2 \in D.$$

The statement that "extension holds for α " or simply " $e(M_1, M_2; \alpha)$ holds" means that, for arbitrary $D \subset M_1$, every map in $\text{Lip}(D, M_2; \alpha)$ extends to a map in $\text{Lip}(M_1, M_2; \alpha)$. The problem of extending Lipschitz (contraction) and Lipschitz-Hölder maps was first considered by MacShane [5] and Banach [1] in the case M_2 is the real line, and it follows from a well-known result of Kirszbraun [4] that $e(H, H; 1)$ holds for H a Hilbert space. A review of other related results and a basic bibliography to the subject is given in [2]. In [3] it was shown that if M_1 is an L^q space, $2 \leq q < \infty$, and M_2 is a Hilbert space H , then $e(L^q, H; \alpha)$ holds for $0 < 2\alpha \leq q/(q-1)$; and also that $e(L^q, H; \alpha)$ holds for $0 < 2\alpha \leq q$ and $1 \leq q \leq 2$. In this paper we generalize these results as follows:

THEOREM 1. Let (M_1, d_1) be a metric space and let (X, μ) and (Y, ν) be two σ -finite measure spaces. Then

(a) $e(L^q(\mu), L^p(\nu); \alpha)$ holds for

- (i) $0 < \alpha \leq q/p$ if $1 < q \leq 2$ and $2 \leq p < \infty$;
- (ii) $0 < \alpha \leq q'/p$ if $2 \leq p < \infty$ and $2 \leq q < \infty$;
- (iii) $0 < \alpha \leq q/p'$ if $1 < p \leq 2$ and $1 < q \leq 2$;
- (iv) $0 < \alpha \leq q'/p'$ if $1 < p \leq 2$ and $2 \leq q < \infty$,

where $1/p + 1/p' = 1/q + 1/q' = 1$. Furthermore the range of α is sharp if the respective L^p spaces are infinite dimensional. Also

(b) $e(M_1, L^p(\nu); \alpha)$ holds for $0 < \alpha \leq 1/p$ when $2 \leq p < \infty$ and for $0 < \alpha < 1/p'$ when $1 < p \leq 2$.

We shall show in Section 2 that this extension theorem is a direct consequence of the following fundamental L^p space inequalities:

THEOREM 2. Let (X, μ) be a σ -finite measure space. Choose a finite set x_1, x_2, \dots, x_n in $L^p(\mu)$ and non-negative numbers c_1, c_2, \dots, c_n such that $\sum c_i = 1$. Then

$$(A) \quad \sum_{i,j=1}^n c_i c_j \|x_i - x_j\|_p^p \leq 2 \sum_{i=1}^n c_i \|x_i\|_p^p, \quad 1 < p \leq 2;$$

$$(B) \quad \sum_{i,j=1}^n c_i c_j \|x_i - x_j\|_p^{p'} \leq 2 \sum_{i=1}^n c_i \|x_i\|_p^{p'}, \quad 2 \leq p < \infty;$$

$$(C) \quad \sum_{i,j=1}^n c_i c_j \|x_i - x_j\|_p^\beta \geq 2 \sum_{i=1}^n c_i \|x_i\|_p^\beta - \sum_{j=1}^n c_j x_j \|_p^\beta, \quad 1 < p \leq 2, \beta \geq p';$$

and

$$(D) \quad \sum_{i,j=1}^n c_i c_j \|x_i - x_j\|_p^\beta \geq 2 \sum_{i=1}^n c_i \|x_i\|_p^\beta - \sum_{j=1}^n c_j x_j \|_p^\beta, \quad 2 \leq p < \infty, \beta \geq p.$$

The connection between these inequalities and the problem of extending Lipschitz-Hölder maps is provided by a general theorem of Minty [6].

Suppose Y is a vector space over the reals and X is a set. A map $\Phi: Y \times X \times X \rightarrow R$ is called a K -function provided that

(i) for each $x_1, x_2 \in X$, Φ is finitely lower semicontinuous and convex on Y ; and

(ii) for any sequence $(y_1, x_1), \dots, (y_n, x_n)$ in $Y \times X$, any $x \in X$ and any sequence c_1, c_2, \dots, c_n of non-negative numbers with $\sum c_i = 1$ one has

$$(1) \quad \sum_{i,j=1}^n c_i c_j \Phi(y_i - y_j; x_i, x_j) \geq 2 \sum_{i=1}^n c_i \Phi(y_i - \sum_{j=1}^n c_j y_j; x_i, x).$$

THEOREM 3 (Minty). Let Y be a linear space, X a space and Φ a K -function on $Y \times X \times X$. If $(y_1, x_1), \dots, (y_n, x_n)$ is a finite sequence in $Y \times X$ such that $\Phi(y_i - y_j; x_i, x_j) \leq 0$ for all $i, j, 1 \leq i, j \leq n$ and if $x \in X$, then there exists a vector $y \in Y$ such that $\Phi(y_i - y; x_i, x) \leq 0$ for all $i, 1 \leq i \leq n$. Moreover, y can be chosen in the convex hull of $\{y_1, y_2, \dots, y_n\}$.

In Section 3 we establish the inequalities of Theorem 2 by application of a multiple-component version of the Riesz-Thorin interpolation theorem [3]. Finally, Section 4 is devoted to the construction of examples which establish the sharpness of the range of α .

2. Application of Minty's criterion for extension. We shall show that inequalities (A) and (D) of Theorem 2 imply part (i) of Theorem 1.

Let f be a Lipschitz function of order α from a subset D of $L^q(\mu)$ into $L^p(\nu)$, where $1 < q \leq 2$, $2 \leq p < \infty$ and $0 < \alpha \leq q/p$. Let $\beta = q/\alpha$ and note that $\beta \geq p$. Define

$$\Phi: L^p(\mu) \times L^q(\nu) \times L^q(\nu) \rightarrow R$$

by

$$\Phi(y; x_i, x_j) = \|y\|_p^\beta - \|x_i - x_j\|_q^\alpha.$$

By virtue of (A) and (D) we have

$$\begin{aligned} \sum_{i,j=1}^n c_i c_j \Phi(y_i - y_j; x_i, x_j) &= \sum_{i,j=1}^n c_i c_j \|y_i - y_j\|_p^\beta - \sum_{i,j=1}^n c_i c_j \|(x_i - x) - (x_j - x)\|_q^\alpha \\ &\geq 2 \sum_{i=1}^n c_i \|y_i\|_p^\beta - \sum_{j=1}^n c_j y_j \|_p^\beta - 2 \sum_{i=1}^n c_i \|x_i - x\|_q^\alpha \\ &= 2 \sum_{i=1}^n c_i \Phi(y_i - \sum_{j=1}^n c_j y_j; x_i, x) \end{aligned}$$

so Φ satisfies (1) and is clearly continuous and convex in y ; hence Φ is a K -function. In order to extend the domain of f to a point $z \in L^q(\mu) \setminus D$ it is enough to show that $\bigcap_{x \in D} S_{f(x)} \neq \emptyset$, where

$$S_{f(x)} = \{y \in L^p(\nu) : \|y - f(x)\|_p \leq \|z - x\|_q^\alpha\}.$$

Fix $x_0 \in D$ and define $S_{f(x)}^0 = S_{f(x)} \cap S_{f(x_0)}$ for $x \in D$. Then $\bigcap_{x \in D} S_{f(x)} = \bigcap_{x \in D} S_{f(x)}^0$ and, since $L^p(\nu)$ is reflexive, $S_{f(x_0)}$ is weakly compact. So we can conclude that $\bigcap_{x \in D} S_{f(x)}^0 \neq \emptyset$ provided the finite intersection property holds. To this end choose a finite set x_1, x_2, \dots, x_n in D , and note that, for $1 \leq i, j \leq n$,

$$\Phi(f(x_i) - f(x_j); x_i, x_j) = \|f(x_i) - f(x_j)\|_p^\beta - \|x_i - x_j\|_q^\alpha \leq 0$$

because

$$\|f(x_i) - f(x_j)\|_p^\beta \leq \|x_i - x_j\|_q^\alpha = \|x_i - x_j\|_q^\alpha.$$

In view of Minty's theorem there exists a $y \in L^p(\nu)$ such that $\Phi(f(x_i) - y; x_i, z) \leq 0$ for $i = 1, 2, \dots, n$. Therefore

$$0 \geq \|f(x_i) - y\|_p^\beta - \|x_i - z\|_q^\alpha \quad \text{or} \quad \|f(x_i) - y\|_p \leq \|x_i - z\|_q^\alpha$$

for $i = 1, 2, \dots, n$ so that $y \in \bigcap_{i=1}^n S_{f(x_i)}$. Moreover, y can be chosen in the convex hull of $\{f(x_1), f(x_2), \dots, f(x_n)\}$. It follows that f can be extended to z with $f(z) = y$ in the closed convex hull of $f(D) = \{f(x) : x \in D\}$. A simple Zorn's lemma argument now shows that f can be extended to all of $L^q(\mu)$ in such a way that the Lipschitz condition is preserved and the range of the extension is contained in the closed convex hull of $f(D)$.

We omit the entirely similar arguments which show that (ii) follows from (B) and (D), (iii) follows from (A) and (C), and (iv) follows from (B) and (C).

In order to see that Theorem 1 (b) follows from (C) and (D), simply note that $\Phi: L^p(\nu) \times M_1 \times M_1 \rightarrow R$ defined by $\Phi(y; x_1, x_2) = \|y\|_p^\beta - d_1(x_1, x_2)$ is a K -function, where $\beta \geq p$ for $2 \leq p < \infty$ and $\beta \geq p'$ for $1 < p \leq 2$.

3. Proof of Theorem 2. We need the following generalization of the Riesz-Thorin interpolation theorem given in [3].

Let $(X_1, \mu_1), (X_2, \mu_2), \dots, (X_n, \mu_n)$ be a finite sequence of σ -finite measure spaces, $P = (p_1, p_2, \dots, p_n)$ an n -tuple of numbers in $[1, \infty]$ and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ a sequence of positive weights. For each r , $1 \leq r \leq \infty$, define $L^{P,r}(\lambda)$ as the linear space of all vectors $f = (f_1, f_2, \dots, f_n)$, $f_k \in L^{p_k}(\mu_k)$, such that

$$\|f\|_{P,r} = \left\{ \sum_{k=1}^n \left(\int_{X_k} |f_k|^{p_k} d\mu_k \right)^{r/p_k} \cdot \lambda_k \right\}^{1/r} < \infty.$$

In case $r = \infty$ we define

$$\|f\|_{P,\infty} = \max_{1 \leq k \leq n} \|f_k\|_{p_k}.$$

Introduce a second sequence $(Y_1, \nu_1), (Y_2, \nu_2), \dots, (Y_n, \nu_n)$ of σ -finite measure spaces and define the space $L^{Q,s}(\eta)$ in a similar way. These spaces are Banach spaces. A vector with measurable components is termed a measurable vector and a vector whose components are simple measurable functions is a simple measurable vector.

THEOREM 4. Let T be a linear transformation from the simple measurable vectors on $X = (X_i)_{i=1}^n$ to the measurable vectors on $Y = (Y_i)_{i=1}^n$. Let $1 \leq P_i$, $Q_i \leq \infty$, $1 \leq r_i \leq \infty$, $1 \leq s_i \leq \infty$ for $i = 1, 2$, and set $1/P = (1-t)/P_1 + t/P_2$, $1/Q = (1-t)/Q_1 + t/Q_2$, $1/r = (1-t)/r_1 + t/r_2$, $1/s = (1-t)/s_1 + t/s_2$, where $0 \leq t \leq 1$. Suppose there exist constants M_1 and M_2 such that

$$(2) \quad \|Tf\|_{Q_i, s_i} \leq M_i \|f\|_{P_i, r_i} \quad \text{for } i = 1, 2$$

and any simple vector f . Then

$$(3) \quad \|Tf\|_{Q,s} \leq M_1^{1-t} M_2^t \|f\|_{P,r}$$

and if $P < \infty$, T can be extended uniquely to $L^{P,r}(\lambda)$.

For a proof of inequalities (A) and (B) we refer the reader to [3]. We proceed to establish (C) and (D).

Fix a positive integer n and choose positive numbers c_1, c_2, \dots, c_n such that $\sum_{i=1}^n c_i = 1$. (We lose nothing by assuming $c_i > 0$). For (X, μ) a σ -finite measure space and $1 \leq p, q \leq \infty$ define $L^{p,q}(c^2)$ to be the space of $(n-1)$ -tuples $(f^1, f^2, \dots, f^{n-1})$ in which f^j is an $(n-j)$ -tuple of elements of $L^p(\mu)$. On this space introduce the norm

$$\|f\|_{p,q} = \left(\sum_{i=1}^{n-1} \sum_{j>i} c_i c_j \|f_{j-i}^i\|_p^q \right)^{1/q}, \quad 1 \leq q < \infty$$

and

$$\|f\|_{p,\infty} = \max_{\substack{1 \leq i \leq n-1 \\ j>i}} \|f_{j-i}^i\|_p.$$

Let $M(c^2)$ denote the subspace of $L^{p,q}(c^2)$ consisting of the simple measurable vectors and define $L^{p,q}(c)$ to be the space of n -tuples (g_1, g_2, \dots, g_n) of p -integrable functions on X with

$$\|g\|_{p,q} = \left(\sum_{i=1}^n c_i \|g_i\|_p^q \right)^{1/q}, \quad 1 \leq q < \infty,$$

$$\|g\|_{p,\infty} = \max_{1 \leq i \leq n} \|g_i\|_p.$$

We let $M(c)$ denote the corresponding space of simple vectors.

Consider the linear operator $T: M(c^2) \rightarrow M(c)$ defined by

$$(Tf)_j = \sum_{i=1}^{n-j} c_{i+j} f_i^j - \sum_{k=1}^{j-1} c_k f_{j-k}^k, \quad 1 \leq j \leq n,$$

it being understood that the summation is zero if the upper index is zero. In the following two lemmas we establish the continuity of T at the end points 2 and ∞ which allows us to apply Theorem 4.

LEMMA 1. $\|Tf\|_{2,2} \leq \|f\|_{2,2}$ for $f \in M(c^2)$.

Proof. Let \langle, \rangle denote the inner product in $L^2(\mu)$. Then

$$\begin{aligned} \|Tf\|_{2,2}^2 &= \sum_{j=1}^n c_j \left\| \sum_{i=1}^{n-j} c_{i+j} f_i^j - \sum_{k=1}^{j-1} c_k f_{j-k}^k \right\|_2^2 \\ &= \sum_{j=1}^n c_j \left\{ \sum_{i=1}^{n-1} c_{i+j} c_{i+j} \langle f_i^j, f_i^j \rangle - \sum_{k=1}^{j-1} \sum_{l=1}^{n-j} c_k c_{l+j} \langle f_{j-k}^k, f_{j-l}^l \rangle - \right. \\ &\quad \left. - \sum_{m=1}^{j-1} \sum_{l=1}^{n-j} c_{i+j} c_m \langle f_i^j, f_{j-m}^m \rangle + \sum_{k,m=1}^{j-1} c_k c_m \langle f_{j-k}^k, f_{j-m}^m \rangle \right\} \\ &= \sum_{j=1}^n \sum_{i,k=1}^{n-j} c_j c_{i+j} c_{k+j} \langle f_i^j, f_k^j \rangle + \sum_{j=1}^n \sum_{i,k=1}^{j-1} c_j c_i c_k \langle f_{j-k}^k, f_{j-i}^i \rangle - \\ &\quad - \sum_{j=1}^n \sum_{k=1}^{j-1} \sum_{l=1}^{n-j} c_j c_k c_{i+j} \{ \langle f_{j-k}^k, f_l^l \rangle + \langle f_l^l, f_{j-k}^k \rangle \} \\ &= \sum_{1 \leq i < j < k \leq n} c_i c_j c_k \{ \langle f_{j-i}^i, f_{k-i}^i \rangle + \langle f_{k-i}^i, f_{j-i}^i \rangle + \langle f_{k-i}^i, f_{j-i}^i \rangle + \\ &\quad + \langle f_{j-i}^i, f_{k-i}^i \rangle - \langle f_{j-i}^i, f_{k-i}^i \rangle - \langle f_{k-i}^i, f_{j-i}^i \rangle \} + \\ &\quad + \sum_{\substack{i,j=1 \\ j>i}}^n \{ c_i c_j c_j \langle f_{j-i}^i, f_{j-i}^i \rangle + c_i c_j c_j \langle f_{j-i}^i, f_{j-i}^i \rangle \}. \end{aligned}$$

Since this last sum is equal to

$$\sum_{\substack{i=1 \\ j>i}}^n c_i c_j \left(1 - \sum_{\substack{k \neq i, j}} c_k\right) \langle f_{j-i}^i, f_{j-i}^j \rangle = \|f\|_{2,2}^2 - \sum_{\substack{i=1 \\ j>i}}^{n-1} \sum_{\substack{k \neq i, j}} c_i c_j c_k \langle f_{j-i}^i, f_{j-i}^j \rangle$$

we have

$$\begin{aligned} \|Tf\|_{2,2}^2 &= \|f\|_{2,2}^2 - \sum_{1 \leq i < j < k \leq n} c_i c_j c_k \{ -\langle f_{j-i}^i, f_{k-i}^i \rangle - \langle f_{k-i}^i, f_{j-i}^j \rangle - \\ &\quad - \langle f_{k-i}^j, f_{k-j}^j \rangle - \langle f_{k-j}^j, f_{k-i}^i \rangle + \langle f_{j-i}^i, f_{k-j}^j \rangle + \\ &\quad + \langle f_{k-j}^j, f_{j-i}^i \rangle + \langle f_{j-i}^j, f_{j-i}^i \rangle + \langle f_{k-j}^j, f_{k-j}^j \rangle + \langle f_{k-i}^i, f_{k-i}^i \rangle \} \\ &= \|f\|_{2,2}^2 - \sum_{1 \leq i < j < k \leq n} c_i c_j c_k \|f_{j-i}^i - f_{k-i}^i + f_{k-j}^j\|_2^2 \\ &\leq \|f\|_{2,2}^2. \end{aligned}$$

LEMMA 2. If $1 \leq r \leq \infty$, then $\|Tf\|_{r,\infty} \leq \|f\|_{r,\infty}$ for $f \in M(c^2)$.

Proof. We have

$$\begin{aligned} \|Tf\|_{r,\infty} &= \max_{1 \leq j \leq n} \left\| \sum_{i=1}^{n-j} c_{i+j} f_i^j - \sum_{k=1}^{j-1} c_k f_{j-k}^k \right\|_r \\ &\leq \max_{1 \leq j \leq n} \left\{ \sum_{i=1}^{n-j} c_{i+j} \|f_i^j\|_r + \sum_{k=1}^{j-1} c_k \|f_{j-k}^k\|_r \right\} \\ &\leq \sum_{k=1}^n c_k \left\{ \max_{\substack{1 \leq j \leq n \\ 1 \leq i \leq n-j}} \|f_i^j\|_r \right\} = \|f\|_{r,\infty}. \end{aligned}$$

In order to apply the interpolation theorem to the operator T , suppose $1 < p \leq 2$, $p' \leq \beta < \infty$; then set $1-t = 2/\beta$ and observe that

$$\begin{aligned} (1-t)/2 + t &= (1+t)/2 = 1 - (1-t)/2 = 1 - 1/\beta \geq 1 - 1/p' \\ &= 1/p \geq (1-t)/p + t/p \geq (1-t)/2 + t/p. \end{aligned}$$

Hence there is a number r , $1 \leq r \leq p$, such that $1/p = (1-t)/2 + t/r$. By the preceding lemmas we have $\|Tf\|_{2,2} \leq \|f\|_{2,2}$ and $\|Tf\|_{r,\infty} \leq \|f\|_{r,\infty}$ for $f \in M(c^2)$. Thus, by Theorem 4,

$$\|Tf\|_{p,\beta} \leq \|f\|_{p,\beta}, \quad 1 < p \leq 2, \quad p' \leq \beta.$$

Now if $2 \leq p < \infty$ and $p \leq \beta < \infty$, set $1-t = 2/\beta$ and choose r , $p < r \leq \infty$ so that $1/p = (1-t)/2 + t/r$. Again, by application of Theorem 4, we have

$$\|Tf\|_{p,\beta} \leq \|f\|_{p,\beta}.$$

If x_1, x_2, \dots, x_n are simple measurable functions on X , then

$$x = (x_1 - x_2, x_1 - x_3, \dots, x_1 - x_n, x_2 - x_3, \dots, x_{n-1} - x_n)$$

is an element of $M(c^2)$, where in our notation $x_i^j = x_j - x_{i+j}$.

We now compute $\|Tx\|_{p,\beta}^\beta$ and $\|x\|_{p,\beta}^\beta$.

$$\begin{aligned} \|Tx\|_{p,\beta}^\beta &= \sum_{j=1}^n c_j \left\| \sum_{i=1}^{n-j} c_{i+j} x_i^j - \sum_{k=1}^{j-1} c_k x_{j-k}^k \right\|_p^\beta \\ &= \sum_{j=1}^n c_j \left\| \sum_{i=1}^{n-j} c_{i+j} (x_j - x_{i+j}) - \sum_{k=1}^{j-1} c_k (x_k - x_j) \right\|_p^\beta \\ &= \sum_{j=1}^n c_j \left\| \sum_{i=j+1}^n c_i (x_j - x_i) + \sum_{i=1}^{j-1} c_i (x_j - x_i) \right\|_p^\beta \\ &= \sum_{j=1}^n c_j \left\| \sum_{i=1}^n c_i (x_j - x_i) \right\|_p^\beta = \sum_{j=1}^n c_j \left\| x_j - \sum_{i=1}^n c_i x_i \right\|_p^\beta, \end{aligned}$$

and

$$\|x\|_{p,\beta}^\beta = \sum_{i=1}^{n-1} \sum_{j>i} c_i c_j \|x_{j-i}^i\|_p^\beta = \sum_{i=1}^{n-1} \sum_{j>i} c_i c_j \|x_i - x_j\|_p^\beta = \frac{1}{2} \sum_{i,j=1}^n c_i c_j \|x_i - x_j\|_p^\beta.$$

Hence we have

$$\sum_{i,j=1}^n c_i c_j \|x_i - x_j\|_p^\beta \geq 2 \sum_{j=1}^n c_j \left\| x_j - \sum_{i=1}^n c_i x_i \right\|_p^\beta,$$

where $\beta \geq p'$ for $1 < p \leq 2$, and $\beta \geq p$ for $2 \leq p < \infty$.

Therefore (C) and (D) hold for all simple functions x_1, \dots, x_n on X , which is clearly sufficient to complete the proof.

Insight into the above argument may be obtained by considering the case $n = 3$. In this situation, $L^{p,q}(c^2)$ is the space of 3-tuples $(f_1, f_2, f_3) = ((f_1^1, f_2^1), f_3^2)$ with weights $c_1 c_2, c_1 c_3, c_2 c_3$, $L^{p,q}(c)$ is the space of 3-tuples (g_1, g_2, g_3) with weights c_1, c_2, c_3 , and the operator T is given in matrix form by

$$T = \begin{pmatrix} c_2 & c_3 & 0 \\ -c_1 & 0 & c_3 \\ 0 & -c_1 & -c_2 \end{pmatrix}.$$

4. Sharpness of Theorem 1. We begin with the construction of some sets in finite dimensional \mathcal{V}^p space which exhibit "nice" geometrical properties.

Corresponding to the 4×8 matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 \end{pmatrix}$$

define an operator T on the set of all matrices over R as follows:

If $B = (b_{ij})$ is an $n \times m$ matrix, then $T(B)$ is the $4n \times 8m$ matrix $(b_{ij}A)$ with the obvious identification between a $n \times m$ matrix of 4×8 matrices and a $4n \times 8m$ matrix. Let $X^1 = A$ and define $X^k = T(X^{k-1})$ for $k > 1$. Note that X^k is a $2^{2k} \times 2^{3k}$ matrix each of whose entries is ± 1 .

We omit the straightforward arguments for the next two lemmas.

LEMMA 3. If $k \geq 1$, then

$$\sum_{i=1}^{2^{2k}} X_{i,j}^k = 0 \text{ or } \pm 2^{2k} \text{ for } 1 \leq j \leq 2^{3k}$$

and the second possibility holds for exactly 2^k choices of j .

LEMMA 4. If $k \geq 1$ and $1 \leq r < s \leq 2^{2k}$, then

$$\sum_{j=1}^{2^{3k}} |X_{r,j}^k - X_{s,j}^k|^p = 2^{3k-1} \cdot 2^{3k}.$$

For $k \geq 1$ define the 2^{2k} vectors $z_1^k, z_2^k, \dots, z_{2^{2k}}^k$ in $\ell_{2^{3k}}^p$ by

$$(z_i^k)_j = 2^{-3k/p} \left(X_{i,j}^k - 2^{-2k} \sum_{r=1}^{2^{2k}} X_{r,j}^k \right), \quad 1 \leq j \leq 2^{3k}.$$

Of course $(z_i^k)_j$ denotes the j -th component of z_i^k . We record the essential properties of these vectors.

LEMMA 5. The points $\{z_i^k\}_{i=1}^{2^{2k}}$ satisfy the following conditions:

(a) Each component of z_i^k is 0 or $\pm 2^{-3k/p}$;

(b) $\sum_{i=1}^{2^{2k}} z_i^k = 0$;

(c) $\|z_i^k\|_p = (1 - 2^{-2k})^{1/p}$;

(d) $\|z_i^k - z_j^k\|_p = 2^{1/p'}$ for $i \neq j$; and

(e) $\min_{z \in \ell_{2^{3k}}^p} \max_{1 \leq i \leq 2^{2k}} \|z - z_i^k\|_p = (1 - 2^{-2k})^{1/p}$.

Proof. Part (a) follows from Lemma 3 and (b) follows from the definition of z_i^k . Also, by Lemma 3 we see that z_i^k has 2^k zero entries, so that $\|z_i^k\|_p = 2^{-3k/p} (2^{2k} - 2^k)^{1/p} = (1 - 2^{-2k})^{1/p}$. By Lemma 4, $\|z_i^k - z_j^k\|_p = 2^{-3k/p} (2^{3k-1})^{1/p} = 2^{1/p'}$.

In order to prove (e) suppose there exists $z \in \ell_{2^{3k}}^p$ such that

$$\max_{1 \leq i \leq 2^{2k}} \|z - z_i^k\|_p < (1 - 2^{-2k})^{1/p} = \max_{1 \leq i \leq 2^{2k}} \|z_i^k\|_p.$$

Let Φ_i be the linear functional on $\ell_{2^{3k}}^p$ such that $\|\Phi_i\| = 1$ and $\Phi_i(z_i^k) = \|z_i^k\|_p$ for $1 \leq i \leq 2^{2k}$. By virtue of (a) we have

$$\Phi_i(x) = \langle x, z_i^k \rangle 2^{3k/p} \cdot 2^{-3k(p-1)/p} (1 - 2^{-2k})^{-1/p'} = \langle x, z_i^k \rangle \cdot a$$

for $x \in \ell_{2^{3k}}^p$. Also, $\Phi_i(z) > 0$ since

$$(1 - 2^{-2k})^{1/p} > \|z - z_i^k\|_p \geq |\Phi_i(z - z_i^k)| \geq (1 - 2^{-2k})^{1/p} - \Phi_i(z).$$

Hence we have the contradiction

$$0 < \sum_{i=1}^{2^{2k}} \Phi_i(z) = a \cdot \sum_{i=1}^{2^{2k}} \langle z, z_i^k \rangle = a \cdot \left\langle z, \sum_{i=1}^{2^{2k}} z_i^k \right\rangle = 0.$$

This proves (e).

Let $e_1^k, e_2^k, \dots, e_{2^{2k}}^k$ denote the 2^{2k} unit vectors in the standard basis for $\ell_{2^{2k}}^p$; that is, $(e_i^k)_j = \delta_{ij}$ for $1 \leq j \leq 2^{2k}$.

LEMMA 6. The following hold for $\{e_i^k\}$:

(f) $\|e_i^k - e_j^k\|_p = 2^{1/p}$, $i \neq j$;

(g) $\|e_i^k\|_p = 1$;

(h) $\lim_{k \rightarrow \infty} \min_{y \in \ell_{2^{2k}}^p} \max_{1 \leq i \leq 2^{2k}} \|e_i^k - y\|_p = 1$.

Proof. Only (h) requires argument. For each $k \geq 1$ the min max is assumed at some point $y = (y_1, y_2, \dots, y_{2^{2k}})$ for which $0 \leq y_i$. Choose s so that $y_s \leq y_j$ for $1 \leq j \leq 2^{2k}$. Then

$$\max_{1 \leq j \leq 2^{2k}} \|y - e_j^k\|_p \geq \{(1 - y_s)^p + \sum_{j \neq s} y_j^p\}^{1/p} \geq \{(1 - y_s)^p + (2^{2k} - 1) y_s^p\}^{1/p}.$$

Now the function $(1 - t)^p + (2^{2k} - 1)t^p$ assumes its minimum for positive t at $t = [(2^{2k} - 1)^{1/(p-1)} + 1]^{-1} = c_k^{-1}$ so if we define $w^k = (c_k^{-1}, c_k^{-1}, \dots, c_k^{-1})$, then

$$\min_{y \in \ell_{2^{2k}}^p} \max_{1 \leq i \leq 2^{2k}} \|e_i^k - y\|_p = \|w^k - e_i^k\|_p = \{(1 - c_k^{-1})^p + (2^{2k} - 1)c_k^{-p}\}^{1/p}$$

and this expression tends to one as $k \rightarrow \infty$.

Suppose $e(L^q, L^p; a)$ holds, where $1 < p, q \leq 2$. Let $d = 2^{1/2p' - 1/q}$ and set $D = \{de_i^k: 1 \leq i \leq 2^{2k}\}$. Define $f: D \rightarrow \ell_{2^{2k}}^p$ by $f(de_i^k) = z_i^k$. Then

$$\begin{aligned} \|f(de_i^k) - f(de_j^k)\|_p &= \|z_i^k - z_j^k\|_p = 2^{1/p'} = d^{a/2} a^{1/2} \\ &= d^a \|e_i^k - e_j^k\|_q^a = \|de_i^k - de_j^k\|_q^a \end{aligned}$$

so $f \in \text{Lip}(D, \ell_{2^{2k}}^p; a)$. By assumption the map f can be extended to zero so as to preserve the Lipschitz condition. According to Lemma 5 (e) the

best choice for $f(0)$ is 0. It must therefore be true that $\|z_i^k\|_p \leq \|de_i^k\|_q^a$, that is, $(1-2^{-2k})^{1/p} \leq 2^{1/p'-a/q}$ and this must hold for integers k since we are assuming that $e(L^q, L^p; a)$ always holds. Therefore $2^{1/p'-a/q} \geq 1$ or $q/p' \geq a$.

For $1 < p \leq 2 \leq q < \infty$ define f by $f(2^{1/ap'-1/q'} z_i^k) = z_i^k$; for $2 \leq p$, $q < \infty$ define f by $f(2^{1/ap'-1/q'} z_i^k) = e_i^k$; and for $1 < q \leq 2 \leq p < \infty$ define $f(2^{1/ap'-1/q} e_i^k) = e_i^k$. In each case extension of f to zero implies sharpness of the corresponding part of Theorem 1.

Notice, for example in the case $1 < q \leq 2 \leq p < \infty$, that we have constructed functions $f \in \text{Lip}(D, L^p; a)$ for which the best choice for extension to the point zero does not lie on the convex hull of $f(D)$.

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Conjugate kernels and convergence of harmonic singular integrals

by

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Introduction. In this paper we shall be concerned with singular integrals having the form P.V. $\Omega(X')|X|^{-m} * f$, where $\Omega(X')$ is a spherical harmonic.

The purpose of this paper is twofold.

First, to show that if $K_\varepsilon(X) = \Omega(X')|X|^{-m}$ for $|X| \geq \varepsilon$ and zero otherwise, there exists a unique radial function $k(|X|)$ belonging to L^1 such that

$$\text{P.V.} \int_{\mathbb{R}^m} \Omega(Y')|Y|^{-m} \varepsilon^{-m} k(\varepsilon^{-1}|X-Y|) dY = K_\varepsilon(X) \quad \text{a.e.}$$

The function $k(|X|)$ is the same for any spherical harmonic $\Omega(X')$ of a fixed degree.

Second, we shall use this representation to study the pointwise convergence of harmonic singular integrals at individual points by giving conditions on $K(f)$ only. The kernel $k(|X|)$ is also studied.

NOTATION

1. $X = (x_1, \dots, x_m)$ will denote a point in the m -dimensional Euclidean space and $dX = dx_1 \dots dx_m$ the element of volume there.

2. Σ will denote the surface of the unit sphere in \mathbb{R}^m , X' any point there and $d\sigma$ the "area" element on Σ .

3. $f * g$ will denote the convolution of f and g , namely

$$\int_{\mathbb{R}^m} f(X-Y)g(Y) dY;$$

$f(\lambda X) = f(\lambda x_1, \dots, \lambda x_m)$ for any real λ and any function f defined on \mathbb{R}^m .

$$4. \int_{\mathbb{R}^m} \exp(-i2\pi \langle X, Y \rangle) f(Y) dY = \hat{f}(X)$$

will be the Fourier transform used here; $\langle X, Y \rangle = \sum_{j=1}^m x_j y_j$.