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A generalization of the mean ergodic theorem

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§ 1. Introduction. Let E be a Banach space and V be a continuous linear operator of E into E; i. e. V be a continuous endomorphism on E. Suppose N is the null space of the operator $I-V$, where I is the identity operator; let R be the range of $I-V$ and let $\bar{R}$ denote the closure of R.

One version of the mean ergodic theorem is the following:

Let $\{\|V^n\|\}$ be uniformly bounded. Let $T_n x = \dfrac{(I+V+\ldots+V^{n-1})x}{n}$. *If $\{T_n x\}$ is weakly relatively compact then $T_n x \to Px$, where the convergence is pointwise and P is the projection of X onto N parallel to $\bar{R}$.*

Viewing the above as pointwise $(C, 1)$ summability of $\{V^n x\}$ to Px suggests generalizations, replacing $(C, 1)$ by (scalar) matrices of the Toeplitz type and results in this direction can be found in Cohen [1]. A partial generalization of Cohen's result is due to Kurtz and Tucker [5] who consider transformations of $\{V^n x\}$ by operator-valued Hausdorff summability matrices, but retaining the condition of uniform boundedness of $\|V^n\|$.

A generalization due to Yosida [9] is to replace, in the case of $(C, 1)$ summability, the uniform boundedness of $\|V^n\|$ by the weaker hypothesis, $\|V^n\|/n \to 0$.

The present paper attempts a general theorem, involving (a) replacing $(C, 1)$ summability by general operator valued matrices of the Toeplitz kind and (b) sacrificing the uniform boundedness of $\|V^n\|$ to a less restrictive growth condition involving both the operators $\{V^n\}$ and the matrix under consideration. The main result in this direction is in Theorem 1 and two particular cases of this situation are given in Section 4.

§ 2. Preliminaries. Let E be a Banach space. Let $A = (A_{nk})$, $n, k = 0, 1, \ldots$ be a matrix of operators A_{nk} each of which is linear and continuous on E into E. If for each convergent $\{x_n\}$ in E, the sequence $\{y_n\}$, $y_n = \sum_{k=0}^{\infty} A_{nk} x_k$, exists and converges with $\lim y_n = \lim x_n$ then A is said to be a *Toeplitz matrix*.

The following lemma gives a characterization of Toeplitz matrices.

LEMMA 1. *The matrix* $A = (A_{nk})$ *is Toeplitz if and only if*

(i) $\|\sum_{k=0}^{\infty} A_{nk}x_k\| \leqslant M$ *for* $\|x_k\| \leqslant 1$, $k = 0, 1, \ldots$ *and* $n, r = 0, 1, \ldots$;

(ii) $\lim_{n\to\infty} A_{nk}x = \theta$ *for each* $k = 0, 1, 2, \ldots$ *and each* $x \in E$

and

(iii) $\sum_{k=0}^{\infty} A_{nk}x$ $(n = 0, 1, \ldots)$ *exists for each* $x \in E$ *and* $\lim_n \sum_{k=0}^{\infty} A_{nk}x = x$.

For a proof of the lemma see Zeller [10].

§ 3. Ergodic Theorem. We now prove our main result.

THEOREM 1. *Let* E *be a Banach space and* V *be a continuous endomorphism of* E. *Suppose that* $A = (A_{nk})$ *is a Toeplitz matrix of continuous linear operators on* E *and that* V *commutes with each* A_{nk}. *Let* $\{\gamma_n\}$ *be an increasing sequence of non-negative reals. Assume that*

(i) $T_n x = \sum_{k=0}^{\infty} A_{nk}V^k x$ *exists for each* $x \in E$ *and each* n *and* $\{T_n x\}$, $n = 0, 1, \ldots$ *is weakly relatively compact;*

(ii) $\sum_{k=0}^{\infty} \|A_{nk} - A_{n,k+1}\|\gamma_{k+1} \to 0$ *as* $n \to \infty$;

(iii) $\|V^n\| \leqslant \gamma_n$.

Then for each $x \in E$, $T_n x \to Px$, *where* P *is the projection of* E *onto the null space* N *of* $(I-V)$.

Proof. First suppose $x \in N$. Then $x = Vx = V^2x = \ldots = V^n x = \ldots$ Thus $T_n x = \sum_{k=0}^{\infty} A_{nk}x$, and since A is Toeplitz, $T_n x \to x = Px$.

Next assume $x \in \bar{R}$; then for given $\varepsilon > 0$ we can find $y \in E$ such that $\|x-(I-V)y\| < \varepsilon$. Writing $z = x - y + Vy$, we get

$$T_n x = \sum_{k=0}^{\infty} A_{nk}V^k(y - Vy + z) = \sum_{k=0}^{\infty} A_{nk}(V^k y - V^{k+1}y) + \sum_{k=0}^{\infty} A_{nk}V^k z$$

$$= \sum_{k=0}^{\infty} (A_{nk} - A_{n,k-1})V^k y + \sum_{k=0}^{\infty} A_{nk}V^k z,$$

(where we put $A_{n,-1} = 0$).

Since $\{T_n x\}$ is weakly relatively compact for each x, the operators $\{T_n\}$ are uniformly bounded. The uniform boundedness of the operators $\{T_n\}$, the hypothesis that A is Toeplitz and satisfies conditions (ii) and (iii) and the fact that $\|z\| < \varepsilon$ show that $T_n x \to \theta = Px$.

Thus the theorem is proved for $x \in N$ and $x \in \bar{R}$. Next we shall show that E admits the algebraic direct sum decomposition $E = N \oplus \bar{R}$.

For each $x \in E$, $\{T_n x\}$ being weakly relatively compact has a weakly convergent subsequence whose limit we shall denote by x_0. We shall prove that $x_0 \in N$. Using the commutativity of V with each A_{nk}, we see that

$$(I-V)T_n x = \sum_{k=0}^{\infty} A_{nk}(V^k - V^{k+1})x \to \theta$$

by (ii) and (iii). Taking limits (through subsequences, if necessary) we see

$$(I-V)x_0 = \theta, \quad \text{i. e.,} \quad x_0 \in N.$$

Next we claim that $x - x_0 \in \bar{R}$. First we can find a continuous linear functional $\hat{x}_0$ on E such that $\langle \bar{R}, \hat{x}_0\rangle = 0$; in addition, if $x - x_0 \notin \bar{R}$, we can choose $\hat{x}_0$ such that $\langle x - x_0, \hat{x}_0\rangle = 1$. Since $\hat{x}_0$ is orthogonal to $\bar{R}$, $\langle (I-V)x, \hat{x}_0\rangle = 0$; i. e., $\langle x, \hat{x}_0\rangle = \langle Vx, \hat{x}_0\rangle$ for each x in E. Using again the commutativity of V with each A_{nk} we now obtain

$$\langle T_n x, \hat{x}_0\rangle = \Big\langle \sum_{k=0}^{\infty} A_{nk}V^k x, \hat{x}_0\Big\rangle = \Big\langle \sum_{k=0}^{\infty} V^k A_{nk}x, \hat{x}_0\Big\rangle$$

$$= \sum_{k=0}^{\infty} \langle V^k A_{nk}x, \hat{x}_0\rangle, \quad \text{by continuity of } \hat{x}_0,$$

$$= \sum_{k=0}^{\infty} \langle A_{nk}x, \hat{x}_0\rangle, \quad \text{since } \langle x, \hat{x}_0\rangle = \langle Vx, \hat{x}_0\rangle,$$

$$= \Big\langle \sum_{k=0}^{\infty} A_{nk}x, \hat{x}_0\Big\rangle, \quad \text{again by the continuity of } \hat{x}_0.$$

Letting $n \to \infty$ (if necessary through the subsequence) and using the fact that the matrix A is Toeplitz, we get that

$$\langle x_0, \hat{x}_0\rangle = \langle x, \hat{x}_0\rangle.$$

Thus $\langle x - x_0, \hat{x}_0\rangle = 0$, setting up a contradiction. Therefore $x - x_0 \in \bar{R}$.

Also, that $N \cap \bar{R} = \{\theta\}$ follows from the fact that for each $x \in N$, $T_n x \to x$, while for each x in $\bar{R}$, $T_n x \to \theta$. Finally, since $x = (x - x_0) + x_0$ it follows that E is the algebraic direct sum of N and $\bar{R}$ and this completes the proof of the theorem.

Remark. One can easily see that E is actually the topological direct sum of N and $\bar{R}$.

§ 4. Two special cases of the matrix A. In this section we shall determine increasing sequences $\{\gamma_n\}$ satisfying condition (ii) of Theorem 1 for two general classes of matrices.

First we observe that $\{\gamma_n\}$ has been determined for Cesàro, Abel and Borel methods in two different contexts in summability theory by Cooke [2] and Lorentz [6] and so we only quote these. For (C, r), $r \geqslant 1$, $\gamma_n = o(n)$ and for $r < 1$, $\gamma_n = o(n^r)$. For the Abel method $\gamma_n = o(n)$ while for the Borel method $\gamma_n = o(\sqrt{n})$.

We shall consider now suitably restricted operator valued Hausdorff methods and scalar valued quasi-Hausdorff methods.

(1) Hausdorff method (H, μ_n). The matrix $H = (H_{nk})$ of this method is given by the lower-semi-matrix $H_{nk} = \binom{n}{k} \Delta^{n-k}\mu_k$, $0 \leqslant k \leqslant n = 0, 1, 2, \ldots$ where μ_n are continuous linear operators on E into E and $\Delta^0\mu_k = \mu_k$, $\Delta^1\mu_k = \mu_k - \mu_{k+1}$ and $\Delta^n\mu_k = \Delta(\Delta^{n-1}\mu_k)$. We assume that H is Toeplitz so that by a known theorem of Kurtz and Tucker [4] there exists a function χ on $[0, 1]$ into $B^+(E, E)$ (¹) such that χ is of bounded semi-variation in the sense of Gowurin and $\mu_n x = \int_0^1 t^n x\, d\chi(t)$, $n = 0, 1, 2, \ldots$, $\chi(0) = 0$ and $\chi(1) = I$ and $\chi(t)x$ is continuous at $t = 0$ for all $x \in E$.

Suppose now that χ is of finite variation (in the usual sense) and let V_0^t denote the variation of χ in $[0, t]$. Then we can show that if $\int_0^1 \frac{dV_0^t}{\sqrt{t(1-t)}} < \infty$ and $\gamma_n = o(\sqrt{n})$, then the corresponding Hausdorff method satisfies condition (ii) of the theorem. The proof basically rests on the known estimate (see Lorentz [6]) that for $0 < t < 1$

$$\sum_{k=0}^{n} |p_{nk}(t) - p_{n,k+1}(t)| \leqslant \frac{A}{\sqrt{n}\,\sqrt{t(1-t)}},$$

where

$$p_{nk}(t) = \binom{k}{n}(1-t)^{n-k}t^k, \quad 0 \leqslant k \leqslant n, \quad p_{n,n+1}(t) = 0$$

and A is an absolute constant independent of n and t.

(2) Quasi-Hausdorff method (H^*, μ_n). The matrix $H^* = (h^*_{nk})$ of this method is given by the upper-semi-matrix

$$h^*_{nk} = \binom{k}{n} \Delta^{k-n}\mu_{n+1}, \quad k \geqslant n,$$

where the μ_n are scalars. By a known result of Ramanujan [7] the matrix H^* is Toeplitz if and only if $\mu_n = \int_0^1 t^n\, dm(t)$, $n = 0, 1, \ldots$ where $m(t)$ is a scalar function of bounded variation in $[0, 1]$, $m(0) = 0$ and $m(1) - m(0+) = 1$. The integral considered is in the Lebesgue–Stieltjes sense.

(¹) For notations and terminology, see Tucker [8].

Let $q_{nk}(t) = \binom{k}{n}(1-t)^{k-n}t^{n+1}$, $k \geqslant n$. Then it follows from a known theorem (see for instance Hardy [3], Theorem 139) that $q_{nk}(t) \leqslant \frac{At}{\sqrt{n}\,\sqrt{1-t}}$ where A is an absolute constant, independent of n and t. Suppose now that $\int_0^1 \frac{d|m|}{\sqrt{1-t}} < \infty$ and that $\gamma_n = o(\sqrt{n})$. Using the above estimate on $q_{nk}(t)$ and the fact that $\gamma_n = o(\sqrt{n})$ one can prove that

$$\text{(a)} \qquad \sum_{k=n}^{\infty} |q_{nk}(t) - q_{n,k+1}(t)|\,\gamma_k \leqslant \frac{C}{\sqrt{1-t}}, \qquad 0 \leqslant t < 1$$

and that for every fixed t, $0 \leqslant t < 1$,

$$\text{(b)} \qquad \sum_{k=n}^{\infty} |q_{nk}(t) - q_{n,k+1}(t)|\,\gamma_k \to 0 \quad \text{as} \quad n \to \infty.$$

Then by an application of Lebesgue's dominated convergence theorem it can be shown that the matrix H^* satisfies condition (ii) of the theorem.

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