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On weighted H^p spaces

by

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Abstract. For $p > 1$ there is a well known isomorphism between the space of harmonic functions $F(x, y)$ in the half space $y > 0$ of R^{n+1} normed by $\sup\{\|F(\cdot, y)\|_p: y > 0\}$ and L^p associating to F its boundary value function $F(\cdot, 0)$ with a substitute result in case $p = 1$. The present paper is concerned with a generalization of this result to weighted L^p norms and more generally weighted Lorentz norms.

To obtain generalizations of corresponding results for H^p spaces of systems of conjugate harmonic functions (in the sense of Stein and Weiss) a criterion for harmonic majorization of positive subharmonic functions in a half space is proved. By means of Kelvin's transformation by reciprocal radii an isomorphism is established between spaces of subharmonic functions in a half space considered earlier and spaces of subharmonic functions in a ball with bounded weighted L^p norms on concentric spheres.

0. Introduction. The main concern of the present paper will be with harmonic functions in the half space

$$R_+^{n+1} = \{(x, y):\ x \in R^n, y > 0\}$$

of R^{n+1}. As usual for $(x, y) \in R^{n+1}$ define $|(x, y)|^2 = \sum_{i=1}^n x_i^2 + y^2$. The Poisson kernel for R_+^{n+1} is

$$P(x, y) = c_n^{-1} y(|x|^2 + y^2)^{-(n+1)/2},$$

where $c_n = 1/2\,\omega_{n+1} = \pi^{(n+1)/2}\big[\Gamma\big((n+1)/2\big)\big]^{-1}$. If $f(1+|\cdot|)^{-(n+1)}$ is integrable on R^n the Poisson integral $P * f$ in R_+^{n+1} is defined by

$$P * f(x, y) = P(\cdot, y) * f(x).$$

It is well known that for $1 < p \leqslant \infty$ the mapping $f \to P * f$ establishes an isomorphism between $L^p(R^n)$ and the space of harmonic functions F in R_+^{n+1} subject to

$$\sup_{y>0} \|F(\cdot, y)\|_p < \infty \tag{1}$$

and normed by the left-hand side of (1). If $p = 1$ this is an isomorphism between the space of totally finite regular Borel measures $\mathscr{M}^1$ and the harmonic functions satisfying (1) (see, e.g., [15], [29]). Stein and Weiss in [29] proved that the pth power of the length of the gradient of a harmonic function in R^n is subharmonic for $p \geqslant (n-2)/(n-1)$. By means of this result they generalized those results about H^p spaces of holomorphic functions F in a half plane which can be proved by harmonic majorization of $|F|^p$ to systems of conjugate harmonic functions F, i.e., gradients of harmonic functions satisfying (1) for $p \geqslant (n-1)/n$.

Recall the definition of Lorentz spaces, e.g., in [17]. For f measurable set

$$\|f\|_{pq}^* = \Big(q/p\int_0^\infty (f^*(t)t^{1/p})^q\,dt/t\Big)^{1/q}$$

where f^* denotes the decreasing rearrangement of f on $(0,\infty)$. $\|f\|_{pq} = \|f^{**}\|_{pq}^*$, where

$$f^{**}(t) = f^{**}(t,r) = \Big(t^{-1}\int_0^t (f^*(s))^r\,ds/s\Big)^{1/r}, \qquad 0<r\leqslant 1,\ r\leqslant q,\ r<p.$$

If w denotes a non-negative measurable function on R^n define

$$\|f\|_{pq,w} = \|fw\|_{pq}$$

and $L_w^{pq} = \{f\colon \|f\|_{pq,w}<\infty\}$. Also let $L_w^p = L_w^{pp}$. (On one occasion it will be convenient to denote L_w^{pq} by $L(p,q,w)$.) In case $w(x) = \omega(|x|)$ or more particularly $w(x) = |x|^a$ the notation $\|.\|_{pq,\omega} L_\omega^{pq}$ or $\|.\|_{pq,a}$, L_a^{pq} respectively will be used. If w is a continuous function which does not vanish except, possibly, at the origin define

$$w\mathscr{M}^1(R^n) = \{\nu\colon \nu = w\mu,\ \mu\in\mathscr{M}^1(R^n)\}, \qquad \|\nu\|[w\mathscr{M}^1] = \|w^{-1}\nu\|,$$

where the norm of a measure $\mu\in\mathscr{M}^1$ is its total variation $|\mu|(R^n)$. Hence if $w(0)=0$ and $\nu\in w\mathscr{M}^1$ then $\nu(\{0\})=0$.

It is well known that the continuity of singular and fractional integral operators between L^p spaces generalizes to continuity between the weighted L^p spaces L_a^p for $-n/p<a<n/p'$, where $1/p+1/p'=1$, (see [27], [28]).

These facts lead to the consideration of (systems of conjugate) harmonic functions F in R_+^{n+1} subject to $\sup_{y>0}\|F(\cdot,y)\|_{p,a}<\infty$. In fact the norm $\|F(\cdot,y)\|_{p,a}$ will be allowed to increase linearly in y as $y\to\infty$ and in place of $|x|^a$ more general weight functions will be considered. In order to obtain more precise results for singular and fractional integrals weighted Lorentz norms defined above will be used.

Let p_0 denote the projection on the y-axis, i.e., $p_0(x,y)=y$ for any $(x,y)\in R^{n+1}$. First conditions are given under which

$$\sup_{y>0}\big((1+y)^{-1}\|F(\cdot,y)\|_{pq,w}\big) = M<\infty \tag{2}$$

implies that F is the sum of the Poisson integral of a function in L_w^{pq} and a constant multiple of p_0 (Proposition 1). Conversely it will be shown that if w is radial, $w(x)=\omega(|x|)$, then $f\in L_\omega^{pq}$ implies that $F=P*f$ satisfies (2) provided there are $a<n/p'$, $\beta<n/p$, $1<p\leqslant\infty$ such that $\omega(\tau)\min(\tau^{-a},\tau^{-a-1})\downarrow$ (= decreasing) and $\omega(\tau)\max(\tau^{\beta},\tau^{\beta+1})\uparrow$ (= increasing) and $0<q\leqslant\infty$ with less general results in case $a=n/p'$, $\beta=n/p$ or $p=1$. There are similar results for the Hardy–Littlewood maximal function $M^\eta(f)$ defined by

$$M^\eta(f)(x) = \sup_{\varepsilon\leqslant\eta}\varepsilon^{-n}\Big|\int_{|t-x|\leqslant\varepsilon} f(t)\,dt\Big|$$

for $\eta>0$. These will yield non-tangential boundedness of the Poisson integral F by a function in the same space as f or a related larger one in case at least one of p, a, β is at an end-point of its permissible range. The results discussed so far imply that $f\to P*f\big((f,z)\to P*f+zp_0\big)$ is a topological isomorphism between L_ω^{pq} ($L_\omega^{pq}\oplus C$ or $\omega^{-1}\mathscr{M}^1\oplus C$) and the space of harmonic functions in R_+^{n+1} normed by the left-hand side of (2).

To prove harmonic majorization of certain subharmonic functions in [18] and [29] use is made of the fact that if s is a non-negative subharmonic function in R_+^{n+1} and $\sup_{y>0}\|s(\cdot,y)\|_p<\infty$ then $s\to 0$ as $y\to\infty$ or $|x|\to\infty$ while y is bounded below by an arbitrary positive number. This does not appear to carry over readily in required generality to non-negative subharmonic functions satisfying $\sup_{y>0}\|s(\cdot,y)\|_{p,w}<\infty$. In the theory of functions of one complex variable, however, there is a well known method of proving harmonic majorization in a half plane by use of the formula for the solution of the Dirichlet problem for a semi-disk and boundary values vanishing on the diameter. This can be extended to R^{n+1} and is used to prove Proposition 3, possibly, the main result of this paper. It gives a criterion for harmonic majorization of subharmonic functions in R_+^{n+1} and also asserts that the least harmonic majorant is the sum of the weak limit of $s(\cdot,y)$ as $y\to 0$ and a constant multiple of p_0. This then permits extension of most of the results of [29] on H^p spaces to H^p spaces with certain radial weight functions. In particular the range $-n/p<a<n/p'$ valid for continuity on L^p of fractional and singular integral operators is enlarged to $-n/p<a<n\big(n/(n-1)-1/p\big)$ for H_a^p (H^p with weight function $|x|^a$). This is of similar significance for fractional integrals of functions in $L_{n/p'}^{p1}$ (see Proposition E) as in the well known case $p=1$, $w=1$ (Theorem H of [29]). Lastly Kelvin's transformation is used to relate some of the sets of subharmonic functions in R_+^{n+1} considered in the preceding sections to certain sets of subharmonic functions in the unit ball of R^{n+1} (Proposition 4).

The Banach space of continuous functions φ such that $\lim_{|x|\to\infty}\varphi(x)=\varphi(\infty)$ exists, that is, the space of functions which are restrictions to R^n of continuous functions on the one-point-compactification R^{n*} of R^n (with the topology of uniform convergence) will be denoted $C(R^{n*})$. Its dual, consisting of the bounded measures (or in another terminology, totally finite regular Borel measures) on R^{n*}, i.e., of the functionals

$$\mu+z\varepsilon_\infty:\ \varphi\to\int\varphi(x)\mu(dx)+z\lim_{|x|\to\infty}\varphi(x),$$

where $\mu\in\mathcal{M}^1(R^n)$, $z\in C$ will be denoted $\mathcal{M}^1(R^{n*})$ ($\mathcal{M}^1(R^{n*})\cong\mathcal{M}^1(R^n)\oplus C$). C_p, e.g., will be used to denote a constant not necessarily the same at each occurrence depending on p and possibly n.

1. Harmonic functions. The following generalizes Lemma 3.6 of [29].

PROPOSITION 1. *Let B be a Banach space such that $(1+|\cdot|)^{-n-1}C(R^{n*})$ is (continuously) contained in B so that its dual B' may be taken to be contained in $(1+|\cdot|)^{n+1}\mathcal{M}^1(R^{n*})$. Suppose F is a harmonic function on R_+^{n+1} such that*

(3) $$\sup_{y>0}\,[(1+y)^{-1}\|F(\cdot,y)\|_{B'}]=M<\infty.$$

Then there exist $\mu\in B'$ and a (complex) number δ such that

(4) $$F(x,y)=P(\cdot,y)*\mu(x)+\delta y$$

and

$$\|\mu\|_{B'}\leqslant\liminf_{y\to0}\|F(\cdot,y)\|_{B'},\qquad\delta=\lim_{y\to\infty}y^{-1}F(\cdot,y),$$

$\delta=0$ if $\lim(y^{-1}\|F(\cdot,y)\|_{B'}=0$. In case $B'=(1+|\cdot|)^{n+1}\mathcal{M}^1(R^{n*})$ it can be assumed that $\mu\in(1+|\cdot|)^{n+1}\mathcal{M}^1(R^n)$. Also at the boundary $y=0$ F tends non-tangentially to the absolutely continuous part f, say, of μ a.e.

COROLLARY. *If F is harmonic in R_+^{n+1}, satisfies (2), where $1<p\leqslant\infty$ $1\leqslant q\leqslant\infty$ or $p=q=1$ and*

(5) $$\|w^{-1}(1+|\cdot|)^{-n-1}\|_{p'q'}<\infty$$

and in case $p=1$ w^{-1} is continuous on R^n then (4) holds with $\mu=f\in L_w^{pq}$ if $p>1$, while $\mu\in w^{-1}\mathcal{M}^1(R^n)$ if $p=1$.

Proof. The hypotheses imply

$$\|F(\cdot,y)(1+|\cdot|)^{-n-1}\|_1\leqslant CM(1+y),\qquad(C=C_B)$$

$$|F(x,y)|\leqslant C\max(y^{-n-1},1)\int\limits_{|t-x|^2+|s-y|^2\leqslant\min(y^2,1)}|F(t,s)|\,dt\,ds$$

$$\leqslant C\max(y^{-n-1},1)\int\limits_{y-\min(y,1)}^{y+\min(y,1)}\int\limits_{|t-x|\leqslant\min(y,1)}|F(t,s)|\,dt\,ds.$$

Hence

(6) $$|F(x,y)|\leqslant C\max(y^{-n},1)(1+|x|)^{n+1}\sup_{|s-y|\leqslant\min(y,1)}\|F(\cdot,s)\|_1$$
$$\leqslant CM\max(y^{-n},1)(1+|x|)^{n+1}(1+y).$$

For $y>\eta>0$ let $W_\eta(x,y)=P(\cdot,y-\eta)*F(\cdot,\eta)(x)$ so that

$$|W_\eta(x,y)|\leqslant C\Big(\int\limits_{|t|\leqslant\min((y-\eta)^{-1},1)}+\int\limits_{|t|>\min((y-\eta)^{-1},1)}\Big)P(t,y-\eta)|F(x-t,\eta)|\,dt.$$

Hence

(7) $$|W_\eta(x,y)|\leqslant CM\Big[\max(\eta^{-n},1),\min\big((y-\eta)^{-n}\big)(1+|x|)^{n+1}(1+y)+$$
$$+(y-\eta)\sup_t\Big(\frac{1+|x-t|}{1+|t|}\Big)^{n+1}\Big]$$

since $(y-\eta)+(y-\eta)^{-1}\geqslant2$. Also by dominated convergence the second term in the sum preceding (7) is $o(y)$ as $y\to\infty$. Consequently for any x

(8) $$\lim_{y\to\infty}y^{-1}W_\eta(x,y)=0.$$

It is easy to see that for $\varepsilon>0$

$$\sup_{\varepsilon<y<\varepsilon^{-1},|x|<\varepsilon}|(\partial/\partial x)^a(\partial/\partial y)^kP(x-\cdot,y)|(1+|\cdot|)^{n+1}\in L^\infty.$$

Thus by dominated convergence $W_\eta(x,y)$ is a harmonic function for $y>\eta$. If $f(1+|\cdot|)^{-n-1}$ is integrable and continuous in an open set Ω of R^n and if K is compact and contained in Ω then there exist a continuous function g supported in Ω and function h such that $f=g+h$ and h vanishes in a compact neighborhood of K. Therefore $P(\cdot,y)*g\to g$ uniformly, while $P(\cdot,y)*h\to0$ uniformly in K by dominated convergence. Thus W_η can be extended to a continuous function for $y\geqslant\eta$ by $W_\eta(x,\eta)=F(x,\eta)$. By the reflection principle the function W^* defined by

$$W^*(x,y)=F(x,y+\eta)-W_\eta(x,y+\eta)$$

for $y\geqslant0$ and $W^*(x,y)=-W^*(x,-y)$ for $y\leqslant0$ is harmonic in R_+^{n+1}. Furthermore by (6), and (7) $W^*(x,y)=O(|(x,y)|^{n+2})$ as $|(x,y)|\to\infty$. Now by the Poisson integral formula for a sphere:

$$W^*(x,y)=\omega_{n+1}^{-1}(1-a^{-2}|(x,y)|^2)\int\limits_{S^n}\frac{W^*(a\sigma)}{|\sigma-a^{-1}(x,y)|^{n+1}}d\sigma,$$

where $S^n=\{(x,y):\ |(x,y)|=1\}$. By differentiation it follows that

$$\sup_{|(x,y)|\leqslant a/2}|D^\beta W^*(x,y)|\leqslant C_\beta a^{-|\beta|}\max_{\sigma\in S^n}|W^*(a\sigma)|,$$

where β is any multi-index with $(n+1)$ components. If $|\beta|=\sum\beta_i>n+2$ this results in

$$|D^\beta W^*(x,y)|\leqslant C_\beta\liminf_{a\to\infty}a^{-|\beta|+n+2}=0.$$

Thus $W^*(x, y) = \sum_{k=1}^{n+2} y^k P_{n+2-k}(x)$ where $P_k(x)$ is a polynomial in $x_1, \ldots, x_n$ of degree k at most. As $W^*(x, y) = O(|y|)$ for $|y| \to \infty$, for all x, P_k must vanish for $2 \leqslant k \leqslant n$, i.e., $W^*(x, y) = yP_{n+1}(x)$. Now $\|(1+|\cdot|)^{-n-1}P_{n+1}\| < \infty$ requires $P_{n+1} = \text{const.} = \delta(\eta)$, say. By (6) and (8)

$$\delta(\eta) = \lim_{y\to\infty} y^{-1}|F(0, y)| \leqslant CM.$$

By hypothesis $P(\cdot - t, y) \in B$ for any $(t, y) \in R_+^{n+1}$. Since the family $\{F(\cdot, \eta): 0 < \eta \leqslant 1\}$ is bounded in B' it is relatively compact with respect to the weak topology of the pairing (B', B) so there is a sequence $\eta_k \to 0$ and $\mu \in B'$ such that $F(\cdot, \eta_k) \to \mu$ weakly and also δ such that $\delta(\eta_k) \to \delta$ as $k \to \infty$ and $|\delta| \leqslant CM$. Hence

$$F(x, y) = \lim_{k\to\infty} F(x, y+\eta_k) = \lim_{k\to\infty} \big(P(\cdot, y) * F(\cdot, \eta_k)(x) + \delta(\eta_k)y\big)$$
$$= P(\cdot, y) * \mu(x) + \delta y.$$

In case $B' = (1+|\cdot|)^{n+1}\mathscr{M}^1(R^{n*})$ the notation will now be changed from μ to μ^* and δ to γ_1. There are $\mu \in (1+|\cdot|)^{n+1}\mathscr{M}^1(R^n)$, $\gamma_0 \in C$ such that $\mu^* = \mu + 1/2\, \omega_{n+1}\gamma_0|\cdot|^{n+1}\varepsilon_\infty$ let $\delta = \gamma_0 + \gamma_1$ then

$$F(x, y) = P(\cdot, y) * \mu(x) + \gamma_0 y \lim_{t\to\infty} [(1+|t|)^{n+1}/(y^2+|x-t|^2)^{(n+1)/2}] + \gamma_1 y$$
$$= P(\cdot, y) * \mu(x) + \delta y.$$

Clearly $\|\mu\|_{B'} \leqslant \|\mu^*\|_{B'}$. (It follows from (4) or Lemma 3 below that in fact $\gamma_0 = 0$, $\mu = \mu^*$). Also for $\mu(1+|\cdot|)^{-n-1} \in \mathscr{M}^1$ it is well known that $P * \mu$ tends to the absolutely continuous part of μ non-tangentially a.e. (cf. [15] Proposition 2.1).

The corollary follows from $L^{pq} = (L^{p'q'})'$ for $1 < p < \infty$, $q \geqslant 1$ (see [12]).

(6) clearly holds for any subharmonic function F satisfying (3). In the special case when $w(x) = |x|^a$ and $s(x, y) \geqslant 0$ is subharmonic in R_+^{n+1} and such that

$$\sup_{y>0} \|s(\cdot, y)\|_{pq,a} = M < \infty$$

more precise estimates needed below can be given. Let $B^n(x, y) = \{t \in R^n: |t - x| < y\}$. Then

$$s(x, y) \leqslant Cy^{-n-1}\int_0^{2y} \|s(\cdot, t)\|_{pq,a}\, dt\, \big\||\cdot|^{-a}\chi_{B^n(x,y)}\big\|_{p'q'}.$$

If $a \geqslant 0$

$$\||\cdot|^{-a}\chi_{B^n(x,y)}\|_{p'q'} \leqslant \||\cdot|^{-a}\chi_{B^n(0,y)}\|_{p'q'} = C\Big(\int_0^{Cy^n} t^{(-a/n+1/p')q'-1}dt\Big)^{1/q'} = Cy^{-a+n/p'}.$$

Also for $|x| > y$

$$\||\cdot|^{-a}\chi_{B^n(x,y)}\|_{p'q'} \leqslant C(|x|-y)^{-a}\|\chi_{B^n(0,y)}\|_{p'q'} \leqslant C(|x|-y)^{-a}y^{n/p'}$$

hence

$$\||\cdot|^{-a}\chi_{B^n(x,y)}\|_{p'q'} \leqslant Cy^{n/p'}(|x|+y)^{-a}$$

and so

(9) $$s(x, y) \leqslant Cy^{-n/p}(|x|+y)^{-a}$$

for $0 \leqslant a < n/p'$ and $a = n/p'$, $q = 1$. On the other hand if $a \leqslant 0$, then

$$s(x, y) \leqslant Cy^{-n-1}(|x|+y)^{-a}\int_0^{2y} \|s(\cdot, t)\|_{pq,a}\,dt\,\|\chi_{B^n(0,y)}\|_{p'1}$$
$$\leqslant CMy^{-n/p}(|x|+y)^{-a}$$

($a < -n/p$ implies $s(0, y) = 0$ for all $y > 0$), i.e., (9) holds in this case likewise.

If $F = P * \mu$, where $\mu \in (1+|\cdot|)^{n+1}\mathscr{M}^1(R^n)$ then μ is absolutely continuous with respect to Lebesgue measure in an open set Ω of R^n iff for some (and hence all) $\eta > 0$ the family of measures $\{F(x, y)dx, 0 < y \leqslant \eta\}$ is uniformly (or equi-) absolutely continuous in Ω, in other words iff the family of functions $\{F(\cdot, y): 0 < y \leqslant \eta\}$ is uniformly locally integrable in Ω. The necessity could be proved by approximating μ in $L^1_{\text{loc}}(\Omega)$ by continuous functions of compact support. The sufficiency follows from the readily proved fact that μ is the weak limit of the measures $P(\cdot, y) * \mu$. (Let $\mathscr{K}(\Omega)$ denote the space of continuous functions whose support is contained in Ω. It follows from [2] bk. 6 chap. 3 sec. 2 no. 5 and chap. 5 sec. 5 no. 2 Theorem 2 c' that a family of measures $\{\mu_i\}$ is uniformly absolutely continuous in Ω iff for any non-negative $g \in \mathscr{K}(\Omega)$ and for any $\varepsilon > 0$ there is a $\delta > 0$ such that $h \in \mathscr{K}(\Omega)$, $|h| \leqslant g$ and $\int |h(x)|\,dx \leqslant \delta$ imply $|\int h d\mu_i| \leqslant \varepsilon$ for all i. Hence also $|\int h d\mu| < \varepsilon$ for any weak limit μ of the family $\{\mu_i\}$.

2. Lemmas on integral operators in weighted L^p spaces. It is well known that singular integral operators and the Hardy–Littlewood maximal operator

$$M(f)(x) = \sup_{\varepsilon>0} \varepsilon^{-n}\int_{|t|\leqslant\varepsilon} |f(x+t)|\,dt$$

preserve L^p_a if $-n/p < a < n/p'$ (see [27]). Let now $w(x) = \omega(|x|)$. It was proved by Chen in [6] that ω increasing and (a) $\omega(\tau)\tau^{-a}$ decreasing for some $a < n/p'$ or dually ω decreasing and (b) $\omega(\tau)\tau^{\beta}$ increasing for some $\beta < n/p$ implies that (c) M is bounded on L^p_ω. This result is generalized below to the simpler statement: (a) and (b) imply (c). While singular and fractional integral operators only take $L^{p1}_{n/p'}$, $L^{p1}_{-n/p}$ into $L^{q\infty}_{-\beta}$ for appropriate q, β Lemmas 2 and 3 say, in particular, that M, $P(\cdot, y)^*$ preserve $L^{p\infty}_{-n/p}$ $(p>1)$ and hence by duality $P(\cdot, y)^*$ also preserves $L^{p1}_{n/p'}$. Also for Poisson integrals the restrictions on ω necessary in the case of singular integrals can be relaxed at infinity.

LEMMA 1. *Suppose H is measurable on $R^n\times R^n$, $0\leqslant H(x,t)\leqslant A|x-t|^{-\lambda}$ and T defined by*

$$T(f)(x)=\int H(x,t)f(t)\,dt$$

satisfies

$$\|Tf\|_{rs}\leqslant A\|f\|_{pq}, \tag{10}$$

where

$$1/p'+1/r=\lambda/n\geqslant 0,\qquad s\geqslant r. \tag{11}$$

If $1<p, r<\infty$ and

$$\omega(\tau)\tau^{-\alpha}\downarrow,\qquad \omega(\tau)\tau^{\beta}\uparrow\quad \text{for some } \alpha<n/p', \beta<n/r \tag{12}$$

then

$$\|Tf\|_{rs,\omega}\leqslant CA\|f\|_{pq,\omega}\qquad (C=C(p,q,r,s,\alpha,\beta)).$$

If (12) *holds with $\alpha=n/p'$, $\beta=n/r$ it is still true that $\|Tf\|_{r\infty,\omega}\leqslant C(A+1)\|f\|_{p1,\omega}$ $(1\leqslant p<\infty)$.*

Proof. Define $K(x,t)=H(x,t)\omega(|x|)\omega(|t|)^{-1}$ and let χ denote the characteristic function of the interval $(0,1)$. Set $K_1(x,t)=K(x,t)\times\chi(2|x|^{-1}|t|)\,K_3(x,t)=K(x,t)\chi(2|x||t|^{-1})$, $K=\sum_{i=1}^{3}K_i$. The K_i give rise to operators S_i:

$$S_if(x)=\int K_i(x,t)f(t)\,dt.$$

Note that

$$K_1(x,t)\leqslant 2^{\lambda}A|x|^{-\lambda}\omega(|x|)\omega(|t|)^{-1}\chi(|x|^{-1}|t|)=2^{\lambda}K_1^*(x,t),\qquad \text{say},$$

$$K_3(x,t)\leqslant 2^{\lambda}A|t|^{-\lambda}\omega(|x|)\omega(|t|)^{-1}\chi(|x||t|^{-1})=2^{\lambda}AK_3^*(x,t),\qquad \text{say}.$$

It follows from (12) that

$$\sup\{\omega(\tau_1)/\omega(\tau_2):\ 1/2\leqslant\tau_1/\tau_2\leqslant 2\}\leqslant C<\infty. \tag{13}$$

Hence since $0\leqslant H(x,t)$ it follows from (10) that

$$\|S_2f\|_{rs}\leqslant CA\|f\|_{pq}. \tag{14}$$

If on the other hand for $i=1,3$ there holds one of

$$\|\varphi_1\|_{p'q'}\leqslant B,\quad \text{where}\quad \varphi_1(t)=\|K_i^*(\cdot,t)\|_{rs}, \tag{15}$$

$$\|\varphi_2\|_{rs}\leqslant B,\quad \text{where}\quad \varphi_2(x)=\|K_i^*(x,\cdot)\|_{p'q} \tag{16}$$

and the exponents are such that at most $\|\cdot\|_{rs}$ in (16) is not necessarily a norm then

$$\|S_if\|_{rs}\leqslant CB\|f\|_{pq}\quad \text{for } i=1,3.$$

This assertion is a fairly obvious generalization of well known results for L^p spaces (see e.g., [30], Lemma 2). In the present case

$$\varphi_1(t)=\|K_1^*(\cdot,t)\|_{r\infty}\leqslant CA\sup_{|x|\geqslant|t|}\omega(|x|)|x|^{n/r-\lambda}\omega(|t|)^{-1}$$

$$\|\varphi_1\|_{p'\infty}\leqslant CA\sup_{\tau\leqslant\sigma}\omega(\sigma)\sigma^{n/r-\lambda}\omega(\tau)^{-1}\tau^{n/p'}.$$

Thus if

$$\omega(\tau)\tau^{-n/p'}\downarrow \tag{17}$$

then (15) and similarly (16) are satisfied for $i=1$. Analogously

$$\|\varphi_2\|_{r\infty}\leqslant CA\sup_{\sigma\leqslant\tau}\omega(\sigma)\sigma^{n/r}\omega(\tau)^{-1}t^{-n/r}\qquad \text{if } \varphi_2(x)=\|K_3^*(x,\cdot)\|_{p'\infty}$$

that is, (16) (and (15) likewise) holds for $i=3$ if

$$\omega(\tau)\tau^{n/r}\uparrow. \tag{18}$$

By the Marcinkiewicz interpolation theorem for Lorentz spaces (see [17]) (and choice of p_0, p_1 close to p and such that $p_0<p<p_1$) it follows that S_1 and S_3 satisfy $\|S_if\|_{rs}\leqslant CA\|f\|_{pq}$. Together with (14) this implies Lemma 1.

Remarks. It follows similarly from (15), (16) that in case $p=\infty$ and

$$\omega(\tau)\tau^{-n}\int_0^{\tau}\omega(\sigma)^{-1}\sigma^{n-1}\,d\sigma\leqslant C$$

(in particular if the first condition of (12) holds for some $\alpha<n$) and the second condition of (12) holds for some $\beta<0$ then $\|Tf\|_{\infty,\omega}\leqslant CA(1+|\beta|^{-1})\|f\|_{\infty,\omega}$.

Lemma 1 applies to fractional integration where $H(x,t)=|x-t|^{-\lambda}$. In the case of singular integrals with kernels bounded on the unit sphere the operator S_2 has to be dealt with differently (cf. [27]). Lemma 1 applies to M for if $\varepsilon(x)$ is a positive function $\chi\big(\varepsilon(x)^{-1}|x-t|\big)=0$ unless $|x-t|<\varepsilon(x)$ so,

$$\varepsilon(x)^{-n}\chi\big(\varepsilon(x)^{-1}|x-t|\big)=|x-t|^{-n}.$$

Lemma 2 below makes a stronger assertion in case $\beta=n/p$, $p>1$.

In the case of fractional integration two different weights $\omega(|t|)$ and $\omega_1(|x|)=\omega(|x|)|x|^{-\varrho}$, $\varrho\geqslant 0$ may be considered, then $K(x,t)=|x-t|^{-\lambda}\omega_1(|x|)\omega(|t|)^{-1}$. If $1/2\leqslant|t||x|^{-1}\leqslant 2$ then $|x-t|\leqslant|x|+|t|\leqslant 3|t|$ and so

$$K_2(x,t)\leqslant C|x-t|^{-\lambda}\omega_1(|x|)\omega(|t|)^{-1}\leqslant C|x-t|^{-\lambda-\varrho}.$$

Hence $1/r+1/p' = (\lambda+\varrho)/n$ is sufficient for (14) while in order that S_1, S_3 be of restricted weak type (p, q) (i.e., bounded from L^{p1} to $L^{q\infty}$) it is sufficient that

$$\sup_{\sigma\geqslant\tau} \omega_1(\sigma)\sigma^{n/r-\lambda}\omega(\tau)^{-1}\tau^{n/p'} < \infty$$

hence (17) along with (18) (proof similar) are sufficient. As before interpolation can be applied if (12) holds.

For the sake of clarity the following definitions are made. Let

$$\text{(19)} \qquad \omega(\tau)\min(\tau^{-a_0}, \tau^{-a_1})\downarrow, \qquad \omega(\tau)\max(\tau^{\beta_0}, \tau^{\beta_1})\uparrow$$

and then

$S^* = \{(p, \omega)\colon 1\leqslant p<\infty,$
(19) with $n/p' = a_0\leqslant a_1\leqslant n/p'+1,\ n/p = \beta_0\leqslant\beta_1\leqslant n/p+1\}$,

$S_0^{*1} = \{(p, \omega)\colon 1<p\leqslant\infty,$
(19) with $a_0<n/p',\ a_0\leqslant a_1\leqslant n/p'+1,\ \beta_0\leqslant n/p,\ \beta_0\leqslant\beta_1<n/p+1\}$,

$S_1^{*1} = \{(p, \omega)\colon 1\leqslant p<\infty,$
(19) with $a_0\leqslant n/p',\ a_0\leqslant a_1<n/p'+1,\ \beta_0<n/p,\ \beta\leqslant\beta_1\leqslant n/p+1\}$,

$S^{*2} = S_0^{*1}\cap S_1^{*1} = \{(p, \omega)\colon 1<p<\infty,$
(19) with $a_0<n/p',\ \beta_0<n/p,\ a_0\leqslant a_1<n/p'+1,\ \beta_0\leqslant\beta_1<n/p+1\}$.

S, S_0^1, S_1^1, S^2 are defined in the same way except that $a_0 = a_1 = a$, $\beta_0 = \beta_1 = \beta$. In order not to introduce more cumbersome notation some fixed a_i, β_i $(i = 0, 1)$ are supposed to be associated with each (p, ω). If (p, ω) is such that, e.g., ω satisfies the defining properties of S for τ in a subinterval I of R_+, write $(p, \omega)\in S$ in I. If, e.g., $(p, \tau^a)\in S^2$ write $(p, a)\in S^2$ so that $(p, a)\in S_0^1$ iff $1<p\leqslant\infty$, $-n/p\leqslant a<n/p'$, $(p, a)\in S_1^1$ iff $1\leqslant p<\infty$ $-n/p<a\leqslant n/p'$.

Lemma 2. *Suppose* $(p, \omega)\in S_0^1$ *then*

$$\|Mf\|_{p\infty,\omega}\leqslant C(n/p'-a)^{-1}\|f\|_{p\infty,\omega},$$

where a *is a possible exponent in the definition of* S_0^1.

Proof. For $\varepsilon(x)>0$ let $\psi_\varepsilon(x, t) = \varepsilon(x)^{-n}\chi\big(\varepsilon(x)^{-1}|x-t|\big)$. It will be sufficient to prove that the integral operator defined by

$$f\to\int\psi_\varepsilon(\cdot, t)\omega(|x|)\omega(|t|)^{-1}f(t)\,dt$$

is bounded in $L^{p\infty}$ with a bound for its norm independent of the measurable function ε. Let $K_1(x, t) = \psi_\varepsilon(x, t)\omega(|x|)\omega(|t|)^{-1}$ if $|x|\geqslant\max\big(2\varepsilon(x), |t|\big)$ or $|t|\geqslant\max\big(2\varepsilon(x), |x|\big)$ and $K_1(x, t) = 0$ otherwise. Then if $K_1(x, t)\neq 0$ either $|t|\geqslant|x|-|x-t|\geqslant|x|/2$ hence $1\leqslant|x|\,|t|^{-1}\leqslant 2$ or $|x|\geqslant|t|-|x-t|\geqslant|t|/2$ hence $1/2\leqslant|x|\,|t|^{-1}\leqslant 1$. Thus by (13) which is a consequence of the hypotheses $K_1(x, t)\leqslant C\psi_\varepsilon(x, t)$.

Next let $K_2(x, t) = \psi_\varepsilon(x, t)\omega(|x|)\omega(|t|)^{-1}$ if $|t|\leqslant|x|\leqslant 2\varepsilon(x)$ and $K_3(x, t) = \psi_\varepsilon(x, t)\omega(|x|)\omega(|t|)^{-1}$ if $|x|\leqslant|t|\leqslant 2\varepsilon(x)$, $= 0$ otherwise. Then

$$\psi_\varepsilon(x, t)\omega(|x|)\omega(|t|)^{-1} = \sum_{i=1}^{3}K_i(x, t)$$

$$K_2(x, t)\leqslant 2^n|x|^{-n}\omega(|x|)\omega(|t|)^{-1}|t|^a|t|^{-a}\leqslant 2^n|x|^{-n+a}|t|^{-a}$$

for $|t|\leqslant|x|$, and $= 0$ otherwise. It follows that (16) holds for K_2 with $p = r$, $q = s = \infty$. $B = C(n/p'-a)^{-1}$. Moreover

$$\begin{aligned} K_3(x, t) &\leqslant \psi_\varepsilon(x, t)\omega(|x|)2^a\omega\big(2\varepsilon(x)\big)^{-1}\varepsilon(x)^a|t|^{-a}\\ &\leqslant 2^a\psi_\varepsilon(x, t)\big(2\varepsilon(x)\big)^{n/p}|x|^{-n/p}\varepsilon(x)^a|t|^{-a}\\ &= 2^{a+n/p}\varepsilon(x)^{a-n/p'}|x|^{-n/p}|t|^{-a}\chi\big(\varepsilon(x)^{-1}|x-t|\big)\end{aligned}$$

so

$$\|K_3(x, \cdot)\|_{p'1}\leqslant C\varepsilon(x)^{a-n/p'}|x|^{-n/p}\int\limits_{|t|\leqslant 2\varepsilon(x)}|t|^{-a-n/p}\,dt\leqslant C(n/p'-a)^{-1}|x|^{-n/p}.$$

Thus (16) holds for K_3 with $p = r$, $q = s = \infty$, $B = C(n/p'-a)^{-1}$. Hence if the operators S_i are defined as in the proof of Lemma 1, S_1 is bounded in $L^{p\infty}$ since M is, S_2 and S_3 are by the proof of Lemma 1.

Lemma 3.

$$\text{(20)} \qquad \|P(\cdot, y)*f\|_{rs,\omega}\leqslant C(y)\|f\|_{pq,\omega}$$

provided one of

(a) $(p, \omega)\in S^{*2}$, $q\leqslant s\leqslant\infty$, (b) $(p, \omega)\in S_1^{*1}$, $q = s = 1$,
(c) $(p, \omega)\in S_0^{*1}$, $q = s = \infty$, (d) $(p, \omega)\in S^*$, $q = 1$, $s = \infty$,

is satisfied. Furthermore in cases (a) *with* $q\geqslant 1$, (b), (c)

$$C(y) = C\bigg[\Big(\frac{n}{p'}+1-a_1\Big)^{-1}y^{\nu_1}+\Big(\frac{n}{p}-\beta_0\Big)^{-1}\big(1+\psi_1(y)\big)\bigg]^{1/q}\times$$
$$\times\bigg[\Big(\frac{n}{p}+1-\beta_1\Big)^{-1}y^{\nu_0}+\Big(\frac{n}{p'}-a_0\Big)^{-1}\big(1+\psi_0(y)\big)\bigg]^{1/q'},$$

where

$$\nu_1 = (a_1-n/p')^+, \qquad \nu_0 = (\beta_1-n/p)^+ \qquad (a^+ = \max(a, 0))$$

and

$$\psi_1(y) = y^{\beta_1-n/p}, \log^+y, (n/p-\beta_0)(n/p-\beta_1)^{-1}$$

according as $\beta_1 > n/p$, $= n/p$ *or* $< n/p$ *and analogously*

$$\psi_0(y) = y^{a_1-n/p'},\ \log^+ y,\ (n/p'-a_0)(n/p'-a_1)^{-1}$$

according as $a_1 > n/p'$, $= n/p'$, $< n/p'$. *In case* (a) *and* $0 < q \leqslant 1$

$$C(y) \leqslant C_{p,r,a,\beta}(1+y^{a_1-n/p'+\varepsilon}+y^{\beta_1-n/p+\varepsilon}) \quad \textit{for any} \quad \varepsilon > 0,$$

while in case (d)

$$C(y) \leqslant C_p(1+y^{\nu}) \textit{ where } \nu = \max(\nu_0, \nu_1).$$

If ω *is continuous and does not vanish in* $[0,1]$ *and* (p,ω) *satisfies* (a) *with* $q \geqslant 1$, (b), (c) *or* (d) *then* $C(y)$ *may be chosen so that* $\lim_{y\to 0} C(y) = 1$. *In any case if* (a) *is satisfied with* $q < \infty$ *and if* $f \in L^{pq}_\omega$ *then*

$$\lim_{y\to 0} \|P(\cdot, y)*f-f\|_{pq,\omega} = 0. \tag{21}$$

Proof. To establish (b) note that it may and will be assumed that $a_1 \geqslant n/p'$, since this does not alter the hypotheses nor the conclusion (all a_i, β_i are supposed to be non-negative). Set

$$K_1^*(x,t) = y(y+|x|)^{-n-1}\omega(|x|)\omega(|t|)^{-1}\chi(|x|^{-1}|t|),$$

$$K_3^*(x,t) = y(y+|x|)^{-n-1}\omega(|x|)\omega(|t|)^{-1}\chi(|x|\,|t|^{-1}).$$

Then

$$\|K_1^*(x,\cdot)\|_{p'\infty} \leqslant C\sup_{\tau\leqslant|x|}\frac{y\,\omega(|x|)}{(y+|x|)^{n+1}}\,\omega(\tau)^{-1}\tau^{n/p'}$$

$$\leqslant C\frac{y\,\omega(|x|)}{(y+|x|)^{n+1}}\sup_{\tau\leqslant|x|}\omega(\tau)^{-1}\max(\tau^{a_0},\tau^{a_1}) \leqslant Cy(y+|x|)^{-n-1}\max(|x|^{a_0}|x|^{a_1}).$$

Let $\varphi_1(x) = 1$ if $|x| \leqslant \min(1, y) = y_1$, say, $= 0$ otherwise, $\varphi_3(x) = 1$ if $|x| \geqslant \max(1, y_3) = y_3$, say, $= 0$ otherwise and $1 = \varphi_1+\varphi_2+\varphi_3$. Denote the double Lorentz norms defined as in (15), (16) by

$$\big\|\,\|K_i^*\|[L^{rs}(x)]\big\|[L^{p'q'}(t)],\quad \big\|\,\|K_i^*\|[L^{p'q'}(t)]\big\|[L^{rs}(x)]$$

respectively. Then

$$\big\|\,\|K_1^*\varphi_1\|[L^{p'\infty}(t)]\big\|[L^{p1}(x)] \leqslant Cy^{-n}y_1^{a_0}\int_0^{Cy_1^n}\tau^{1/p-1}\,d\tau \leqslant Cy_1^{a_0-n/p'} \leqslant C$$

$$\big\|\,\|K_1^*\varphi_3\|[L^{p'\infty}(t)]\big\|[L^{p1}(x)] \leqslant Cy_3^{a_1-n/p'}|B^n(0,y_3)|^{n/p}+y\int_{Cy_3^n}^{\infty}\tau^{(a_1-n-1)/n+1/p-1}\,d\tau$$

$$\leqslant C\left(\frac{n}{p'}+1-a_1\right)^{-1}y^{a_1-n/p'}.$$

Similarly $\big\|\,\|K_1^*\varphi_2\|[L^{p'\infty}(t)]\big\|[L^{p1}(x)] \leqslant C$ (if $y \geqslant 1$ this follows from the estimate for $|x| \leqslant y_1$, if $y \leqslant 1$ from that for $|x| > y_3$). Hence by addition

$$\big\|\,\|K_1^*\|[L^{p'\infty}(t)]\big\|[L^{p1}(x)] \leqslant C\big(1+(n/p'+1-a_1)^{-1}y^{a_1-n/p'}\big).$$

Next

$$\|K_3^*(x,\cdot)\|_{p'\infty} \leqslant Cy\,\omega(|x|)\sup_{\tau\geqslant|x|}\frac{\omega(\tau)^{-1}\tau^{n/p'}}{(y+\tau)^{n+1}}$$

$$\leqslant Cy\,\omega(|x|)\sup_{\tau\geqslant|x|}\frac{\max(\tau^{\beta_0},\tau^{\beta_1})\tau^{n/p'}}{\omega(\tau)\max(\tau^{\beta_0},\tau^{\beta_1})(y+\tau)^{n+1}}$$

so

$$\|K_3^*(x,\cdot)\|_{p'\infty} \leqslant Cy\min(|x|^{-\beta_0}, |x|^{-\beta_1})\sup_{\tau\geqslant|x|}\frac{\max(\tau^{\beta_0},\tau^{\beta_1})\tau^{n/p'}}{(y+\tau)^{n+1}}. \tag{22}$$

Since

$$\frac{d}{d\tau}\log\left[\frac{\tau^{\beta_1+n/p'}}{(y+\tau)^{n+1}}\right] = (\beta_1+n/p')\tau^{-1}+\frac{n+1}{y+\tau} \leqslant 0$$

$$\text{for } \tau \geqslant \frac{\beta_1+n/p'}{n/p+1-\beta^1}\,y = C(\beta_1,p)y,$$

say, which is $\geqslant C(\beta_0, p)y$ it follows that if $|x| \geqslant C(\beta_1,p)y$ then

$$\sup_{\tau\geqslant|x|}\frac{\max(\tau^{\beta_0},\tau^{\beta_1})\tau^{n/p'}}{(y+\tau)^{n+1}} \leqslant \frac{\max(|x|^{\beta_0},|x|^{\beta_1})|x|^{n/p'}}{(y+|x|)^{n+1}}. \tag{23}$$

while if $|x| \leqslant C(\beta_1,p)y$ then

$$\sup_{\tau\geqslant|x|}\frac{\max(\tau^{\beta_0},\tau^{\beta_1})\tau^{n/p'}}{(y+\tau)^{n+1}} \tag{24}$$

$$= \max\left(\sup_{\tau\geqslant|x|}\frac{\tau^{\beta_0+n/p'}}{(y+\tau)^{n+1}},\ \sup_{\tau\geqslant|x|}\frac{\tau^{\beta_1+n/p'}}{(y+\tau)^{n+1}}\right)$$

$$\leqslant \frac{C(\beta_0,p)^{\beta_0+n/p'}}{[1+C(\beta_0,p)]^{n+1}}\,y^{\beta_0-n/p-1}+\frac{C(\beta_1,p)^{\beta_1+n/p'}}{[1+C(\beta_1,p)]^{n+1}}\,y^{\beta_1-n/p-1}.$$

Let now $\lambda_1(x) = 1$ if $|x| \leqslant C(\beta_1,p)y$, $= 0$ otherwise and $1 = \lambda_1+\lambda_2$. Then if $C(\beta_1,p)y \leqslant 1$.

$$\|\min(|\cdot|^{-\beta_0}, |\cdot|^{-\beta_1})\lambda_1\|_{p1} \leqslant C(n/p-\beta_0)^{-1}[C(\beta_1,p)y]^{n/p-\beta_0} \tag{25}$$

while if $C(\beta_1,p)y \geqslant 1$ this is at most

$$\begin{cases} C\left[\left(\dfrac{n}{p}-\beta_0\right)^{-1}+\left(\beta_1-\dfrac{n}{p}\right)^{-1}\right], \\ C\left[\left(\dfrac{n}{p_0}-\beta_0\right)^{-1}+\log C(\beta_1,p)y\right], \\ C\left[\left(\dfrac{n}{p}-\beta_0\right)^{-1}+\left(\dfrac{n}{p}-\beta_1\right)^{-1}y^{n/p+1-\beta_1}\right] \end{cases} \tag{26}$$

according as $\beta_1 > n/p$, $= n/p$ or $< n/p$. If $C(\beta_1, p)y \leqslant 1$ it follows from (22)–(25) that

$$\big\| \|K_3^*\|[L^{p'\infty}(t)]\big\| [L^{p1}(x)] = M_3 \text{ (say)}$$

$$\leqslant C\left[\frac{C(\beta_0, p)^{\beta_0+n/p'}}{(1+C(\beta_0, p))^{n+1}} y^{\beta_0-n/p} + \frac{C(\beta_1, p)^{\beta_1+n/p'}}{(1+C(\beta_1, p))^{n+1}} y^{\beta_1-n/p}\right]\times$$

$$\times \|\min(|\cdot|^{-\beta_0}, |\cdot|^{-\beta_1})\lambda_1\|_{p1} + Cy\left\|\frac{|\cdot|^{n/p'}}{(y+|\cdot|)^{n+1}}\lambda_2\right\|_{p1}$$

$$\leqslant C\left(\frac{n}{p}-\beta_0\right)^{-1}\left(1+C(\beta_1, p)^{n+\beta_1-\beta_0}(1+C(\beta_1, p))^{-n-1}y^{\beta_1-\beta_0}\right)$$

while if $C(\beta_1, p)y \geqslant 1$ then by (26)

$$M_3 \leqslant C\left[\left(\frac{n}{p}-\beta_0\right)^{-1} + \left(\beta_1 - \frac{n}{p}\right)^{-1}\right]y^{\beta_1-n/p}, \qquad C\left[\left(\frac{n}{p}-\beta_0\right)^{-1} + \log^+ y\right],$$

$$C\left[\left(\frac{n}{p}-\beta_0\right)^{-1} + \left(\frac{n}{p}-\beta_1\right)^{-1}\right]$$

according as $\beta_1 > n/p$, $= n/p$ or $< n/p$.

This proves (b) along with the corresponding estimate for $C(y)$ in this case. (c) is obtained from (b) by duality. (a) in case $q \geqslant 1$ is obtained from (b) and (c) by the complex method of interpolation ([4]). In case $0 < q \leqslant 1$ (a) follows from the Marcinkiewicz interpolation theorem by choosing p_0, p_1 such that $p_1 < p < p_0$ and, e.g., $(p_0, \omega) \in S_0^{*1}$, $(p_1, \omega) \in S_1^{*1}$. To prove (d) observe that if $H(x, t) \geqslant 0$ is bounded by $\Phi(|x-t|)$ instead of $A|x-t|^{-\lambda}$ where Φ is decreasing and satisfies (13) i.e., $\Phi(\tau/2) \leqslant C\Phi(\tau)$ then the proof of Lemma 1 shows that

$$\sup_{\tau \leqslant \sigma} \omega(\sigma)\sigma^{n/q}\Phi(\sigma)\omega(\tau)^{-1}\tau^{n/p'} \leqslant C, \qquad \sup_{\tau \leqslant \sigma} \omega(\tau)\tau^{n/q}\omega(\sigma)^{-1}\sigma^{n/p'}\Phi(\sigma) \leqslant C$$

is sufficient for $\|Tf\|_{q\infty,\omega} \leqslant C\|f\|_{p1,\omega}$. Also if $\Phi_y(|x|) = C_n^{-1}\min(1, y/|x|)|x|^{-n}$ then $\Phi_y(\tau/2) \leqslant 2^{n+1}\Phi_y(\tau)$ and $P(x, y) \leqslant \Phi_y(|x|)$. (d) follows provided it can be shown that $(p, \omega) \in S^*$ and $\tau \leqslant \sigma$ imply

$$\omega(\sigma)\sigma^{-n/p'}\min(1, y/\sigma)\omega(\tau)^{-1}\tau^{n/p'} \leqslant 1 + y^{\alpha_1-n/p'} \tag{27}$$

and

$$\omega(\tau)\tau^{n/p}\omega(\sigma)^{-1}\sigma^{-n/p}\min(1, y/\sigma) \leqslant 1 + y^{\beta_1-n/p}. \tag{28}$$

But

$$\omega(\sigma)\sigma^{-n/p'} = \omega(\sigma)\sigma^{-n/p'}\min(\sigma^{-n/p'}, \sigma^{-\alpha_1})\max(\sigma^{n/p'}, \sigma^{\alpha_1})$$

$$\leqslant \omega(\tau)\min(\tau^{-n/p'}, \tau^{-\alpha_1})\max(1, \sigma^{\alpha_1-n/p'}).$$

This is $\leqslant \omega(\tau)\tau^{-n/p'}$, $\omega(\tau)\tau^{-n/p'}\sigma^{\alpha_1-n/p'}$ or $\leqslant \omega(\tau)\tau^{-\alpha_1}\sigma^{\alpha_1-n/p'} \leqslant \omega(\tau)\tau^{-n/p'}\times$ $\times\sigma^{\alpha_1-n/p'}$ according as $\sigma \leqslant 1$, $\tau \leqslant 1 \leqslant \sigma$ or $\tau \geqslant 1$. Thus the left-hand side of (27) is at most $\min(1, y/\sigma)$ for $\sigma \leqslant 1$ and $\leqslant \min(1, y/\sigma)\sigma^{\alpha_1-n/p'}$ for $\sigma \geqslant 1$ and (27) follows. (28) follows from (27) by replacing ω by ω^{-1} and p' by p.

To prove the last part of the lemma observe that if ω and ω^{-1} are continuous in $[0, 1]$, $\omega(\tau)\tau^{-\alpha_1}\downarrow$, $\omega(\tau)\tau^{\beta_1}\uparrow$ for $\tau \geqslant 1$ and ψ is defined by $\psi(\varepsilon) = \sup_{|\tau-\sigma|\leqslant\varepsilon} \dfrac{\omega(\tau)}{\omega(\sigma)}$ then $\lim_{\varepsilon\to+0} \psi(\varepsilon) = 1$. Also by Minkowski's inequality for integrals $\|P(\cdot, y)*f\|_{pq} \leqslant \|f\|_{pq}$ whenever $\|\cdot\|_{pq}$ is a norm. Hence if

$$K(x, t) = P(x-t, y)\omega(|x|)\omega(|t|)^{-1}, \qquad K_\varepsilon(x, t) = K(x, t)\chi(\varepsilon^{-1}|x-t|)$$

then

$$\sup\left\{\left\|\int K_\varepsilon(\cdot, t)f(t)\,dt\right\|_{pq} : \|f\|_{pq} \leqslant 1\right\} \leqslant \psi(\varepsilon).$$

If $K'_\varepsilon = K - K_\varepsilon$ then $K'_\varepsilon(x, t) \leqslant C\varepsilon^{-1}y\varepsilon(\varepsilon+|x-t|)^{-n-1}\omega(|x|)\omega(|t|)^{-1}$. Therefore by what has already been proved

$$\lim_{y\to 0}\sup\left\{\left\|\int K'_\varepsilon(\cdot, t)f(t)\,dt\right\|_{ps} : \|f\|_{pq} \leqslant 1\right\} = C\varepsilon^{-1}\lim_{y\to 0} y = 0$$

and so

$$\limsup_{y\to 0}\sup\{\|P(\cdot, y)*f\|_{ps,\omega} : \|f\|_{pr,\omega} \leqslant 1\} \leqslant \psi(\varepsilon).$$

If ε is made to tend to 0 it follows that

$$\limsup_{y\to 0}\sup\{\|P(\cdot, y)*f\|_{ps,\omega} : \|f\|_{pr,\omega} \leqslant 1\} \leqslant 1.$$

If $q < \infty$ the continuous functions φ of compact support disjoint from $\{0\}$ are dense in L_ω^{pq}. It therefore suffices to prove (21) for such a function φ. But then $P(\cdot, y)*\varphi \to \varphi$ uniformly and for $|x| \leqslant 1/2\inf\{|y| : y \in \operatorname{supp}\varphi\} = \delta$, say, $P(\cdot, y)*\varphi(x) \leqslant Cy\,\delta^{-n-1}\|\varphi\|_1$ while for $|x| \geqslant 2\sup\{|y| : y \in \operatorname{supp}\varphi\}$, $P(\cdot, y)*\varphi(x) \leqslant Cy|x|^{-n-1}\|\varphi\|_1$ which implies (21).

LEMMA 4. *If M^η is defined by*

$$M^\eta f(x) = \sup_{\varepsilon\leqslant\eta}\varepsilon^{-n}\left|\int_{|t-x|\leqslant\varepsilon} f(t)\,dt\right|$$

then

$$\|M^\eta f\|_{rs,\omega} \leqslant C\|f\|_{pq,\omega} \tag{29}$$

provided one of

(a) $(p, \omega) \in S^2$, $q \leqslant s \leqslant \infty$, (b) $(p, \omega) \in S_0^1$, $q = s = \infty$, (c) $(p, \omega) \in S$, $q = 1$, $s = \infty$

holds in the interval $(0, \eta)$ and ω satisfies (13).

Proof. If $\varepsilon(x)$ is a positive function on R^n

$$\varepsilon(x)^{-n}\chi\big(\varepsilon(x)^{-1}|x-t|\big)\,\omega(|x|)\,\omega(|t|)^{-1}\leqslant C\varepsilon(x)^{-n}\chi\big(\varepsilon(x)^{-1}|x-t|\big)\quad\text{for}\quad |x|\geqslant 2\eta$$

and for $|x|\leqslant 2\eta$ it vanishes unless $|t|\leqslant 3\eta$. Hence if $\omega^*(\tau)=\omega(\tau)$ for $\tau\leqslant 3\eta$ and $=\omega(3\eta)$ for $\tau\geqslant 3\eta$ then $(p,\omega^*)\in S^2$, S_0^1 or S as the case may be. Also

$$\varepsilon(x)^{-n}\chi\big(\varepsilon(x)^{-1}|x-t|\big)\,\omega(|x|)\,\omega(|t|)^{-1}\leqslant C\varepsilon(x)^{-n}\chi\big(\varepsilon(x)^{-1}|x-t|\big)\,\omega^*(|x|)\omega^*(|t|)^{-1}$$

and so the lemma follows from Lemmas 1 and 2 and the remark pertaining to to the maximal operator M after the proof of Lemma 1.

Define

$$\Gamma_k^\eta(x)=\{(t,y)\colon\ |t-x|\leqslant ky<k\eta\}\qquad \big(\Gamma_k(x)=\Gamma_k^\infty(x)\big).$$

LEMMA 5. *Let* $F(x,y)=P(\cdot,y)*f(x)$ *and*

$$F^{*\eta}(x)=\sup\{|F(t,y)|\colon\ (t,y)\in\Gamma_k^\eta(x)\}.$$

Then $\|F^{*\eta}\|_{ps,\omega}\leqslant C(\eta)\|f\|_{pq,\omega}$ *provided one of the following conditions is satisfied*

(a) $(p,\omega)\in S^{*2}$, $q\leqslant s\leqslant\infty$, (b) $(p,\omega)\in S_0^{*1}$, $q=s=\infty$,

(c) $(p,\omega)\in S^*$, $q=1$, $s=\infty$.

If $\mu=\max\big((\alpha_1-\alpha_0),(\beta_1-\beta_0)\big)$ *then if* (a) *holds and* $\varepsilon>0$, $C(\eta)\leqslant C_\varepsilon(1+$ $+\eta^{\mu+\varepsilon})$ *while if* (b) *or* (c) *hold* $C(\eta)\leqslant C(1+\eta^\mu)$ $(\leqslant C(1+\log^+\eta)$ *if* $\alpha_0=\alpha_1$ $=n/p'$ $\beta_0=\beta_1)$.

Proof. It can be assumed without loss of generality that $f\geqslant 0$. Then

$$F(t,y)\leqslant C_kF(x,y)\quad\text{for } (t,y)\in\Gamma_k(x)\quad(\text{see, e.g., [29], (3.16)}),\tag{30}$$

hence

$$\begin{aligned}F^{*\eta}(x)&\leqslant C_k\sup_{y\leqslant\eta}F(x,y)\\&\leqslant C_k\Big[M^\eta f(x)+\sup_{y\leqslant\eta}\int_{|t|\geqslant\eta}P(t,y)f(x-t)\,dt\Big]\\&\leqslant C_k[M^\eta f(x)+P(\cdot,\eta)*f(x)].\end{aligned}$$

Furthermore if, e.g., $\omega(\tau)\min(\tau^{-\alpha_0},\tau^{-\alpha_1})\downarrow$ then for $1\leqslant\tau\leqslant\sigma\leqslant\eta$ $\omega(\tau)\tau^{-\alpha_0}$ $=\omega(\tau)\tau^{-\alpha_1}\tau^{\alpha_1-\alpha_0}\geqslant\omega(\sigma)\sigma^{-\alpha_0}(\tau/\sigma)^{\alpha_1-\alpha_0}\geqslant\omega(\sigma)\sigma^{-\alpha_0}\eta^{\alpha_0-\alpha_1}$. Clearly if in Lemma 4 the condition $\omega(\tau)\tau^{-\alpha}\downarrow$ is replaced by $\omega(\sigma)\sigma^{-\alpha}\leqslant A\omega(\tau)\tau^{-\alpha}$ and $\omega(\tau)\tau^{\beta}\uparrow$ by $\omega(\tau)\tau^\beta\leqslant A\omega(\sigma)\sigma^\beta$ for $\tau\leqslant\sigma\leqslant\eta$ the conclusion is the same except that the right-hand side of (29) is multiplied by A. In the present case this yields $\|M^\eta f\|_{ps,\omega}\leqslant C(1+\eta^\mu)\|f\|_{pq,\omega}$. Together with Lemma 3 this proves Lemma 5 (since $\nu\leqslant\mu$).

The next lemma will be of significance in connection with Proposition 3 below.

LEMMA 6. *For* $y\geqslant 1$

$$\omega(y)\|\chi(y^{-1}|\cdot|)P(\cdot,y)*f\|_{ps}+\|\chi(y|\cdot|^{-1})\omega(|\cdot|)P(\cdot,y)*f\|_{ps}\leqslant C(y)\|f\|_{pq,\infty}$$

provided one of (a), …, (d) *of Lemma 3 holds, and in case* (a) *with* $q\geqslant 1$, (b), (c)

$$\begin{aligned}C(y)\leqslant[(n/p'+1-\alpha_1)^{-1}y^{\alpha_1-n/p'}+1]^{1/q}[(n/p'-\alpha_0)^{-1}y^{\alpha_1-n/p'}+\\+(n/p+1-\beta_1)^{-1}+\psi(y)]^{1/q'},\end{aligned}$$

where

$$\psi(y)=(\alpha_1-n/p')^{-1},\log^+y,(n/p'-\alpha_1)^{-1}y^{\alpha_1-n/p'}\tag{31}$$

according as $\alpha_1<n/p'$, $=n/p'$ or $>n/p'$.

In case (a) *(and* $0<q<1$*)* $C(y)\leqslant C_\varepsilon(y^{\alpha_1-n/p'+\varepsilon}+1)$ *for any* $\varepsilon>0$ *while* $C(y)\leqslant C_p(y^{\alpha_1-n/p'}+1)$ *in case* (d), *in particular, if in addition* $\alpha_1<n/p'$ $(\alpha_1\leqslant n/p'$ *in case* (b) *or* (d)) *then* $C(y)$ *is bounded.*

Proof. To establish case (b), for $y\geqslant 1$, set

$$\begin{aligned}K^y(x,t)&=\omega(y)y(y+|x-t|)^{-n-1}\omega(|t|)^{-1}\chi(y^{-1}|x|),\\K_y(x,t)&=y(y+|x-t|)^{-n-1}\omega(|x|)\,\omega(|t|)^{-1}\chi(y|x|^{-1}).\end{aligned}$$

It is sufficient to show that K^y,K_y satisfy (15) or (16) with $r=p$, $q=s$ $=1$. Consider first K^y. Set

$$K_1^y(x,t)=K^y(x,t)\chi(1/2y^{-1}|t|),\qquad K_2^y=K^y-K_1^y$$

so that

$$K_1^y(x,t)\leqslant Cy^{-n+\alpha_1}\min(|t|^{-\alpha_0},|t|^{-\alpha_1})$$

hence

$$\|K_1^y(x,\cdot)\|_{p'\infty}\leqslant Cy^{-n+\alpha_1}\sup_{|t|\leqslant 2y}\min(|t|^{-\alpha_0+n/p'}|t|^{-\alpha_1+n/p'})\leqslant Cy^{-n+\alpha_1}$$

(it was again assumed that $\alpha_1\geqslant n/p'$, cf. start of proof of Lemma 3). As a result

$$\big\|\|K_1^y\|[L^{p'\infty}(t)]\big\|[L^{p1}(x)]\leqslant Cy^{-n/p'+\alpha_1}.$$

On the other hand

$$K_2^y(x,t)\leqslant Cy|t|^{-n-1}\omega(y)\,\omega(|t|)^{-1}\leqslant Cy^{1-\beta_1}|t|^{-n-1+\beta_1}$$

hence $\|K_2^y(x,\cdot)\|_{p'\infty}\leqslant Cy^{1-\beta_1}\sup\limits_{|t|\geqslant 2y.}|t|^{-1-n/p+\beta_1}\leqslant Cy^{-n/p}$

and since K_2^y vanishes for $|x|\geqslant y$

$$\big\|\|K_2^y\|[L^{p'\infty}(t)]\big\|[L^{p1}(x)]\leqslant C.$$

Let now

$$K_y^1(x,t)=K_y(x,t)\chi(2|x|^{-1}|t|),\ K_y^3(x,t)=K_y(x,t)\chi(2|x|\,|t|^{-1}),$$
$$K_y^2=K_y-K_y^1-K_y^3.$$

Thus

$$K_y^2(x,t) \leqslant CP(x-t,y), \quad K_y^1(x,t) \leqslant Cy\,|x|^{a_1}(y+|x|)^{-n-1}\min(|t|^{-a_0},|t|^{-a_1})$$

and so

$$\big\|\,\|K_y^1\|[L^{p'\infty}(t)]\big\|[L^{p1}(x)] \leqslant C[(n/p'+1-a_1)^{-1}y^{a_1-n/p'}+1].$$

Finally $K_y^3(x,t) \leqslant y\,|t|^{-n-1+\beta_1}|x|^{-\beta_1}$, so

$$\|K_y^3(x,\cdot)\|_{p'\infty} \leqslant Cy\,|x|^{-\beta_1}\sup_{|t|\geqslant|x|}|t|^{-n/p-1+\beta_1} = Cy\,|x|^{-n/p-1}$$

and $\big\|\,\|K_y^3\|[L^{p'\infty}(t)]\big\|[L^{p1}(x)] \leqslant C$.

Case (c) follows similarly from

$$\big\|\,\|K_1^y\|[L^{p'1}(t)]\big\|[L^{p\infty}(x)] \leqslant C(n/p'-a_0)^{-1}y^{a_1-n/p'},$$

$$\big\|\,\|K^y\|[L^{p'1}(t)]\big\|[L^{p\infty}(x)] \leqslant C(n/p+1-\beta_1)^{-1},$$

$$\big\|\,\|K_y^1\|[L^{p'1}(t)]\big\|[L^{p\infty}(x)] \leqslant C(n/p'-a_0)^{-1}y^{a_1-n/p}+C\psi(y)$$

(see (31)) and

$$\big\|\,\|K_y^3\|[L^{p'1}(t)]\big\|[L^{p\infty}(x)] \leqslant C(n/p+1-\beta_1)^{-1}.$$

(a) now follows by interpolation and (d) is proved similarly.

The following will be needed below.

(32) $$\|\omega(|\cdot|)^{-1}(1+|\cdot|)^{-n-1}\|_{p'1} < \infty \quad \text{if} \quad (p,\omega)\in S_0^{*1}.$$

For

$$\|\omega(|\cdot|)^{-1}(1+|\cdot|)^{-n-1}\|_{p'1} \leqslant \omega(1)\Big(\big\||\cdot|^{-a_0}\chi(|\cdot|)\big\|_{p'1}+\big\||\cdot|^{\beta_1-n-1}(1-\chi(|\cdot|))\big\|_{p'1}\Big) < \infty.$$

It follows similarly that

(33) $$\|\omega(|\cdot|)^{-1}(1+|\cdot|)^{-n-1}\|_{p'\infty} < \infty \quad \text{if} \quad (p,\omega)\in S^*.$$

Hence if $f\in L_\omega^{pq}$, p,ω,q satisfying (a), (b), (c) or (d) of Lemma 3 then $f(1+|\cdot|)^{-n-1}$ is integrable. It follows from $\lim_{y\to\infty}P(\cdot,y)=0$ by dominated convergence that $\lim_{y\to\infty}y^{-1}P(\cdot,y)*f=0$. By the usual density argument if $(p,\omega)\in S_1^{*1}$ then $\lim_{y\to\infty}y^{-1}\|P(\cdot,y)*f\|_{p1,\omega}=0$.

$\omega>0$, $\omega(\tau)\tau^{-a}\downarrow$; $\omega(\tau)\tau^{\beta}\uparrow$ imply, as is well known, that $\log\omega$ is (locally) Lipschitzian, hence if $(p,\omega)\in S^*$ then ω must be continuous. In case $(1,\omega)\in S^*$ ω^{-1} is continuous at 0 for in this case $a_0=0$ hence ω^{-1} is increasing.

By means of the preceding lemmas, in case $w(x)=\omega(|x|)$, $(p,\omega)\in S^*$ the conclusion of Proposition 1 can be strengthened and extended to $q<1$. For the purposes of the following proposition let H_ω^{pq} denote the space of harmonic functions F in R_+^{n+1} satisfying

(34) $$\|F\|[H_\omega^{pq}] = \sup_{y>0}(1+y)^{-1}\|F(\cdot,y)\|_{pq,\omega} < \infty.$$

Proposition 2. *Suppose $F\in H_\omega^{pq}$. If $\eta>0$ the convergence of F in Γ_k^η to the boundary value function f (guaranteed by Proposition 1) is dominated by a function $f^{*\eta}+|\delta|p_0$ such that $\|f^{*\eta}\|_{ps,\omega} \leqslant C(\eta)\|F\|[H_\omega^{pq}]$ provided one of the following conditions is satisfied:*

(a) $(p,\omega)\in S^{*2}$, $0<q=s\leqslant\infty$, (b) $(p,\omega)\in S_1^{*1}$, $q=1$, $s=\infty$,

(c) $(p,\omega)\in S_0^{*1}$, $q=s=\infty$, (d) $(p,\omega)\in S^*$, $q=1$, $s=\infty$.

Hence $\|F(\cdot,y)-f\|_{pq,\omega}\to 0$ if (a) *and* $q<\infty$. *In cases* (a), (b), $p>1$, ($p=1$), (c) *and if the constant function $1\in L_\omega^{pq}$ the mapping*

$$(f,\delta)\to P*f+\delta p_0$$

is a topological isomorphism between $L_\omega^{pq}\oplus C((\omega^{-1}\mathscr{M}^1)\oplus C)$ and H_ω^{pq}. In case $1\notin L_\omega^{pq}$ it is an isomorphism between L_ω^{pq} $(\omega^{-1}\mathscr{M}^1)$ and H_ω^{pq}.

Proof. To prove the first part, by consideration of $F-\delta p_0$, if necessary, it is sufficient to consider the case $\delta=0$. Let

$$F^{*\eta}(x,y_1) = \sup\{|F(t,y)|:\ (t,y-y_1)\in\Gamma_k^\eta(x)\},\quad f^{*\eta}=F^{*\eta}(\cdot,0).$$

If $q\geqslant 1$ by (32), (33) and the corollary to Proposition 1 $F=P*f(P*\mu$ if $p=1)$, where $\|f\|_{pq,\omega}\leqslant\liminf_{y\to 0}\|F(\cdot,y)\|_{pq,\omega}$. The first assertion therefore follows from Lemma 5. If $q<1$ (hence case (a)) the remaining hypotheses are also satisfied for q replaced by 1. Hence by the proof of Proposition 1

$$F(x,y)=P(\cdot,y-y_1)*F(\cdot,y_1) \quad \text{for} \quad 0<y_1<y,$$

so by Lemma 5

$$\|F^{*\eta-y_1}(\cdot,y_1)\|_{pq,\omega}\leqslant C(\eta)\|F(\cdot,y_1)\|_{pq,\omega}.$$

Also $F^{*\eta-y_1}(\cdot,y_1)$ is decreasing as function of y_1 hence by the Fatou property of the (quasi-) norm $\|\cdot\|_{pq}$

$$\|f^{*\eta}\|\leqslant C(\eta)\liminf_{y\to 0}\|F(\cdot,y)\|_{pq,\omega}.$$

(If $\{f_n\}$ is a sequence of measurable functions such that $f_n\uparrow f$ and λ_f denotes the distribution function of f it follows that $\lambda_{f_n}\uparrow\lambda_f$ hence $f_n^*\uparrow f^*$. Hence by Fatou's lemma $\|f\|_{pq}=\lim_{n\to\infty}\|f_n\|_{pq}$). In case $p>1$, $q=s<\infty$ it follows from dominated convergence that $[(F(\cdot,y)-f)\omega]^{**}\to 0$ and hence again by dominated convergence $\lim_{y\to 0}\|F(\cdot,y)-f\|_{pq,\omega}=0$. The assertion concerning the topological isomorphism now follows from Proposition 1 and Lemma 3.

Remark. The existence of boundary values $F(\cdot,0)$ can also be deduced from Calderón's theorem ([3]) which asserts the equivalence of non-tangential boundedness and convergence of harmonic functions a.e. (Still, the proof of this theorem in [3] requires the weak compactness of bounded subsets of the dual of a Banach space.) For in any case $f^{*\eta}<\infty$ a.e.

3. Harmonic majorization. This section is devoted to the proof of the following proposition and corollary.

PROPOSITION 3. *Suppose U is a non-negative subharmonic function in R_+^{n+1}. Then U has a harmonic majorant iff*

$$(35)\qquad \sup_{0<y\leqslant 1} \|U(\cdot, y)(1+|\cdot|)^{-n-1}\| = M_0 < \infty$$

and

$$(36)\qquad \sup_{1\leqslant y<\infty} \Big(y^{-n-1}\int_{|x|\leqslant y} U(x, y)\,dx + \int_{|x|>y} U(x, y)|x|^{-n-1}dx\Big) = M_1 < \infty.$$

In this case the weak limit of $U(\cdot, y)$ as $y\to 0$ exists in $(1+|\cdot|)^{n+1}\mathscr{M}^1(R^n)$ and if this limit is denoted μ

$$(37)\qquad \lim_{y\to 0}\|U(\cdot, y)(1+|\cdot|)^{-n-1}\|_1 = \|\mu(1+|\cdot|)^{-n-1}\|.$$

Furthermore $y^{-n-1}\int_{|x|\leqslant y} U(x, y)\,dx$ converges as $y\to\infty$ and if the limit is written as $(\omega_n/n)\delta$ the least harmonic majorant of U is given by

$$(38)\qquad P*\mu+\delta p_0, \qquad \big(\|\mu(1+|\cdot|)^{-n-1}\|\leqslant CM_0,\ |\delta|\leqslant CM_1\big).$$

COROLLARY. *Suppose $U\geqslant 0$ is subharmonic in R_+^{n+1} and such that*

$$(39)\qquad \sup_{0<y\leqslant 1}\|U(\cdot, y)\|_{pq,\omega} = M_0 < \infty,$$

$$(40)\qquad \sup_{1\leqslant y<\infty}\big(\omega(y)\|\chi(y^{-1}|\cdot|)U(\cdot, y)\|_{pq} + \|\chi(y|\cdot|^{-1})U(\cdot, y)\|_{pq,\omega}\big) = M_1 < \infty,$$

where p, q, ω satisfy (a), (b), (c) *or* (d) *of Lemma* 3 *then $\lim_{y\to\infty} y^{-n-1}\int_{|x|\leqslant y} U(x, y)\,dx = (\omega_n/n)\delta$ exists, $\delta\leqslant CM_1$ and the least harmonic majorant is $P*\mu+\delta p_0$ where μ is the weak limit of $U(\cdot, y)$ in $L^{pq}\big(\omega^{-1}\mathscr{M}^1(R^n)\big)$. Moreover*

$$(41)\qquad \|\mu\|_{pq,\omega} = \lim_{y\to 0}\|U(\cdot, y)\|_{pq,\omega}$$

provided (a) *with $q<\infty$ or* (b) *is satisfied or* (c′) *$q=\infty$, $(p, \omega)\in S_0^{*1}$ in $[1, \infty)$ and ω is continuous in $[0, 1]$, $\omega(0)\neq 0, \infty$. Also $U(\cdot, y)$ converges to the absolutely continuous part $U(\cdot, y)$ of μ a.e. as $y\to 0$.*

The proof requires a few lemmas all of which are well known for $n=1$ and subharmonic functions vanishing on the boundary line $y=0$ (see [13] p. 112, [32] pp. 188–194, pp. 149–153 and also [7] pp. 1–9).

For $\sigma\in S_+^n = S^n\cap R_+^{n+1}$ let θ, σ' be defined by

$$\sigma = (\sigma'\cos\theta, \sin\theta),\ \sigma'\in S^{n-1}, \qquad 0\leqslant\theta\leqslant\pi/2$$

and define

$$(42)\qquad m(U, r) = \int_{S_+^n} U(r\sigma)\sin\theta\,d\sigma.$$

For the sake of completeness a proof of the following known result (see Theorem 2 of [9]) will be given.

LEMMA 7. *Suppose U is subharmonic in the domain*

$$R(r_1, r_2) = \{(x, y)\colon r_1^2 < |x|^2+y^2 < r_2^2,\ y>0\}$$

upper semi-continuous in its closure and vanishes for $y=0$. Then $r^{-1}m(U; r)$ is a convex function of r^{-n-1} in $[r_1, r_2]$. If $r_1=0$ then $r^{-1}m(U; r)$ is increasing.

Proof. Suppose a function h is harmonic in $R(r_1, r_2)$ and continuous in its closure and vanishes when $y=0$, then there are constants c_0, c_1 such that

$$m(h; r) = c_0 r^{-n} + c_1 r.$$

For since $y = r\sin\theta$ is harmonic, if $r_1<r_3\leqslant r_4<r_2$, then by Green's formula

$$\int_{S_+^n}\Big[\big(h(r\sigma)\sin\theta - (\partial h(r\sigma)/\partial r)\,r\sin\theta\big)\Big]r^n d\sigma\Big|_{r=r_3}^{r=r_4}$$

$$= \iint_{r_3\leqslant|(x,y)|\leqslant r_4}\big[h(x, y)\Delta y - \big(\Delta h(x, y)\big)y\big]\,dx\,dy.$$

In other words $r^n\big(m(h; r) - r\,dm(h; r)/dr\big)$ equals a constant $(n+1)c_0$, say, or

$$(d/dr)\big(r^{-1}m(h; r)\big) = -c_0(n+1)r^{-n-2}$$

hence $r^{-1}m(h; r) = c_0 r^{-n-1}+c_1$. By continuity this last result holds for $r_1\leqslant r\leqslant r_2$. To deduce convexity of $m(U; r)$ with respect to the family of functions $c_0r^{-n}+c_1 r$, $c_0, c_1\in R$, observe that for r_3, r_4 as above there is a sequence of continuous functions $\{\varphi_k\}$ on the boundary $\partial R(r_3, r_4)$ of $R(r_3, r_4)$ vanishing for $y=0$ which tends decreasingly to U. For let $\{\varphi_k^*\}$ be a decreasing sequence of continuous functions tending to U on $\partial R(r_3, r_4)$ (which exists by upper semi-continuity of U), ψ a continuous function $\geqslant U$ on $\partial R(r_1, r_2)$ and vanishing on $[-r_4, -r_3]$ and $[r_3, r_4]$ and let Ψ be the solution of the Dirichlet problem in $R(r_1, r_2)$ with boundary values ψ, then $\varphi_k = \min(\varphi_k^*, \Psi)$ is a possible choice for φ_k. Application of the result for harmonic functions to the solutions of the Dirichlet problem Φ_k say, for the boundary values φ_k, again the maximum principle for subharmonic functions applied to $U-\Phi_k$ and passage to the limit as $k\to\infty$ finish the proof of the first part of the lemma.

Also if $r_1=0$ then by upper semi-continuity of U $r^{-1}m(U; r) = o(r^{-1})$ as $r\to 0$, i.e., $r^{-1}m(U; r) = o\big((r^{-n-1})^{1/(n+1)}\big)$, hence since $r^{-1}m(U; r)$ is convex as a function of r^{-n-1}, $r^{-1}m(U; r)$ is bounded near 0 and decreasing as a function of r^{-n-1}, i.e., $r^{-1}m(U; r)$ is increasing.

Let $z = (x, y)$, $w = (t, v)$ denote points in $R^n \times R_+ = R_+^{n+1}$. The Green's function for R_+^{n+1} may then be written

$$G(z, w) = [(n-1)\omega_{n+1}]^{-1}(|z-w|^{-n+1} - |z-\overline{w}|^{-n+1}),$$

where $\overline{w} = (t, -v)$.

LEMMA 8. *Let $m = m(G(\cdot, w); \cdot)$ where G is the Green's function for the half space R_+^{n+1}. Then*

$$(43)\qquad r^{-1}m(r) = (n+1)^{-1}v|w|^{-n-1} \quad or \quad = (n+1)^{-1}vr^{-n-1}$$

$$according\ \ as\ \ r \leqslant |w|\ \ or\ \ r \geqslant |w|.$$

If $U(x, y) = P(\cdot, y) * \mu(x)$, where $\mu \in (1+|\cdot|)^{n+1}\mathscr{M}^1$ then

$$(44)\qquad r^{-1}m(U; r) = (n+1)^{-1}\Big(r^{-n-1}\int_{|t|\leqslant r}\mu(dt) + \int_{|t|>r}|t|^{-n-1}\mu(dt)\Big).$$

Consequently $\lim_{r\to\infty} r^{-1}m(U; r) = 0$ *and for* $\mu \geqslant 0$ $r^{-1}m(U; r)$ *is decreasing and a concave function of* r^{-n-1}. (The last statement is well known to be true, see [8]).

Proof. Let u be a continuous function on the boundary of

$$B_+^n = \{z = (x, y)\colon |z| < 1, y > 0\}$$

and vanish for $y = 0$. By the reflection principle and the Poisson integral formula for a sphere the function harmonic in B_+^{n+1}, continuous in the closure $\mathrm{cl}(B_+^{n+1})$ of B_+^{n+1} and equal to u on ∂B_+^n equals

$$(45)\qquad \int_{S_+^n} K(z, \sigma)u(\sigma)d\sigma, \quad \text{where } K(z, \sigma) = \omega_{n+1}^{-1}(1-|z|^2)\times$$

$$\times(|\sigma-z|^{-n-1} - |\sigma-\bar{z}|^{-n-1})$$

and $\bar{z} = (x, -y)$. It follows that

$$(46)\qquad (\partial/\partial y)h(0) = 2(n+1)\omega_{n+1}^{-1}r^{-1}m(h; r)$$

for any function h harmonic in rB_+^n continuous in its closure and vanishing for $y = 0$. Hence for $r < |w|$

$$2\omega_{n+1}^{-1}v|w|^{-n-1} = (\partial/\partial y)G(0, w) = 2(n+1)\omega_{n+1}^{-1}r^{-1}m(h; r),$$

i.e., (43) holds for $r < |w|$. Since if $z' = |z|^{-1}z$

$$|z-w| \geqslant C|w|\,|z'-w'| \quad \text{for} \quad |z| \geqslant |w|/2$$

and $|z'-w'|^{-n+1}$ is integrable over S^n it follows from dominated convergence that m is a continuous function of r even at $|w|$. For $r \geqslant 2|w|$, say, $m(r) \leqslant Cr^{-n}|w|$, hence by Lemma 7 $r^{-1}m(r) = C_w r^{-n-1}$ and (43) now follows by continuity. Also

$$P(x-t, y) = (\partial/\partial v)G(z, w)|_{v=0}$$

and for $r \geqslant \frac{1}{2}|t|$, say,

$$P(r\sigma'\cos\theta - t, r\sin\theta)\sin\theta \leqslant C_n|t|^{-n}(|\sigma'\cos\theta - t'| + \sin\theta)^{-n-1}\sin^2\theta$$
$$\leqslant C_n|t|^{-n}|\sigma'\cos\theta - t'|^{-n+1}.$$

Hence if $m_t = m(P(\cdot - t, \cdot); \cdot)$, by dominated convergence differentiation of (43) yields

$$(47)\qquad r^{-1}m_t(r) = (n+1)^{-1}|t|^{-n-1} \ \text{ if } \ r \leqslant |t|, \qquad (n+1)^{-1}r^{-n-1} \text{ if } r \geqslant |t|.$$

This latter function is easily seen to be a concave function of r^{-n-1} and a decreasing function of r, hence so is $\int m_t\mu(dt)$ for $\mu \geqslant 0$. (44) follows from (47) by Fubini's theorem. Obviously

$$\lim_{r\to\infty}\int_{|t|\geqslant r}|t|^{-n-1}\mu(dt) = 0.$$

On the other hand

$$\limsup_{r\to\infty} r^{-n-1}\int_{|t|\leqslant r}\mu(dt) \leqslant \limsup_{r\to\infty} r^{-n-1}\int_{|t|\leqslant r\varepsilon}\mu(dt) + \limsup_{r\to\infty} r^{-n-1}\int_{r\varepsilon\leqslant|t|\leqslant r}\mu(dt)$$
$$= \limsup_{r\to\infty} r^{-n-1}\int_{|t|\leqslant r\varepsilon}\mu(dt) \leqslant \varepsilon^{n+1}\int(1+|t|)^{-n-1}\mu(dt)$$

for any $\varepsilon > 0$ and hence Lemma 8 is completely proved.

LEMMA 9. *For a non-negative function U subharmonic in R_+^{n+1} and upper semicontinuous in $\mathrm{cl}(R_+^{n+1})$ (implying locally bounded above) there exists a function h harmonic in R_+^{n+1} and continuous in $\mathrm{cl}(R_+^{n+1})$ and at least equal to U if and only if*

$$\int U(x, 0)(1+|x|)^{-n-1}dx < \infty \quad and \quad \limsup_{r\to\infty} r^{-1}m(U; r) < \infty.$$

In this case $r^{-1}m(U; r)$ converges, and if the limit is denoted γ the least harmonic majorant is given by $P * U(\cdot, 0) + 2(n+1)\omega_{n+1}^{-1}\gamma p_0$.

Proof. It follows from (45) and the Poisson integral formula for a half space that the solution of the Dirichlet problem for B_+^{n+1} and continuous boundary values u is given by

$$(48)\qquad \int_{|t|\leqslant 1}P(x-t, y)u(t)dt - \int_{S_+^n}\int_{|t|\leqslant 1}P(\sigma'\cos\theta - t, \sin\theta)K(z, \sigma)u(t)\,dt\,d\sigma +$$

$$+\int_{S_+^n}K(z, \sigma)u(\sigma)d\sigma.$$

Now by the mean value theorem

$$K(\sigma, z) = \omega_{n+1}^{-1}(1-|z|^2)2(n+1)(\sin\theta)y\xi^{-(n+3)/2},$$

where ξ is between $|\sigma-\bar{z}|^2$ and $|\sigma-z|^2$ hence

$$(1-|z|)^2 \leqslant \xi \leqslant (1+|z|)^2.$$

As a result

(49) $$2(n+1)\omega_{n+1}^{-1}\frac{1-|z|^2}{(1+|z|)^{n+3}}y\sin\theta \leqslant K(z,\sigma) \leqslant 2(n+1)\omega_{n+1}^{-1}\frac{1-|z|^2}{(1-|z|)^{n+3}}y\sin\theta.$$

It follows from (48), (49) and Lemma 8, (44) that the least harmonic majorant h_r of U in rB^n satisfies

(50) $$\int\limits_{|t|\leqslant r}\frac{yU(t,0)}{(|x-t|^2+y^2)^{(n+1)/2}} - \frac{1-|z|^2/r^2}{(1-|z|/r)^{n+3}}r^{-n-1}\int U(t,0)\,dt\,y+(n+1)\frac{1-|z|^2/r^2}{(1+|z|/r)^{n+3}}r^{-1}m(U;r)y$$
$$\leqslant(\omega_{n+1}/2)h_r(z)\leqslant\int\limits_{|t|\leqslant r}\frac{yU(t,0)}{(|x-t|^2+y^2)^{(n+1)/2}} - \frac{1-|z|^2/r^2}{(1+|z|/r)^{n+3}}r^{-n-1}\int U(t,0)\,dt\,y+(n+1)\frac{1-|z|^2/r^2}{(1-|z|/r)^{n+3}}r^{-1}m(U;r)y.$$

Also it is well known that the kernel for solving the Dirichlet problem is positive. It follows from the first inequality that if U has a harmonic majorant h' then

$$\int\limits_{|t|\leqslant a}U(t,0)\,(1+|t|^2)^{-(n+1)/2}dt\leqslant(\omega_{n+1}/2)h'(0,1)$$

for any $a\geqslant 0$, hence

$$\int U(t,0)\,(1+|t|^2)^{-(n+1)/2}dt\leqslant(\omega_{n+1}/2)h'(0,1)$$

and more directly that $r^{-1}m(U;r)$ is bounded for $r\to\infty$. On the other hand if the two conditions of the lemma are satisfied then it follows from the second inequality of (50) that the family $\{h_r\}$ of harmonic functions which increase with r is locally bounded hence convergent to the least harmonic majorant h. (50) then implies

$$\frac{2(n+1)}{\omega_{n+1}}y\limsup_{r\to\infty}\frac{m(U;r)}{r}\leqslant h(x,y)-P(\cdot,y)*U(\cdot,0)\leqslant\frac{2(n+1)}{\omega_{n+1}}y\liminf_{r\to\infty}\frac{m(U;r)}{r}.$$

It follows that $r^{-1}m(U;r)$ converges to γ, say, and

$$h(x,y)=P(\cdot,y)*U(\cdot,0)(x)+2(n+1)\omega_{n+1}^{-1}\gamma y.$$

LEMMA 10. *Suppose $U\geqslant 0$ is subharmonic in R_+^{n+1}, then U has a harmonic majorant in R_+^{n+1} if and only if*

$$\limsup_{y\to\infty}\|U(\cdot,y)(1+|\cdot|)^{-n-1}\|=M<\infty\quad\text{and}\quad\limsup_{r\to\infty}r^{-1}m(U;r)<\infty.$$

In this case the weak limit μ, say, of $U(\cdot,y)$ as $y\to 0$ exists in $(1+|\cdot|)^{n+1}\mathcal{M}^1$ and the least harmonic majorant is given by $P*\mu+2(n+1)\omega_{n+1}^{-1}\gamma p_0$, where $\gamma=\lim_{r\to\infty}r^{-1}m(U;r)$. In particular any positive harmonic function in R_+^{n+1} has the form $P*\mu+cp_0$, $\mu\geqslant 0$, $c\geqslant 0$. (The last assertion is well known, see [8]).

Proof. That the conditions are sufficient follows from a modification of the second inequality of (50) applied to the function $U_\eta(x,y)=U(x,y+\eta)$:

(51) $$(\omega_{n+1}/2)U(x,y+\eta)\leqslant(n+1)\frac{1-|z|^2/r^2}{(1+|z|/r)^{n+3}}r^{-1}m_\eta(r)y-\frac{1-|z|^2/r^2}{(1+|z|/r)^{n+3}}\Big(r^{-n-1}\int\limits_{|t|\leqslant r}U(t,\eta)\,dt+\int\limits_{|t|>r}|t|^{-n-1}U(t,\eta)\,dt\Big)+P(\cdot,y)*U(\cdot,\eta)(x),$$

where $z=(x,y)$ and $m_\eta=m(U_\eta,\cdot)$. It will be shown first of all that $\lim_{\eta\to 0}m_\eta(r)=m(r)=m_0(r)$.

$$\int U(r\sigma'\cos\theta,r\sin\theta+\eta)\,d\sigma'\leqslant C(r\sin\theta)^{-n-1}\iiint\limits_{|t|^2+|v|^2\leqslant r^2\sin^2\theta}U(r\sigma'\cos\theta+t,r\sin\theta+v+\eta)\,dt\,dv\,d\sigma'$$
$$\leqslant C(r\sin\theta)^{-n-1}\int_0^{2r\sin\theta}\int\limits_{|t|\leqslant 2r}U(t,v+\eta)\int\limits_{|t-r\sigma'\cos\theta|\leqslant r\sin\theta}d\sigma'\,dt\,dv.$$

There is a constant C such that for any $a\in R^n$

(52) $$\int\limits_{|\sigma'-a|\leqslant b}d\sigma'\leqslant C\min(b^{n-1},1)$$

hence

$$\int U(r\sigma'\cos\theta,r\sin\theta+\eta)\,d\sigma'\leqslant Cr^{-1}(1+r)^{-n+1}M(\sin\theta)^{-1}$$

(for η, θ sufficiently small depending on r). Hence by dominated convergence applied to the integral

$$\int_0^{\pi/2}\int\limits_{S^{n-1}}U(r\sigma'\cos\theta,r\sin\theta+\eta)\,d\sigma'(\cos\theta)^{n-1}\sin\theta\,d\theta$$

it follows that $\lim_{\eta\to 0}m_\eta(r)=m(r)$.

By considering also a lower bound for the least harmonic majorant in $(0,\eta)+rB_+^{n+1}$ similar to that of (50) the existence of $\lim_{r\to\infty} r^{-1}m(U;r)$ can be deduced similarly as in the proof of Lemma 9. A somewhat different argument which gives more information about the function $m(U;r)$ runs as follows. Assuming first of all that U is continuous in R_+^{n+1}, let $U_\eta = U_\eta^{(1)}+U_\eta^{(2)}$ where

$$U_\eta^{(1)}(x,y) = U_\eta(x,y)-P(\cdot,y)*U(\cdot,\eta)(x).$$

Also set $m_\eta^{(i)} = m(U_\eta^{(i)};\cdot)$ for $i=1,2$. By weak compactness of the set of measures $\{U(x,\eta)dx\}_{0<y\leqslant 1}$ in $(1+|\cdot|)^{n+1}\mathscr{M}^1(R^{n*})$ there is a sequence $\eta_k\to 0$ and a measure $\mu^*\in(1+|\cdot|)^{n+1}\mathscr{M}^1(R^{n*})$ such that $U(\cdot,\eta_k)\to\mu^*$ weakly as $k\to\infty$.

Let

$$\mu^* = \mu+c|\cdot|^{n+1}\varepsilon_\infty, \qquad \mu\in\mathscr{M}^1(R^n)$$

(it will be seen that $c=0$).

Hence

$$\lim_{\eta\to 0} r^{-1}m_\eta^{(2)}(r) = (n+1)^{-1}\Big(r^{-n-1}\int_{|t|\leqslant r}\mu(dt)+\int_{|t|>r}|t|^{-n-1}\mu(dt)+c\Big) = r^{-1}m^{(2)}(r),$$

say, so $m_\eta^{(1)}$ converges to $m-m^{(2)}=m^{(1)}$, say, as $\eta\to 0$ through the sequence $\{\eta_k\}$ Since $\lim_{r\to\infty} r^{-1}m^{(2)}(r) = c/(n+1)$ and $\limsup_{r\to\infty} r^{-1}m(r)>\infty$, hence $\limsup_{r\to\infty} r^{-1}m^{(1)}(r)<\infty$ and since by Lemma 7 $r^{-1}m_\eta^{(1)}(r)$ is increasing and a convex function of r^{-n-1} so is $r^{-1}m^{(1)}(r)$. Hence, in particular, $\lim_{r\to\infty} r^{-1}m^{(1)}(r)$ exists and so does $\lim_{r\to\infty} r^{-1}m(r)=\gamma$.

If U is not necessarily continuous let φ be the characteristic function of a bounded set in $\mathrm{cl}(R_+^{n+1})$ of measure 1 and define $\check\varphi_\varepsilon(z)=\varepsilon^{-n}\check\varphi(\varepsilon^{-1}z)$ then $\check\varphi_\varepsilon*U$ is continuous and subharmonic in R_+^{n+1} and

$$\check\varphi_\varepsilon*U\to U \quad\text{as}\quad \varepsilon\to 0$$

boundedly on compact subsets of R_+^{n+1}. It follows as in the proof of (37) to be given below that

$$\|[U(\cdot,y+\eta)-U(\cdot,\eta)](1+|\cdot|)^{-n-1}\|_1\to 0.$$

Furthermore it follows from the continuity of the translation operator in L^1 and

$$\big|(1+|x-t|)^{-n-1}-(1+|x|)^{-n-1}\big|\leqslant C(1+|x|)^{-n-2}|t| \quad\text{if}\quad |t|\leqslant 1$$

that

$$\|[U(\cdot-t,y+\eta)-U(\cdot,\eta)](1+|\cdot|)^{-n-1}\|\to 0 \quad\text{as}\quad (t,y)\to 0.$$

Hence $P*((\varphi_\varepsilon*U)(\cdot,\eta))\to P*U(\cdot,\eta)$ boundedly on bounded subsets of R_+^{n+1} for $\eta>0$.

$$r^{-1}m_{\eta,\varepsilon}^{(1)}(r) = r^{-1}m(\check\varphi_\varepsilon*U(\cdot,\eta)-P*[(\check\varphi_\varepsilon*U)(\cdot,\eta)];r)$$

is increasing and also a convex function of r^{-n-1}, moreover $m_{\eta,\varepsilon}^{(1)}\to m_\eta^{(1)}$ hence $r^{-1}m_\eta^{(1)}(r)$ shares these properties Now the argument is the same as before.

If $\eta_k\to 0$ and then $r\to\infty$ in (51) it follows that

$$U(x,y)\leqslant P(\cdot,y)*\mu(x)+(2/\omega_{n+1})\big((n+1)\gamma+c\big)y$$

and, in fact, since the right-hand side is the (increasing) limit of a sequence of least harmonic majorants it is the least harmonic majorant. It follows now from the last part of Lemma 3 that

$$\limsup_{\eta\to 0}\|U(\cdot,\eta)(1+|\cdot|)^{-n-1}\|\leqslant\|\mu(1+|\cdot|)^{-n-1}\|$$

and since

$$\|\mu(1+|\cdot|)^{-n-1}\|+c\leqslant\liminf_{k\to\infty}\|U(\cdot,\eta_k)(1+|\cdot|)^{-n-1}\|$$

c must be zero. μ is unique since if μ' is any weak limit then $P*(\mu-\mu')=0$ by the minimum property of μ,μ', hence since $\mu-\mu'$ is the weak limit of $P(\cdot,y)*(\mu-\mu')$ as $y\to 0$, μ' equals μ and so $U(\cdot,y)$ is weakly convergent as $y\to 0$.

Conversely suppose U has a harmonic majorant h. By Lemma 8 the least harmonic majorant of U_η in R_+^{n+1} is $P*U(\cdot,\eta)+\big(2(n+1)/\omega_{n+1}\big)\gamma_\eta y$, where $\gamma_\eta=\lim_{r\to\infty} r^{-1}m_\eta(r)$. It follows that

$$\limsup_{\eta\to 0}\big(\|U(\cdot,\eta)(1+|\cdot|^2)^{-(n+1)/2}\|+2(n+1)\omega_{n+1}^{-1}\gamma_\eta\big)\leqslant h(0,1)$$

hence

$$\limsup_{\eta\to 0}(\|U(\cdot,\eta)(1+|\cdot|)^{-n-1}\|<\infty \quad\text{and}\quad \limsup_{\eta\to 0}\gamma_\eta<\infty.$$

If $\{\eta_k\}$ denotes a sequence such that $\eta_k\to 0$ and $U(\cdot,\eta_k)\to\mu^*\in(1+|\cdot|)^{n+1}\mathscr{M}^1(R^{n*})$ weakly and $\mu^*=\mu+c\varepsilon_\infty$, $\mu\in(1+|\cdot|)^{n+1}\mathscr{M}^1(R^n)$ and $m^{(1)}$, $m^{(2)}$ are as before it follows from $m_\eta^{(2)}(r)\geqslant 0$ that

$$\liminf_{\eta\to 0}\gamma_\eta\geqslant\lim_{k\to\infty} r^{-1}m^{(1)}(U_{\eta_k};r) = r^{-1}m^{(1)}(r)$$

hence

$$\limsup_{r\to\infty} r^{-1}m(U;r) = \limsup_{r\to 0} r^{-1}m^{(1)}(r)+c\leqslant\liminf_{\eta\to 0}\gamma_\eta+c<\infty$$

(in fact $c=0$, and $\gamma_\eta=\gamma$ as follows from Proposition 3).

LEMMA 11. *Suppose $U \geqslant 0$ is subharmonic in R_+^{n+1} and has a harmonic majorant. Let h be its least harmonic majorant and ν the measure given by the Riesz decomposition theorem (see, e.g., [19] p. 132), then*

$$U(z) = h(z) - \int_{R_+^{n+1}} G(z, w)\nu(dw).$$

In particular

$$\int_{|w|<\varepsilon} v\nu(dw) < \infty, \qquad \int_{|w|>\varepsilon} v|w|^{-n-1}\nu(dw) < \infty \qquad (w = (t, v)) \text{ for any } \varepsilon > 0.$$

Also

$$(53)\qquad r^{-1}m(U; r) = (\omega_{n+1}\gamma/2) + (n+1)^{-1}\times$$
$$\times\Big(r^{-n-1}\int_{|t|\leqslant r}\mu(dt) + \int_{|t|>r}|t|^{-n-1}\mu(dt) - r^{-n-1}\int_{|w|\leqslant r} v\nu(dw) - \int_{|w|>r} v|w|^{-n-1}\nu(dw)\Big).$$

Proof. For $a > 0$ let $G_a(z, w)$ denote the Green's function for aB_+^{n+1}, i.e.,

$$G_a(z, w) = [(n-1)\omega_{n+1}]^{-1}\times$$
$$\times[|z-w|^{-n+1} - |z-\bar{w}|^{-n+1} - a^{n-1}|w|^{-n+1}(|z-w_a^*|^{-n+1} - |z-\bar{w}_a^*|^{-n+1})],$$

where $w_a^* = a^2|w|^{-2}w$, for $n > 1$. The case $n = 1$ is similar (see [32]). Clearly $\lim_{a\to\infty} G_a(z, w) = G(z, w)$. Also it is well known that $G_a(z, w)$ is an increasing function of a. Now

$$U_\eta(z) = -\int_{S_+^n}\frac{\partial G_a}{\partial r}(z, r\sigma)\Big|_{r=a} U(a\sigma)\,d\sigma + \int_{|t|\leqslant a}\frac{\partial G_a}{\partial v}\big(z, (t,v)\big)\Big|_{v=0} U(t, \eta)\,dt$$
$$-\int_{v\geqslant\eta} G_a\big(z, w-(0,\eta)\big)\nu(dw).$$

By Lemma 9 the first two integrals in the last equation tend to the least harmonic majorant h_η of U_η in R_+^{n+1}. It follows that

$$\int_{v\geqslant\eta} G\big(z, w-(0,\eta)\big)\nu(dw) = h_\eta(z) - U_\eta(z).$$

Again since the Green's function G_D of a domain D increases as D expands it follows from the monotone convergence theorem that

$$\int_{R_+^{n+1}} G(z, w)\nu(dw) = h(z) - U(z).$$

Proof of Proposition 3. (35), (36) are necessary by the last part of Lemma 10 and Lemmas 3 and 6 since $\big(1, (1+|\cdot|)^{-n-1}\big)\in S_1^{*1}$, with $\alpha_0 = \alpha_1 = 0$, $\beta_0 < n_1$, $\beta_1 = n+1$.

Let now (35), (36) be satisfied. To prove that U has a harmonic majorant it suffices to show that $\limsup_{r\to\infty} r^{-1}m(U; r) < \infty$. But

$$r^{-1}m(U; r) = \int_0^{\pi/2}\int_{S^{n-1}} U(r\sigma'\cos\theta, r\sin\theta)\sin\theta\cos^{n-1}\theta\,d\sigma'\,d\theta.$$

Also

$$U(r\sigma'\cos\theta, r\sin\theta) \leqslant C(r\sin\theta)^{-n-1}\int_{|t-r\sigma'\cos\theta|^2+|v-r\sin\theta|^2\leqslant(r\sin\theta)^2/4} U(t, v)\,dt\,dv.$$

Hence

$$r^{-1}m(U; r) \leqslant Cr^{-n-2}\int_0^{3r/2}\int_{|t|\leqslant 3r/2} U(t, v)\int_{(2v)/(3r)\leqslant\sin\theta\leqslant(2v)/r}\times$$
$$\times(\sin\theta)^{-n}(\cos\theta)^{n-1}\int_{|\sigma'-t/(r\cos\theta)|\leqslant\sin\theta/(2\cos\theta)} d\sigma'\,d\theta\,dt\,dv$$

since

$$|v - r\sin\theta| \leqslant (r\sin\theta)/2 \quad\text{implies}\quad v/(3r) \leqslant (\sin\theta)/2 \leqslant v/r.$$

So by (52)

$$r^{-1}m(U; r) \leqslant Cr^{-n-2}\int_0^{3r/2}\int_{|t|\leqslant 3/(2r)} U(t, v)\int_{2v/(3r)\leqslant\sin\theta\leqslant 2v/r}(\sin\theta)^{-n}\times$$
$$\times\min\Big(\big(\sin\theta/(2\cos\theta)\big)^{n-1}, 1\Big)\cos^{n-1}\theta\,d\theta\,dt\,dv$$
$$\leqslant Cr^{-n-2}\int_0^{3r/2}\int_{|t|\leqslant 3/(2r)} U(t, v)\int_{(2v)/(3r)\leqslant\sin\theta\leqslant 2v/r}\theta^{-1}\,d\theta\,dt\,dv$$
$$\leqslant Cr^{-n-2}\int_0^{3r/2}\int_{|t|\leqslant 3r/2} U(t, v)\,dt\,dv$$
$$\leqslant Cr^{-n-2}\int_0^1\int_{|t|\leqslant 3r/2} U(t, v)\,dt\,dv + Cr^{-n-2}\int_1^{3r/2}\Big(\int_{|t|\leqslant v} U(t, v)\,dt +$$
$$+\int_{v\leqslant|t|\leqslant 3r/2} U(t, v)\,dt\Big)\,dv$$
$$\leqslant CM_0r^{-1} + Cr^{-1}\int_1^{3r/2}\Big(v^{-n-1}\int_{|t|\leqslant v} U(t, v)\,dt +$$
$$+\int_{|t|\geqslant v} U(t, v)|t|^{-n-1}\,dt\Big)\,dv \leqslant CM_0r^{-1} + CM_1.$$

Define

$$n(U; y) = y^{-n}\int_{|x|\leqslant y} U(x, y)\,dx.$$

If $h = P*\mu+\delta p_0$ is the least harmonic majorant of U it follows from Lemma 6 that $y^{-1}n(P*\mu; y)\to 0$ as $y\to\infty$ (since this holds for the measures of compact support, which form a dense subset of $(1+|\cdot|)^{n+1}\mathcal{M}^1(R^n)$ in the weak topology). Also $n(p_0; y) = (\omega_n/n)y$. It therefore remains to show that $y^{-1}n(g; y)\to 0$ where

$$g(z) = \int_{R_+^{n+1}} G(z, w)\nu(dw)$$

(see Lemma 11). This will follow from

LEMMA 12. *Let $z = (x, y)$, $w = (t, v)$ then*

$$y^{-n-1}\int_{|x|\leqslant y} G(z, w)\,dx \leqslant Cv|w|^{-n-1} \quad \textit{and also} \leqslant Cvy^{-n-1}. \tag{54}$$

For then by Fubini's theorem

$$y^{-n-1}\int_{|x|\leqslant y} g(z)\,dx \leqslant Cy^{-n-1}\int_{|w|\leqslant y/2} v\nu(dw)+C\int_{|w|>y/2} v|w|^{-n-1}\nu(dw)\to 0 \quad \text{as } y\to\infty$$

as follows from Lemma 11.

Proof of Lemma 12. $G(z, w)\leqslant Cyv|z-w|^{-n-1}$ hence if $y\geqslant 2v$, $|t|\geqslant 2y$ then

$$y^{-n-1}\int G(z, w)\,dx \leqslant Cy^{-n-1}\int_{|x|\leqslant y} vy|t-x|^{-n-1}dx \leqslant Cv|t|^{-n-1}\leqslant Cv|w|^{-n-1}.$$

If $y\geqslant 2v$, $|t|\leqslant 2y$, then

$$y^{-n-1}\int_{|x|\leqslant y} G(z, w)\,dx \leqslant Cvy^{-2n-1}\int_{|x|\leqslant y} dx \leqslant Cvy^{-n-1}\leqslant Cv|w|^{-n-1}.$$

If $|w|/3\leqslant y\leqslant 2v$ then

$$y^{-n-1}\int_{|x|\leqslant y} G(z, w)\,dx \leqslant Cy^{-n-1}\int_{|x|\leqslant y} |x-t|^{-n+1}dx \leqslant Cy^{-n}\leqslant Cv|w|^{-n-1}.$$

If $y\leqslant |w|/3$ then $|x|\leqslant y$ implies $|z|\leqslant 2y\leqslant 2|w|/3$ hence $|z-w|\geqslant w/3$ and

$$y^{-n-1}\int_{|x|\leqslant y} G(z, w)\,dx \leqslant Cvy^{-n}|w|^{-n-1}\int_{|x|\leqslant y} dx \leqslant Cv|w|^{-n-1}.$$

Proof of the corollary. By (32), (33) (39) implies (35) and similarly for $y\geqslant 1$

$$y^{-n-1}\int_{|x|\leqslant y} U(x, y)\,dx+\int_{|x|>y} U(x, y)|x|^{-n-1}dx$$
$$\leqslant Cy^{-n/p-1}\|\chi(y^{-1}|\cdot|)U(\cdot, y)\|_{pq}+|\chi(y|\cdot|^{-1})|\cdot|^{-n-1}\omega|\cdot|^{-1}\|_{p'q'}\|\chi(y|\cdot|^{-1})U(\cdot, y)\|_{pq,\infty}$$
$$\leqslant C\omega(1)^{-1}[\omega(y)y^{\beta_1-n/p-1}\|\chi(y^{-1}|\cdot|)U(\cdot, y)\|_{pq}+$$
$$+\|\chi(y|\cdot|^{-1})|\cdot|^{\beta_1-n-1}\|_{p'q'}\|\chi(y|\cdot|^{-1})U(\cdot, y)\|_{pq,\infty}]$$
$$\leqslant C\omega(1)^{-1}y^{\beta_1-n/p-1}[\omega(y)\|\chi(y^{-1}|\cdot|)U(\cdot, y)\|_{pq}+\|\chi(y|\cdot|^{-1})U(\cdot, y)\|_{pq,\infty}].$$

So by Proposition 3 there are μ, δ such that (38) holds. For $q\geqslant 1$ a weak limit μ', say, exists in $L_\omega^{pq}\subset(1+|\cdot|)^{n+1}\mathcal{M}^1(R^n)$ hence by uniqueness $\mu = \mu'\in L_\omega^{pq}$ (Since $\omega^{-1}\mathcal{M}^1(R^{n*})\subset(1+|\cdot|)^{n+1}\mathcal{M}^1(R^{n*})$ and since by Proposition 3 the weak limit of $U(\cdot, y)$ in $(1+|\cdot|)^{n+1}\mathcal{M}^1(R^{n*})$ is in $(1+|\cdot|)^{n+1}\times\mathcal{M}^1(R^n)$ i.e., a measure on R^n so must be the weak limit of $U(\cdot, y)$ in $\omega^{-1}\mathcal{M}^1(R^{n*})$. Hence if $p=1$ $\mu\in\omega^{-1}\mathcal{M}^1(R^n)$. Now by Proposition 4 below there is a well known conformal mapping I of R_+^{n+1} onto $B^{n+1} = \{\zeta\in R^{n+1}: |\zeta|<1\}$ such that if U is a subharmonic function in R_+^{n+1} satisfying (35) then If defined by

$$If(\zeta) = 2^{n-1}|\zeta+(0, 1)|^{-n+1}f(I^{-1}\zeta)$$

is subharmonic in B^{n+1} and has a harmonic majorant there. Now by Littlewood's theorem on subharmonic functions in a disk ([20]) extended to subharmonic functions in a ball in R^n by Privalov in [24] (see also [26]) and since by a similar proof Littlewood's theorem holds for approach along the images of the straight lines perpendicular to $y=0$ under the mapping I it follows that $\lim\limits_{y\to 0} U(x, y) = U(x, 0)$ a.e., where $U(\cdot, 0)$ is the absolutely continuous part of μ. Thus by dominated convergence in case (a) and $q<\infty$ it follows that $\|U(\cdot, y)-U(\cdot, 0)\|_{pq,\infty}\to 0$. If (b) or (c′) (also if (a), $q\geqslant 1$) holds (41), and, in particular (37) follows from the last part of Lemma 3 and weak convergence:

$$\limsup_{y\to 0}\|U(\cdot, y)\|_{pq,\infty}\leqslant\limsup_{y\to 0}\|P(\cdot, y)*\mu\|_{pq,\infty}\leqslant\|\mu\|_{pq,\infty}\leqslant\liminf_{y\to 0}\|U(\cdot, y)\|_{pq,\infty}.$$

Remarks. If (a), $q<\infty$ or (b) with $p>1$ holds, then

$$\lim_{y\to 0}\|U(\cdot, y)-U(\cdot, 0)\|_{pq,\infty} = 0. \tag{55}$$

In the first case this has just been shown. In the latter case for any $\varepsilon>0$ there is a compact set $K\subset R^n\sim\{0\}$ such that $U(\cdot, y)\to U(\cdot, 0)$ uniformly in K and if $\chi_{\sim K}$ denotes the characteristic function of the complement $\sim K$ of K then $\|\chi_{\sim K}U(\cdot, 0\|_{pq,\infty}<\varepsilon$ hence if $\mu(dx) = f(x)\,dx$ then (if without loss of generality, $\delta = 0$)

$$\|U(\cdot, y)\chi_{\sim K}\|_{pq,\infty}\leqslant\|\chi_{\sim K}P(\cdot, y)*f\|_{pq,\infty}$$
$$\leqslant\|\chi_{\sim K}P(\cdot, y)*\chi_K\|_{pq,\infty}\|f\chi_K\|_\infty+\|\chi_{\sim K}P(\cdot, y)*(f\chi_K)\|_{pq,\infty}.$$

Let $N(K, \varrho)$ denote the ϱ-neighborhood of K, i.e., $N(K, \varrho) = \{x: \inf\limits_{y\in K}|x-y|<\varrho\}$ and let χ_ϱ denote the characteristic function of $N(K, \varrho)\sim K$ then $P(\cdot, y)*\chi_K\leqslant 1$ and for $x\notin N(K, \varrho)$

$$P(\cdot, y)*\chi_K(x)\leqslant C_{K,\varrho}y(1+|x|)^{-n-1}.$$

Hence

$$\limsup_{y\to 0}\|\chi_{\sim K}P(\cdot, y)*\chi_K\|_{pq,\omega} \leqslant \|\chi_\varrho\|_{pq,\omega}+\limsup_{y\to 0} y\, C_{K,\varrho}\|(1+|\cdot|)^{-n-1}\|_{pq,\omega} = \|\chi_\varrho\|_{pq,\omega}$$

which tends to 0 as $\varrho\to 0$, hence

$$\limsup_{y\to 0}\|\chi_{\sim K}P(\cdot, y)*\chi_K\|_{pq,\omega} = 0.$$

It follows that

$$\limsup_{y\to 0}\|U(\cdot, y)\chi_{\sim K}\|_{pq,\omega} \leqslant \|\chi_{\sim K}P(\cdot, y)*(f\chi_{\sim K})\|_{pq,\omega} \leqslant C\varepsilon \quad \text{(by Lemma 3).}$$

By uniform convergence in K $\|\chi_K[U(\cdot, y)-U(\cdot, 0)]\|_{py,\omega}\to 0$, hence altogether

$$\limsup_{y\to 0}\|U(\cdot, y)-U(\cdot, 0)\|_{pq,\omega} \leqslant \|U(\cdot, 0)\chi_{\sim K}\|_{pq,\omega}+\limsup_{y\to 0}\|U(\cdot, y)\chi_{\sim K}\|_{pq,\omega} \leqslant (C+1)\varepsilon.$$

Since ε is arbitrary (55) follows. In case $1<p, q<\infty$ (55) can be proved without use of Littlewood's theorem. For in this case L^{pq} is uniformly convex (see [11]) and hence weak convergence and (41) imply (55) (see, e.g., [10] p. 141). Also in case $p=1$ (and (b)) if μ is absolutely continuous by the same proof as before

$$\|U(\cdot, y)-U(\cdot, 0)\|_{1,\omega}\to 0.$$

The proof of Proposition 3 contained the following criterion: a non-negative subharmonic function U in R_+^{n+1} has a harmonic majorant if and only if (35) is satisfied and

$$\sup_{r\geqslant 1} r^{-n-2}\iint_{|x|^2+y^2=r^2} U(x, y)\,dx\,dy<\infty.$$

Hence

$$\sup_{r\geqslant 1} r^{-n-2}\int_0^r \|U(\cdot, y)\|_{pq,\omega}\,dy\,\|\chi_{B^n(0,r)}\|_{p'q',\omega^{-1}}<\infty$$

and so

$$\sup_{1\leqslant y<r}\|U(\cdot, y)\|_{pq,\omega}\leqslant M_1 r^{n+1}\|\chi_{B^n(0,r)}\|^{-1}_{p'q',\omega^{-1}}$$

along with (4), (5) (for $y\leqslant 1$) are sufficient. In particular

$$\sup_{0<y\leqslant 1}\|U(\cdot, y)\|_{pq,\omega}<\infty \quad\text{and}$$

$$\|U(\cdot, y)\|_{pq,\omega}\leqslant M_1 y^{n+1-\beta_1-n/p'} = M_1 y^{n/p+1-\beta_1} \quad \text{for}\quad y\geqslant 1,$$

where p, q, ω satisfy (a), (b), (c), or (d) are sufficient.

4. H^p spaces. As in [29] let now $F(x, y) = (u(x, y), v_1(x, y), \ldots, v_n(x, y))$ be an $(n+1)$-tuple of conjugate harmonic functions in R_+^{n+1} (the same methods apply to higher gradients, see [5]). Define

$$|F(x, y)| = (|u(x, y)|^2+\sum_{i=1}^n |v_i(x, y)|^2)^{\frac12}.$$

Attention will be restricted to the case $n>1$. If $n=1$ it is clear that the role of $(n-1)/n$ may be played by any real number in the interval $(0, 1)$.

PROPOSITION A. *Suppose $np/(n-1)$, $\omega^{(n-1)/n}$, $nq/(n-1)$, $ns/(n-1)$ satisfy* (a), (b) (c) *or* (d) *of Lemma* 3 *and*

$$\sup_{0<y\leqslant 1}\|F(\cdot, y)\|_{pq,\omega}+\sup_{1\leqslant y<\infty}[\omega(y)\|\chi(y^{-1}|\cdot|)F(\cdot, y)\|_{pq}+\|\chi(y|\cdot|^{-1})F(\cdot, y)\|_{pq,\omega}] \leqslant M<\infty$$

or

$$\|F(\cdot, y)\|_{pq,\omega}\leqslant M(1+y)^{n/p+(1-\beta_1)n/(n-1)}. \tag{56}$$

Then F has non-tangential boundary values $F(\cdot, 0)\in L^{pq}_\omega$ a.e. In case (a) *if $q<\infty$ F converges also in L^{pq}_ω. If $F^{*\eta}$ is defined as in Lemma* 5 *starting from F then*

$$\|F^{*\eta}\|_{ps,\omega}\leqslant C(\eta)^{n/(n-1)}M,$$

where $C(\eta)$ can be found from Lemma 5. *If*

$$\delta = n/\omega_n \lim_{y\to\infty} y^{-n-1}\int_{|x|\leqslant y}|F(x, y)|^{(n-1)/n}dx$$

then in all cases except when $p=(n-1)/n$

$$|F(x, y)|^{(n-1)/n}\leqslant P(\cdot, y)*|F(\cdot, 0)|^{(n-1)/n}(x)+\delta y. \tag{57}$$

Proof. By the result of Stein and Weiss in [29] $|F|^{(n-1)/n}$ is subharmonic. It follows from the assumptions that $|F|^{(n-1)/n}$ satisfies the hypotheses of Proposition 3. Hence there exists $\mu\in L(np/(n-1), nq/(n-1), \omega^{(n-1)/n})\times$ $\times(\mu\in\omega^{-(n-1)/n}\mathcal{M}^1(R^n))$ such that

$$|F(x, y)|^{(n-1)/n}\leqslant P(\cdot, y)*\mu(x)+\delta y$$

hence by Lemma 5 F is non-tangentially bounded a.e. at the boundary $y=0$ and hence by Calderón's theorem non-tangential boundary values exist a.e. If $p>(n-1)/n$ then μ is absolutely continuous with respect to Lebesgue measure and so $\mu(dx) = |F(x, 0)|^{(n-1)/n}dx$. If (56) is satisfied the assertions follow from the last remark of Section 3.

In the special case $\omega(|x|) = |x|^\alpha$ it follows from (9) that

$$0\leqslant\delta\leqslant C\lim_{y\to\infty} y^{-\alpha(n-1)/n-(n-1)/p-1} = 0.$$

Hence Proposition A with condition (56) yields the following

COROLLARY. *Suppose* $F \in H_a^{pq}$ (*i.e.* $\sup_{y>0} \|F(\cdot, y)\|_{pq,a} = \|F\|[H_a^{pq}] < \infty$) *and one of*

(58a) $(n-1)/n < p < \infty, \quad -n/p < a < n\big(n/(n-1) - 1/p\big), \quad 0 < q = s \leqslant \infty,$

(58b) $(n-1)/n < p < \infty, \quad a = -n/p$ *or* $p = \infty, \quad 0 \leqslant a < n^2/(n-1)$ *and* $q = s = \infty,$

(58c) $(n-1)/n < p < \infty, \quad a = n\big(n/(n-1) - 1/p\big)$ *or* $p = (n-1)/n, \quad -n \leqslant a \leqslant n/(n-1), \quad q = (n-1)/n, \quad s = \infty,$

then F *has non-tangential boundary values* $F(\cdot, 0) \in L_a^{pq}$ *a.e. Moreover if*

$$F^*(x) = \sup\{|F(t, y)|:\ (t, y) \in \Gamma_k(x)\} \quad \text{then} \quad \|F^*\|_{ps,a} \leqslant C\|F\|[H_a^{pq}].$$

If $p > (n-1)/n$ *then*

(59) $$|F(x, y)|^{(n-1)/n} \leqslant P(\cdot, y) * |F(\cdot, 0)|^{(n-1)/n}(x).$$

Remark. As for Theorem C of [29] asserting convexity of the function $y \to \|F(\cdot, y)\|_p^{(n-1)/n}$ if $F \in H^p$, $p \geqslant (n-1)/n$ it is clear that the same proof works for H^{pq} whenever $\|\cdot\|_{p^*q^*}$ is a norm for $p^* = np/(n-1)$, $q^* = nq/(n-1)$, i.e., $p > (n-1)/n$, $q \geqslant (n-1)/n$ or $p = q = (n-1)/n$. The well known fact that if $u(x', x'') = u(x_1, \ldots, x_m, x_{m+1}, \ldots, x_n) = v(x_1, \ldots, x_m) = v(x')$ u is subharmonic in $D \times R^{n-m}$, where D is open in R^m, if and only v is subharmonic in D can also be proved as follows. Since upper semi-continuity of u is equivalent to upper semi-continuity of v u is subharmonic if and only if in addition for all $x' \in D$ and r sufficiently small

$$u(x_0', 0) \int_{|x-(x_0', 0)| \leqslant r} dx \leqslant \int_{|x-(x_0', 0)| \leqslant r} u(x)\, dx \quad \text{or}$$

$$v(x_0') \int_{|x'-x_0'| \leqslant r} \int_{|x''|^2 \leqslant r^2 - |x'-x_0'|^2} dx''\, dx' \leqslant \int_{|x'-x_0'| \leqslant r} v(x') \int_{|x''| \leqslant r^2 - |x'-x_0'|^2} dx''\, dx'.$$

Hence if $\varphi(t) = (1+t^2)^{(n-m)/2}$ this condition can be written

$$v(x_0') \int_{|x'-x_0'| \leqslant r} \varphi(|x'-x_0'|/r)\, dx' \leqslant \int_{|x'-x_0'| \leqslant r} v(x')\varphi(|x'-x_0'|/r)\, dx'$$

which (under the assumption of upper semi-continuity) is equivalent to subharmonicity of v (see, e.g., [16] p. 17).

It appears sufficient to restrict attention to the weights $|x|^a$ from now on.

PROPOSITION B. *Suppose* $F \in H_{a_1}^{p_1 q_1}$, $F(\cdot, 0) \in L_{a_2}^{p_2 q_2}$ *where* p_1, q_1, a_1 *and* p_2, q_2, a_2 *satisfy* (58) *and in addition* $p_1 > (n-1)/n$ *then* $F \in H_{a_2}^{p_2 q_2}$.

This follows from (59) and Lemma 3.

Theorem E of [29] asserts that if μ is a finite measure such that each of its M. Riesz transforms $\nu_k = R_k \mu$ is also a finite measure then all the measures $\mu, \nu_1, \ldots, \nu_n$ are absolutely continuous with respect to Lebesgue measure. As usual let $\mathcal{D}$ denote the space of C^∞-functions of compact support in R^n. For any set E $\mathcal{D}(E)$ will denote the subspace of functions of $\mathcal{D}$ whose support is contained in E. $\mathcal{D}$ is provided with the usual inductive limit topology. The (vector-valued) Riesz transform of $\varphi \in \mathcal{D}$ is defined by

$$R\varphi(x) = \lim_{\varepsilon \to +0} c_n^{-1} \int_{|x-t| > \varepsilon} \frac{x-t}{|x-t|^{n+1}} \varphi(t)\, dt.$$

It is well known that $R\varphi \in C^\infty$. For a measure μ such that

(60) $$\|(1+|\cdot|)^{-n}\mu\| < \infty$$

its Riesz transform in the distribution sense R is defined by

(61) $$(R\mu, \varphi) = -(\mu, R\varphi),$$

where φ is any element of $\mathcal{D}$.

It will be convenient to define two function spaces A_0, A_1. Let A_0 denote the space of continuous functions φ such that $\|\varphi\|[A_0] = \sup(1+|x|)^n|\varphi(x)| < \infty$ and let A_1 denote the space of continuously differentiable functions φ such that

$$\|\varphi\|[A_1] = \|\varphi\|_1 + \sup\{(1+|x|)^{n+1}|(\partial/\partial x_i)\varphi(x)|:\ 1 \leqslant i \leqslant n,\ x \in R^n\} < \infty$$

and such that

$$\max_{1 \leqslant i \leqslant n} (1+|x|)^{n+1}|(\partial/\partial x_i)\varphi(x)| \to 0 \quad \text{as} \quad |x| \to \infty.$$

The latter condition implies that $\mathcal{D}$ is dense in A_1.

$$\begin{aligned}
|R\varphi(x)| &= C\left|p.v. \int \frac{\varphi(x-t)t}{|t|^{n+1}}\, dt\right| \\
&\leqslant C \int_{|t| \leqslant |x|/2} |\varphi(x-t) - \varphi(x)|\,|t|^{-n}\, dt + C \int_{|t| \geqslant |x|/2} |\varphi(x-t)|\,|t|^{-n}\, dt \\
&\leqslant C \max_{|x-u| \leqslant |x|/2} |\operatorname{grad}\varphi(u)| \int_{|t| \leqslant |x|/2} |t|^{-n+1}\, dt + C|x|^{-n}\|\varphi\|_1 \\
&\leqslant C\left(\frac{|x|}{(1+|x|)^{n+1}} + |x|^{-n}\right)\|\varphi\|[A_1] \leqslant C\|\varphi\|[A_1]\,|x|^{-n}.
\end{aligned}$$

Also

$$|R\varphi(x)| \leqslant C \int_{|t| \leqslant 1} |\varphi(x-t) - \varphi(x)|\,|t|^{-n}\, dt + C\|\varphi\|_1 \leqslant C\|\varphi\|[A_1]$$

thus $|R\varphi(x)| \leqslant C\|\varphi\|[A_1](1+|x|)^{-n}$ or $\|R\varphi\|[A_0] \leqslant C\|\varphi\|[A_1]$ hence $R\mu$ defined by (61) is a distribution and can be extended continuously to A_1

so that (61) is valid for any $\varphi \in A_1$. If $\varphi = P(\cdot, y) \in A_1$, then $R\varphi = Q(\cdot, y) \times \times \big(Q(x, y) = c_n^{-1} x (y^2 + |x|^2)^{-(n+1)/2}$, the conjugate Poisson kernel$\big)$. Hence if $R\mu$ is again a measure $\nu = (\nu_1, \ldots, \nu_n)$ such that $\|\nu(1+|\cdot|)^{-n}\| < \infty$ then by continuity

$$P(\cdot, y) * \nu = Q(\cdot, y) * \mu \tag{62}$$

$$\big(P(\cdot, y) * \nu(x) = (\nu, P(x - \cdot, y)), \qquad Q(\cdot, y) * \mu(x) = (\mu, Q(x - \cdot, y))\big).$$

Let (61) be satisfied and let the restriction of $R\mu$ in the distribution sense to an open set Ω be a measure ν. Let K be a compact subset of Ω, $\psi \in \mathcal{D}(\Omega)$ and such that $\psi = 1$ on K. Then for $\varphi \in \mathcal{D}(K)$

$$\begin{aligned}\big(R(\psi\mu), \varphi\big) &= -(\psi\mu, R\varphi) = -(\mu, \psi R\varphi)\\ &= -\big(\mu, R(\psi\varphi)\big) + \big(\mu, R(\psi\varphi)\big) - \psi R\varphi\\ &= (\nu, \psi\varphi) + \big(\mu, R(\psi\varphi) - \psi R\varphi\big).\end{aligned}$$

Also

$$\begin{aligned}\big(\mu, R(\psi\varphi) - \psi R\varphi\big) &= C \int\int [\psi(x-t) - \psi(x)]\varphi(x-t)|t|^{-n-1} t \, dt \mu(dx)\\ &= C(-\gamma, \varphi),\end{aligned}$$

where

$$\gamma(x) = \int [\psi(x) - \psi(u)]\, |x-u|^{-n-1}(x-u)\mu(du) \in A_0$$

hence

$$\big(R(\psi u), \varphi\big) = (\varphi\nu - \gamma, \varphi).$$

It is thus apparent that in the case of periodic measures Theorem E of [29] implies a local version since the function corresponding to γ will be in L^1 of the fundamental cure. (For more details on Poisson integrals and Riesz transforms of periodic functions and measures see [31].)

In the present case, however, γ may not be integrable. It would be sufficient to establish an extension of Theorem E to finite measures such that $\|(1+|\cdot|)^{-a} R\mu\| < \infty$ for some $a > 0$, which by the above will imply a local version: if (60) and the restriction of $R\mu$ to the open set Ω is a measure ν then μ, ν are absolutely continuous in Ω. The proof of Theorem E of [29], however also implies that the Riesz transform of μ in the distribution sense is equal to the Riesz transform in the function sense a.e. if the former is integrable (which can be proved without the use of H^p spaces for more general singular integrals).

PROPOSITION C. *Let μ be a measure on R^n satisfying (60) such that the Riesz transform of μ in the distribution sense $R\mu$ is a measure $\nu = (\nu_1, \nu_2, \ldots \ldots, \nu_n)$ such that $\|\nu(1+|\cdot|)^{-n}\| < \infty$ then μ and ν are absolutely continuous with respect to Lebesgue measure and if $\nu(dx) = g(x)dx$, $\mu(dx) = f(x)dx$ then g is the Riesz transform of f a.e.*

Proof. $\big(P(\cdot, y) * \mu(x), Q(\cdot, y) * \mu(x)\big)$ is an $(n+1)$-tuple of conjugate harmonic functions. By (62)

$$F(x, y) = \big(P(\cdot, y) * \mu(x), P(\cdot, y) * \nu(x)\big)$$

hence if $\omega(\tau) = (1+\tau)^{-n}$ it follows from Lemma 3 with $a_0 = a_1 = \beta_0$, $\beta_1 = n$ that

$$\|F(\cdot, y)\|_{1,\omega} \leqslant C(y)\,(\|\mu\omega\| + \|\nu\omega\|) \quad \text{where} \quad C(y) = C(1 + \log^+ y).$$

Also by (condition (56) of Proposition A where $\beta_1 = n-1$ $\big($for $\omega^{(n-1)/n} = (1+|\cdot|)^{-n+1}\big)$ the fact that $\log y = O(y^{n/(n-1)})$ as $y \to \infty$ is sufficient for the conclusion of Proposition A. So $F^{*1} \in L^1_\omega$, hence the family of functions $\{F(\cdot, y)\colon 0 < y \leqslant 1\}$ is uniformly integrable (locally) and so μ, ν are absolutely continuous with respect to Lebesgue measure by the last paragraph of Section 1. If then f, g are defined by μ, ν as in the statement of the proposition $Q(\cdot, y) * \mu = Q(\cdot, y) * f$ tends to g a.e. as $y \to 0$. If R_y denotes the truncated Riesz kernel: $R_y(x) = c_n^{-1}|x|^{-n-1}x$ for $|x| > y = 0$ otherwise then (see [35])

$$|(Q(\cdot, y) - R_y) * f| \leqslant CP(\cdot, y) * |f|.$$

By means of Lemma 5 it follows as in the L^p case that $g = Rf$ a.e.

As in [29] for $F \in H^{pq}_a$ define the fractional integral of order λ of F by

$$I_\lambda F(x, y) = F_\lambda(x, y) = [\Gamma(\lambda)]^{-1} \int_0^\infty F(x, y+u) u^{\lambda-1} du. \tag{63}$$

LEMMA 13. *If $F \in H^{pq}_a$, $\lambda > 0$*

$$(n-1)/n \leqslant p < \infty, \tag{64}$$

$$-n/p < a \leqslant n\big(n/(n-1) - 1/p\big), \tag{65}$$

$$a + n/p - \lambda \geqslant 0 \tag{66}$$

and at most one of (65), (66) is not strict and if moreover $q = (n-1)/n$ if at least one of (64), (65), (66) is not strict then (63) defines a system of conjugate harmonic functions.

Proof. The condition that at least one of (65), (66) is strict is equivalent to the validity of at most one equality in $\lambda \leqslant a + n/p \leqslant n^2/(n-1)$, i.e., to

$$\lambda < n^2/(n-1).$$

It follows from (9) applied to $|F|^{(n-1)/n}$ that

$$|F(x, y+u)|u^{\lambda-1} \leqslant CM(y+u)^{-n/p}(|x| + y + u)^{-a} u^{\lambda-1} \qquad (M = \|F\|[H^{pq}_a]).$$

Hence if $a + n/p - \lambda > 0$ then (63) is locally uniformly convergent, which proves the assertion in this case.

If $\lambda = a+n/p$ let $s(x, y) = |F(x, y)|^{(n-1)/n}$. Then

$$(67)\qquad \int_0^\infty |F(x, y+u)|u^{\lambda-1}du \leqslant \sup_{u>0}|F(x, y+u)u^{\lambda}|^{1/n}\int_0^\infty s(x, y+u)u^{\lambda(n-1)/n-1}du.$$

By (9)

$$\sup_{u>0}|F(x, y+u)u^{\lambda}| \leqslant CM\sup_{u>0}(|x|+y+u)^{-a}(y+u)^{-n/p}u^{\lambda}$$
$$\leqslant CM[\sup_{u\leqslant y+|x|}(|x|+y)^{-a}(y+u)^{-n/p}u^{\lambda}+\sup_{u\geqslant|x|+y}u^{-a-n/p+\lambda}]$$
$$\leqslant CM[(|x|+y)^{-a+\lambda}y^{-n/p}+(|x|+y)^{-a-n/p+\lambda}]$$
$$\leqslant CM(|x|+y)^{-a+\lambda}y^{-n/p}.$$

For $0<\mu<n$ and $f\geqslant 0$

$$\int_0^\infty P*f(x, y)y^{\mu-1}dy = C_\mu|\cdot|^{-n+\mu}*f(x)$$

(see (6.4) of [29]). Hence

$$(68)\qquad \int_0^\infty s(x, y+u)u^{\lambda(n-1)/n-1}du \leqslant C\int_{R^n}s(x-t, y)|t|^{-n+\lambda(n-1)/n}dt.$$

The last integral is at most equal to

$$C\int_{|t|\leqslant 2(y+|x|)}s(x-t, y)|t|^{-n+\lambda(n-1)/n}dt+$$
$$+C\int_{|t|\geqslant 2(y+|x|)}s(x-t, y)|x-t|^{a(n-1)/n}|t|^{-n+(\lambda-a)(n-1)/n}dt.$$

Since further for any two functions f, g in L^1+L^∞, $\int fg\leqslant\int f^*g^*$ where as before f^*, g^* denote the decreasing rearrangements of f, g on $(0, \infty)$ (see [37] II p. 124) the last sum is at most

$$C(y+|x|)^{\lambda(n-1)/n}\sup_{|t|\leqslant 3(y+|x|)}s(t, y)+C\int_0^\infty s^*_{a,y}(\tau)\tau^{(\lambda-a)(n-1)/n^2-1}d\tau,$$

where $s^*_{a,y}$ denotes the decreasing rearrangement of $s(\cdot, y)|\cdot|^{a(n-1)/n}$ for by (64) $(\lambda-a)(n-1)/n = (n-1)/p\leqslant n$ and so the above power of $|t|$ is decreasing. By (9) again the last sum is

$$\leqslant CM^{(n-1)/n}(y+|x|)^{\lambda(n-1)/n}\max\big((y+|x|)^{-a(n-1)/n}, y^{-a(n-1)/n}\big)y^{-(n-1)/p}+$$
$$+C\|F(\cdot, y)\|^{(n-1)/n}_{p,(n-1)/n}$$
$$\leqslant CM^{(n-1)/n}(1+|x|/y)^{\max[1/p,\lambda/n](n-1)}.$$

Thus $\int_0^\infty|F(x, y+u)|u^{\lambda-1}du$ is locally bounded in R^{n+1}_+. Since the systems of conjugate harmonic functions $\int_0^k F(x, y+u)u^{\lambda-1}du$, $k = 1, 2, \ldots$, tend boundedly to $F_\lambda(x, y)$ the latter is a system of conjugate harmonic functions.

Remark. In case $a+n/p-\lambda = 0$ it is sufficient to require

$$1/q = \max(1, 1/p, 1/p+a/n)$$

in place of $1/q = n/(n-1)$ and $\lambda<n/q$. This follows from consideration of $|F|^q$ in place of $|F|^{(n-1)/n}$.

The fractional integration theorem in weighted norms (see [28]) may be stated as follows. With the notation $a^+ = \max(a, 0)$ suppose (i) $a^+/n\leqslant 1/p'$, (ii) $\beta^+\leqslant n/r$, $a+\beta\geqslant 0$, $0<\lambda<n$, $1/r-1/p = (-\lambda+a+\beta)/n$, $0<q\leqslant s\leqslant\infty$ except that $q = 1$, $s = \infty$ if equality holds in (i) or (ii) then

$$\||\cdot|^{-n+\lambda}*f\|_{rs,-\beta}\leqslant C\|f\|_{pq,a}.$$

This slightly more general version for L^{pq} spaces of the theorem given in [28] follows from Lemma 1 by the last remark after the proof of that lemma (here $\omega(|t|) = |t|^a\omega_1(|x|) = |x|^{-\beta+a}\omega(|x|)$). This result can be generalized to weighted H^p spaces as follows.

PROPOSITION D. *Let $p\geqslant(n-1)/n$, $0<\lambda<n^2/(n-1)$, $a+\beta\geqslant 0$,*

$$1/r-1/p = (-\lambda+a+\beta)/n$$

and

$$(69)\qquad n/(n-1)-1/p\geqslant a^+/n,\quad 1/r\geqslant\beta^+/n,$$

$0<q\leqslant s$ except that if equality holds in one of the inequalities of (69) then $q = (n-1)/n$, $s = \infty$ and if F_λ is defined by (63) then

$$\|F_\lambda\|[H^{rs}_{-\beta}]\leqslant C\|F\|[H^{pq}_a].$$

Proof. Observe that $a+n/p = n/r-\beta+\lambda\geqslant\lambda$ hence (66) holds. Also (69) implies

$$(70)\qquad \begin{aligned} n/r-\beta<n/r-\beta+\lambda &= a+n/p\leqslant n^2/(n-1),\\ n/p+a &= n/r+\lambda-\beta>n/r-\beta\geqslant 0.\end{aligned}$$

Also

$$\sup_{u>0}|F(x, y+u)u^{\lambda}|^{(n-1)/n} = \sup_{u>0}u^{\lambda(n-1)/n}s(x, y+u)$$
$$\leqslant C\sup_{u>0}\int u^{\lambda(n-1)/n+1}s(t, y)(|x-t|^2+u^2)^{-(n+1)/2}dt$$
$$\leqslant C\int s(t, y)\sup_{u>0}u^{\lambda(n-1)/n+1}(|x-t|^2+u^2)^{-(n+1)/2}dt.$$

Moreover

$$\sup_{u>0} u^{\lambda(n-1)/n+1}(|x|^2+u^2)^{-(n+1)/2} = C|x|^{\lambda(n-1)/n-n} \quad \text{for} \quad \lambda(n-1)/n \leqslant n$$

hence by (67), (68)

(71) $$|F_\lambda(x,y)| \leqslant C_\lambda\big(|\cdot|^{-n+\lambda(n-1)/n} * s(\cdot, y)(x)\big)^{n/(n-1)}.$$

Now the assertion follows from the fractional integration theorem in weighted norms.

Remarks. If $F(x, y)$ is the Poisson integral of a function $f \in L_a^{pq}$ and α, p, q, λ satisfy the hypotheses of the fractional integration theorem in weighted norms then consideration of $P*|f|$ instead of $|F|^{(n-1)/n}$ use of the fact that $P(\cdot, y)*|f|(x)$ is bounded for a.e. x and interchange of the order of integration yield

$$F_\lambda(x,y) = P(\cdot, y)*(I_\lambda f)(x) \qquad \text{a.e.}$$

Here $I_\lambda f = \gamma_\lambda^{-1}|\cdot|^{n-\lambda}*f$, $\gamma_\lambda = \pi^{n/2}2^\lambda \Gamma(\lambda/2)\big[\Gamma\big((n-\lambda)/2\big)\big]^{-1}$.

(71) implies

$$\big(F_\lambda^*(x)\big)^{(n-1)/n} \leqslant C \sup_{|v|\leqslant ky}\big(I_{\lambda(n-1)/n}s(\cdot, y)\big)(x-v) \leqslant CI_{\lambda(n-1)/n}\big(\sup_{|v|\leqslant ky} s(\cdot, \ -v, y)\big)(x)$$

or

(72) $$\big(F_\lambda^*(x)\big)^{(n-1)/n} \leqslant CI_{\lambda(n-1)/n}\big((F^*)^{(n-1)/n}\big)(x).$$

In case $1/r = \beta^+/n$ but $n/(n-1)-1/p > \alpha^+/n$ the assertion $I_\lambda\colon H_a^{p(n-1)/n} \to H_{-\beta}^{r\infty}$ can be improved to $I_\lambda\colon H_a^{pq} \to H_{-\beta}^{r\infty}$, where q is only required to satisfy

$$(n-1)/n \leqslant q \leqslant 1, \quad q \leqslant p, \quad 0<\lambda<n/q, \quad 1/q-1/p \geqslant \alpha^+/n$$

which follows by consideration of $|F|^q$.

In case $r = \infty$ it follows from the assumptions that necessarily $q = (n-1)/n$ (or in the situation of the preceding remark $q \leqslant 1$ at any rate), hence since the functions in L_a^{pq} of compact support are dense in L_a^{pq} it follows from (72) that $|x|^{-\beta}F_\lambda^*(x) \to 0$ as $|x| \to \infty$. Also

$$\{(x,y)\colon |x| \geqslant R \text{ or } y \geqslant R/k\} \subset \bigcup_{|x|\geqslant R} \Gamma_k(x)$$

hence any (x, y) with $|x| < R$, $y > R/k$ belongs to some $\Gamma_k(x_1)$ with $|x_1| \geqslant R$. Since also by (69) $-\beta \geqslant 0$ it follows that

$$|x|^{-\beta}|F_\lambda(x,y)| \leqslant |x_1|^{-\beta}|F_\lambda(x,y)| \leqslant |x_1|^{-\beta}F_\lambda^*(x_1).$$

Hence if R is allowed to tend to ∞ it follows that

$$\lim_{|x|+y\to\infty} |x|^{-\beta}F_\lambda(x,y) = 0.$$

Proposition D can also be proved by use of the semi-group property of I_λ as in [29] (see also [37]). For simplicity, suppose

(73) $$\alpha^+ < n\big(n/(n-1)-1/p\big), \qquad \beta^+ < n/r$$

(taking α^+, β^+ in place of α, β amounts to requiring $p > (n-1)/n$, $r < \infty$). Then

$$\|F_\lambda(\cdot, y)\|_{rq,-\beta} \leqslant \big\|\big(F^*(\cdot, y)|\cdot|^{\alpha}\big)^{1/n}\big(I_\lambda s(\cdot, y)\big)|\cdot|^{-\beta-\alpha/n}\big\|_{rq}$$
$$\leqslant C\|F^*\|_{pq,\alpha}^{1/n}\|I_\lambda s(\cdot, y)\|_{p_1,q_1,-\beta-\alpha/N},$$

where $1/p_1 = 1/r-1/np$, $1/q_1 = (n-1)/nq$. If

(74) $$1/r-1/np > (\beta+\alpha/n)^+/n$$

the fractional integration theorem in weighted norms yields

$$\|I_\lambda s(\cdot, y)\|_{p_1,q_1,-\beta-\alpha/N} \leqslant C\|s(\cdot, y)\|_{np/(n-1),nq/(n-1),(n-1)\alpha/n} \leqslant C\big(\|F\|[H_a^{pq}]\big)^{(n-1)/n}.$$

Let now N be a positive integer and for $m = 0, 1, \ldots, N$

$$1/p_m = 1/r_{m-1} = (1-m/N)/p+(m/N)/q,\ \alpha_m = -\beta_{m-1}$$
$$= (1-m/N)\alpha-(m/N)\beta.$$

(73), (74) for $p_m, r_m, \alpha_m, \beta_m$ require $\alpha_m < n\big(n/(n-1)-1/p+(m/N)(\lambda-\alpha-\beta)/n\big)$ or

(75) $$\alpha^+ < n\big(n/(n-1)-1/p\big)+(m/N)\lambda,$$

(76) $$(-\alpha_{m+1})^+ < n/p_{m+1},$$

(77) $$1/p_{m+1}-1/(np_m) > (-\alpha_{m+1}+\alpha_m/n)^+/n.$$

(75) is true for all $m \geqslant 0$ by (73). (76) is satisfied by hypothesis for $m = -1$ (since $(-\alpha)^+ < n/p$, i.e., $p < \infty$ and also $\alpha > -n/p$ which is implied by the present hypotheses) and for $m = N-1$ (since $\beta^+ < n/r$). Also as $N \to \infty$ the terms of (77) for $m = 0$ tend to $-(n-1)/(np)$ and $\big(-(n-1)\alpha/n^2\big)^+$ respectively while for $m = N-1$ they tend to $(n-1)/(nr)$, $(n-1)\beta^+/n^2$ and thus by hypothesis (77) is satisfied for N sufficiently large. Hence it is satisfied for $m = 0, \ldots, N-1$ provided N is sufficiently large. So

$$\|F_\lambda\|[H_{-\beta}^{rq}] = \|F_\lambda\|[H_{\alpha_N}^{p_N q}] \leqslant C\|F_{(N-1)\lambda/N}\|[H_{\alpha_{N-1}}^{p_{N-1}q}] \leqslant \ldots \leqslant C\|F\|[H_a^{pq}].$$

By change of the order of integration it follows that $(F_\lambda)_\mu = F_{\lambda+\mu}$ provided both are well defined by virtue of Lemma 13. (It seems that this argument does not work for all α, β with $\alpha^+ = n\big(n/(n-1)-1/p\big)$, $\beta^+ = n/r$ covered by Proposition D.)

It has been proved in [29] that if f and its Riesz transforms are integrable then $I_\lambda f, I_\lambda R_1 f, \ldots, I_\lambda R_n f \in L^{n/(n-\lambda)}$ for $0 < \lambda < n$. By use of the enlarged range for α in the case of H^{pq}_α spaces it will follow that an analogous statement holds not only for $f\in L^1_\alpha$, $-n\leqslant\alpha\leqslant 0$ but also for $f\in L^{p1}_{n/p'}$. This requires an observation about Riesz transforms or more general singular integrals of functions in $L^{p1}_{n/p'}$.

It is well known that singular integral operators with bounded kernels preserve L^p_α for $1<p<\infty$, $-n/p<\alpha<n/p'$ (see [27]). The following lemma is concerned with the case when $\alpha=-n/p$ or $\alpha=n/p'$.

LEMMA 14. *Suppose $K(x)=|x|^{-n}\Omega(x)$ is a singular integral kernel. i.e., $\Omega(\lambda x)=\Omega(x)$ for $\lambda>0$, $x\neq 0$ and Ω has mean value zero on S^{n-1}. Also suppose Ω is bounded (for simplicity). Let the singular integral operator p.v.$K*$ be defined a.e. by*

$$p.v.K*f(x)=p.v.\int K(x-y)f(y)\,dy=\lim_{\varepsilon\to 0+}\int_{|x-y|>\varepsilon}K(x-y)f(y)\,dy$$

and suppose $\|p.v.K*f\|_{p\infty}\leqslant C_p\|f\|_{p1}$ for some p, $1\leqslant p<\infty$. Then also

$$\|p.v.K*f\|_{p\infty,\alpha}\leqslant C_p\|f\|_{p1,\alpha}\quad\text{for}\quad -n/p\leqslant\alpha\leqslant n/p'.$$

Proof. It is sufficient to prove that the kernel

$$K'(x,t)=|x-t|^{-n}\big|1-|x|^\alpha|t|^{-\alpha}\big|$$

gives rise to an integral operator which is bounded from L^{p1} to $L^{p\infty}$ (see [27]). As in the proof of Lemma 1 let

$$K_1'(x,t)=K'(x,t)\chi(2|x|^{-1}|t|),\qquad K_3'(x,t)=K'(x,t)\chi(2|x|\,|t|^{-1}),$$
$$K_2'=K'-K_1'-K_3'.$$

It is easy to see that K_1', K_3' (in place of K_i^*) satisfy (16) with $p=r$, $q=1$, $s=\infty$. Also K_2' can be estimated as in [27]. In fact a somewhat different argument for K_2' might run as follows

$$|K_2'(x,t)|\leqslant C|t|^{-1}|x-t|^{-n+1}\quad\text{for}\quad \tfrac12\leqslant|t|/|x|\leqslant 2,\ =0\quad\text{otherwise.}$$

Hence

$$\sup_x\|K_2'(x,\cdot)\|_1\leqslant C,\qquad \sup_t\|K_2'(\cdot,t)\|_1\leqslant C.$$

Hence it follows that the integral operator T_2 defined by K_2' is bounded in L^∞ and L^1, hence by the Riesz interpolation theorem $\|T_2f\|_p\leqslant C\|f\|_p$ (also $\|T_2f\|^*_{p\infty}\leqslant\|T_2f\|_p$, $\|f\|_p\leqslant\|f\|_{p1}$).

Another lemma will be needed to prove the next proposition.

LEMMA 15. *Suppose $F=(F_0,F_1,\ldots,F_n)\in H^{ps}_\alpha$ $(0<s\leqslant\infty)$, where $F_0(x,y)=P(\cdot,y)*f_0(x)$ $f_0\in L^{pq}_\alpha$ if $p>1$ while $f_0=\mu\in|\cdot|^{-\alpha}\mathcal{M}^1(R^n)$ if $p=1$ and $-n/p\leqslant\alpha\leqslant n/p'$, $1\leqslant p<\infty$ and $q=1$ if $\alpha=-n/p$ or n/p' or $p=1$. Then for $\alpha>-n/p$ $i=1,\ldots,n$*

$$F_i(x,y)=Q_i(\cdot,y)*f_0(x)$$

while if $\alpha=-n/p$ there are constants C_i such that

$$F_i(x,y)=Q_i(\cdot,y)*f_0(x)+C_i.$$

If $s<\infty$ then $C_i=0$ for all i.

Proof. Consider

$$G(x,y)=F(x,y)-\big(P(\cdot,y)*f_0(x),\,Q(\cdot,y)*f_0(x)\big)$$

which is in $H^{p\infty}_\alpha$ by hypothesis and Lemma 14. G is the gradient of a harmonic function u, say, such that $(\partial/\partial y)u=0$ (since the first component of G is 0), hence $(\partial/\partial y)G=0$ and hence $G(x,y)=g(x)$, where g is harmonic in R^n. By (9) if $\alpha<n/p'$ then $|g(x)|\leqslant Cy^{-n/p}(|x|+y)^{-\alpha}$ which tends to zero as $y\to\infty$ for $\alpha+n/p>0$ while g is bounded for $\alpha+n/p=0$ (set $y=|x|$). So g vanishes if $\alpha+n/p>0$ and equals a constant $C=(C_1,\ldots,C_n)$ if $\alpha=-n/p$. If $s<\infty$ this constant must be zero. For then $F(\cdot,y)\neq 0$ implies that the decreasing rearrangement of $F(\cdot,y)|\cdot|^{-n/p}$ evaluated at τ near 0 is at least equal to $C_{F,y}\tau^{-1/p}$ hence $\|F(\cdot,y)\|_{ps,-n/p}=\infty$, so $F(0,y)=0$ for all $y>0$. Also by dominated convergence

$$\lim_{y\to\infty}|Q(\cdot,y)*f_0(0)|\leqslant\lim_{y\to\infty}C_n^{-1}\int|f_0(t)|\,|t|(y^2+|t|^2)^{-(n+1)/2}\,dt=0$$

$(f_0\in L^{p1}_{-n/p})$. Thus it follows that $C=0$.

In the remaining case $\alpha=n/p'$ observe that g is a system of conjugate harmonic functions in R^{n+1}_+ and by means of (9) applied to $|g|^{(n-1)/n}$ it can be proved as above that $g=0$. (Alternatively the hypotheses imply $g\in L^{ps}\cap L^\infty$ since g is independent of y hence g being harmonic must vanish).

PROPOSITION E. *Suppose $f\in L^{p1}_\alpha$, $Rf\in L^{pq}_\alpha$, $p=1$, $q\geqslant 1$, $-n\leqslant\alpha\leqslant 0$ or $1<p<\infty$ $\alpha=n/p'$, $0<\lambda<n$, $1/r=1/p+(-\lambda+\alpha+\beta)/n$, $\alpha+\beta\geqslant 0$, $(-n/r'<)\beta<n/r$ then $I_\lambda f$, $RI_\lambda f\in L^{rq}_{-\beta}$.*

Proof. Let

$$F(x,y)=\big(P(\cdot,y)*f(x),\,Q(\cdot,y)*f(x)\big).$$

By Lemmas 3 and 14 $F\in H^{p\infty}_\alpha$. Since its boundary values (f,Rf) (see the last part of the proof of Proposition C) belong to L^{pq}_α F must be in H^{pq}_α by Proposition B, hence by proposition D $F_\lambda\in H^{rq}_{-\beta}$. By the first remark after Proposition D,

$$(F_\lambda)_0(x,y)=P(\cdot,y)*(I_\lambda f)(x).$$

Hence by Lemma 15 the boundary values of F_λ are $I_\lambda f$, $RI_\lambda f$ and these are in L^{rq}_α.

REMARK. If in proposition E $p=q=1$ then it can be shown that

$$P(\cdot,y)*Rf(x)=Q(\cdot,y)*f(x).\tag{78}$$

Also $L^{p1}_{n/p'} \subset L^1$ continuously since for any measurable function h

$$\int |h(x)|\,dx = \int |f(x)|\,|x|^{n/p'}|x|^{-n/p'}dx \leqslant C\int_0^\infty (f|\cdot|^{n/p'})^*(\tau)\,\tau^{-1/p'}d\tau = C\|f\|_{p1,n/p'}.$$

(Besides $f^*(\tau)\tau^{1/p'} \leqslant C\sup_\tau (f|\cdot|^{n/p'})^*(\tau)$ implies $L^{p\infty}_{n/p'} \subset L^{1\infty}$, hence interpolation gives $L^{pq}_{n/p'} \subset L^{1q}$ for $1 \leqslant q \leqslant \infty$). Hence by the same result (78) holds for any p, $1 \leqslant p < \infty$ if $q = 1$. Thus

$$F(x, y) = \big(P(\cdot, y)*f(x), P(\cdot, y)*Rf(x)\big).$$

Hence if $q = 1$ by the first remark after Proposition D

$$F_\lambda(x, y) = P(\cdot, y)*(I_\lambda f, I_\lambda Rf).$$

Thus $I_\lambda(Rf) = R(I_\lambda f)$. This proves the following

COROLLARY. *If in Proposition E $q = 1$ then $I_\lambda f$, $RI_\lambda f \in L^{q1}_\beta$.*

5. Relations to subharmonic functions inside a sphere. The Poisson kernel for the unit ball B^{n+1} of R^{n+1} is

$$\mathscr{P}(\zeta, \tau) = \omega_{n+1}^{-1}(1-|\zeta|^2)\,(1-2\zeta\cdot\tau+|\zeta|^2)^{-(n+1)/2}.$$

For the sake of conciseness the following definitions analogous to those of S^*, etc. in Section 2, are made. Let ν be a positive function on $(0, \infty)$ such that

(79) $\quad \nu(\lambda)\lambda^{-a}\downarrow,\ \nu(\lambda)\lambda^{\beta}\uparrow$ for $\lambda \leqslant 1$, $\quad \nu(\lambda)\lambda^{a}\uparrow,\ \nu(\lambda)\lambda^{-\beta}\downarrow$ for $\lambda \geqslant 1$

and define

$$T^* = \{(p, \nu):\ 1 \leqslant p < \infty,\ (79) \text{ with } a = n/p',\ \beta = n/p\},$$

$$T^{*1}_0 = \{(p, \nu):\ 1 < p \leqslant \infty,\ (79) \text{ with } a < n/p',\ \beta \leqslant n/p\},$$

$$T^{*1}_1 = \{(p, \nu):\ 1 \leqslant p < \infty,\ (79) \text{ with } a \leqslant n/p',\ \beta < n/p\},$$

$$T^{*2} = T^{*1}_0 \cap T^{*1}_1 = \{(p, \nu):\ 1 < p < \infty,\ a < n/p',\ \beta < n/p\}.$$

Slightly more generally, e.g., instead of requiring $\nu(\lambda)\lambda^{-a}\downarrow$ for $\lambda \leqslant 1$ it will be required that there is a constant C such that for $\lambda \leqslant \lambda' \leqslant 1$ $\nu(\lambda)\lambda^{-a} \geqslant C\nu(\lambda')\lambda'^{-a}$. Let L^{pq} (quasi-) norms of functions on S^n be defined with respect to euclidean surface measure on S^n and let $L^{pq}_\nu(S^n) = \{f:\ \|f\|_{pq,\nu} = \|f\nu(\tan(\varphi/2))\| < \infty\}$ where now

(80) $$\sigma = (\sigma'\sin\varphi, \cos\varphi), \quad 0 \leqslant \varphi \leqslant \pi,\ \sigma' \in S^{n-1}.$$

As usual set $x\cdot t = \sum_{i=1}^{n+1} x_i t_i$ for $x, t \in R^{n+1}$. The Hardy–Littlewood maximal function is defined by

$$Mf(\sigma) = \sup_{-1\leqslant\delta<1}\Big(\int_{(\tau,\sigma)\geqslant\delta} |f(\tau)|\,d\tau \Big/ \int_{(\tau,\sigma)\geqslant\delta} d\tau\Big).$$

In analogy with the results of Section 2 there holds

LEMMA 16. (*For the case $\nu = 1$ see* [25]). *Suppose $f \in L^{pq}_\nu(S^n)$ and one of* (a) $(p, \nu) \in T^{*2}$, $0 < q = s \leqslant \infty$ (b) $(p, \nu) \in T^{*1}_0$, $q = s = \infty$, (c) $(p, \nu) \in T^{*1}_1$, $q = s = 1$ (d) $(p, \nu) \in T^*$, $q = 1$, $s = \infty$ *and define F by*

$$F(\varrho\sigma) = \int_{S^n} \mathscr{P}(\varrho\sigma, \tau) f(\tau)\,d\tau$$

then

(81) $$\|F(\varrho\,\cdot)\|_{ps,\nu} \leqslant C_{pq,\nu}\|f\|_{pq,\nu}.$$

If (a), (b) *or* (d) *holds then*

(82) $$\|Mf\|_{ps,\nu} \leqslant C_{pq,\nu}\|f\|_{pq,\nu}.$$

Proof. It is sufficient to assume f vanishes for $\varphi > \pi/2$ for the conditions on ν are invariant under the transformation $\lambda \to \lambda^{-1}$ resulting from $\varphi \to \pi - \varphi$. If $\Phi = \cos^{-1}(\tau\cdot\sigma)$ (= the geodesic distance between σ and τ on S^n) then

(83) $$1 - 2\varrho(\sigma\cdot\tau) + \varrho^2 = (1-\varrho)^2 + 4\varrho\sin^2(\Phi/2).$$

The mapping T: $\varrho(\sigma'\sin\varphi, \cos\varphi) \to (\sigma'\varphi, 1-\varrho)$ is a diffeomorphism from $\{\varrho\sigma:\ 1/2 \leqslant \varrho \leqslant 1,\ 0 \leqslant \varphi \leqslant 3\pi/4\}$ onto $\{(x, y):\ |x| \leqslant 3\pi/4,\ 0 \leqslant y \leqslant 1/2\}$ such that if (80) and similarly $\tau = (\tau'\sin\theta, \cos\theta)$ and $x = \sigma'\varphi$, $t = \tau'\theta$

(84) $$C_1|x-t| \leqslant \sin(\Phi/2) \leqslant C_2|x-t|.$$

Define $Tf = f\circ T^{-1}$. Observe that for $0 \leqslant \theta \leqslant 3\pi/4$

$$\nu\big(\tan(\theta/2)\big) \leqslant C\nu(\theta) = C\nu(|t|) \quad \text{and} \quad \nu(|t|) \leqslant C'\nu\big(\tan(\theta/2)\big).$$

Since T maps the closed spherical balls

$$K_\varrho = \{(\varrho\sigma'\sin\varphi, \varrho\cos\varphi):\ \sigma' \in S^{n-1},\ 0 \leqslant \varphi \leqslant 3\pi/4\}$$

diffeomorphically onto the balls $\{(x, 1-\varrho):\ |x| \leqslant 3\pi/4\}$ it follows that the ratio between the image under T of the volume n-form on K_ϱ defined by surface area on S^n and the volume form dx defined by Lebesgue measure on $\{(x, 1-\varrho):\ x \in R^n\}$ is bounded above and below by positive constants for $\frac{1}{2} \leqslant \varrho \leqslant 1$. Hence $F(\varrho\,\cdot) \in L^{pq}_\nu(S^n)$ if and only if $TF(\cdot, 1-\varrho) \in L^{pq}(R^n)$, where $TF(x, y)$ is set equal to zero for $|x| > 3\pi/4$, and there is a constant C such that

$$C^{-1} \leqslant \|F(\varrho\,\cdot)\|_{pq,\nu}/\|TF(\cdot, 1-\varrho)\|_{pq,\nu} \leqslant C.$$

Furthermore it follows now from (83), (84) that there exists $C>0$ such that

$$C^{-1}TF(x,1-\varrho)\leqslant C_n^{-1}\int_{R^n}(1-\varrho)[(1-\varrho)^2+|x-t|^2]^{-(n+1)/2}Tf(t)\,dt \leqslant CTF(x,1-\varrho)$$

for $\frac{1}{2}\leqslant\varrho\leqslant 1$. For $\varrho\leqslant\frac{1}{2}$ or $\varphi\geqslant 3\pi/4$, $\mathscr{P}(\varrho\sigma,\tau)\leqslant C$ hence

$$\int_{S^n}\mathscr{P}(\varrho\sigma,\tau)|f(\tau)|\,d\tau\leqslant C\|f\|_1\leqslant C\|f\|_{pq,\nu}.$$

Thus (81) is equivalent to

$$\sup_{1/2\leqslant\varrho<1}\|\chi_{B^n(0,3\pi/4)}P(\cdot,1-\varrho)*Tf\|_{ps,\nu}\leqslant C_{pq,\nu}\|Tf\|_{pq,\nu}$$

which is contained in Lemma 3.

The proof of (82) is similar.

The next lemma follows similarly as did Proposition 2.

Lemma 17. *The mapping $f\to F=\int\mathscr{P}(\cdot,\tau)f(\tau)\,d\tau$ is a topological isomorphism between $L^{pq}(S^n)$ $([\nu(\tan(\varphi/2))]^{-1}\mathscr{M}(S^n)$, where $\mathscr{M}(S^n)$ denotes the space of Radon measures on S^n, in case $p=1)$ and the space of harmonic functions F in B^{n+1} provided with the (quasi-) norm $\sup_{0\leqslant\varrho<1}\|U(\varrho\,\cdot)\|_{pq,\nu}$ if* (a), (b) *or* (c) *of Lemma* 16 *holds.*

It is well known that the transformation $f(x)\to|x|^{-n+1}f(|x|^{-2}x)$ takes harmonic functions in a domain $D\subset R^{n+1}$ into harmonic functions in $\{|x|^{-2}x\colon x\in D\}$ (see, e.g., [1] p. 160). Let now the mapping I from $\mathrm{cl}(R_+^{n+1})$ to the closure of the unit ball in R^{n+1} be defined by inversion in the sphere of radius 2 and center at $(0,-2)$ followed by translation by $(0,1)$:

$$I(x,y)=4\frac{(x,y+2)}{|(x,y+2)|^2}+(0,1)\quad\text{so that for } |\zeta|\leqslant 1,$$

$$I^{-1}\zeta=4\frac{\zeta+(0,1)}{|\zeta+(0,1)|^2}-(0,2).$$

Also define

(85) $$(If)(\zeta)=2^{n-1}|\zeta+(0,1)|^{-n+1}f(I^{-1}\zeta)$$

and so

$$(I^{-1}f)(z)=2^{n-1}|z+(0,2)|^{-n+1}f(Iz)$$

so that I, I^{-1} map the class of harmonic functions in a domain D onto the class of harmonic functions in $I(D)$, $I^{-1}(D)$ respectively.

Let $t=r\tau'$, $r=|t|$, $I(t)=(\tau'\sin\theta,\cos\theta)$. It is easy to see that therefore $\tan(\theta/2)=r/2$ hence

$$\begin{aligned}dt&=r^{n-1}dr\,d\sigma'=2^{n-1}(\tan(\theta/2))^{n-1}(\cos(\theta/2))^{-2}d\theta\,d\sigma'\\&=2^{n-1}(\tan(\theta/2))^{n-1}(\cos(\theta/2))^{-2}(\sin\theta)^{-n+1}(\sin\theta)^{n-1}d\theta\,d\sigma'\\&=(1+\tan^2(\theta/2))^n d\sigma\end{aligned}$$

or

(86) $$d\sigma=(1+|t|^2/4)^{-n}dt.$$

Moreover

$$|\tau+(0,1)|=2\cos(\theta/2)=2(1+\tan^2(\theta/2))^{-1/2}.$$

Lemma 18. *The image under I of the harmonic function $P(\cdot,y)*\mu(x)+cy$ in R_+^{n+1} $(\mu\in(1+|\cdot|)^{n+1}\mathscr{M}^1(R^n))$ is*

$$\int\mathscr{P}(\zeta,\tau)I\mu(d\tau)+2^nc(1-|\zeta|^2)|\zeta+(0,1)|^{-n-1},$$

where $I\mu$ is defined by

(87) $$\int\varphi(\sigma)I\mu(d\sigma)=\int\varphi(Ix)\,(1+|x|^2/4)^{-(n+1)/2}\mu(dx)$$

for φ continuous on S^n (as a consequence $I\mu(\{(0,-1)\})=0$) (for functions this definition agrees with (85)).

Proof. If g is continuous and of compact support in R^n then $P*g$ is the unique harmonic function G in R_+^{n+1} which is extended continuously to $\mathrm{cl}(R_+^{n+1})$ by $G(x,0)=g(x)$ and which satisfies $G(z)=O(|z|^{-n})$ as $|z|\to\infty$. On the other hand

$$H(\zeta)=\int_{S^n}\mathscr{P}(\zeta,\tau)Ig(\tau)\,d\tau$$

is the solution of the Dirichlet problem for continuous boundary values Ig which vanish in a neighborhood of $(0,-1)$ hence (e.g., by the reflection principle)

$$H(\zeta)=O(|\zeta+(0,1)|)\quad\text{as}\quad\zeta\to(0,-1)$$

but also

$$IG(\zeta)=O(|\zeta+(0,1)|^{-n+1}|G(I^{-1}\zeta)|)=O(|\zeta+(0,1)|).$$

It follows that $H=IG$. Hence the lemma is proved for $c=0$ and $\mu(dx)=g(x)\,dx$. Any measure $\mu\in(1+|\cdot|)^{n+1}\mathscr{M}^1(R^n)$ is the weak limit with respect to the pairing with $(1+|\cdot|)^{-n-1}C_0$ (C_0 denoting the space of continuous functions vanishing at ∞ on R^n) of a sequence of continuous functions of compact support $\{g\}$. Since $\mathscr{P}(\zeta,\cdot)$ is a continuous function on S^n for any $|\zeta| < 1$ it suffices to show that $I\mu$ is the weak limit of $\{Ig\}$ with respect to the pairing $(\mathscr{M}(S^n), C(S^n))$. For then

$$I(P*\mu)(\zeta) = \lim_{n\to\infty} I(P*g)(\zeta) = \lim_{n\to\infty} \int_{S^n} P(\zeta,\tau)Ig_n(\tau)\,d\tau = \int_{S^n} \mathscr{P}(\zeta,\tau)I\mu(d\tau)$$

It also suffices to consider $\mu \geqslant 0$ and $g \geqslant 0$. In this case weak convergence implies that

$$\lim_{R\to\infty} \sup_n \int_{|x|\geqslant R} g(x)\,(1+|x|)^{-n-1}dx = 0$$

hence

$$\lim_{\varepsilon\to 0} \sup_n \int_{\tau_{n+1}\leqslant -1+\varepsilon} Ig(\tau)\,d\tau = 0.$$

Therefore it suffices to show that $\lim_{n\to\infty} (Ig_n, \psi) = (I\mu, \psi)$ for continuous ψ on S^n vanishing near $(0, -1)$. But

$$\int_{S^n} \psi(\sigma)Ig(\sigma)\,d\sigma = 2^{n-1}\int_{S^n} |\sigma+(0,1)|^{-n+1}\psi(\sigma)g(I^{-1}\sigma)\,d\sigma$$
$$= \int_{R^n} (1+|x|^2/4)^{(n-1)/2}\psi(Ix)g(x)\,(1+|x|^2/4)^{-n}dx$$
$$\to \int_{R^n} \psi(Ix)\,(1+|x|^2/4)^{-(n+1)/2}\mu(dx) = \int \psi(x)I\mu(dx).$$

It remains to show that the image of the function $p_0: (x, y)\to y$ is the Poisson integral of the measure of mass $2^n\omega_{n+1}$ concentrated at $(0, -1)$. But by (85) if $\zeta = (\xi, \eta)$ then

$$(Ip_0)(\zeta) = 2^{n-1}|\zeta+(0,1)|^{-n+1}p_0(I^{-1}\zeta)$$
$$= 2^n|\zeta+(0,1)|^{-n-1}\big(2(\eta+1)-|\zeta+(0,1)|^2\big)$$
$$= 2^n|\zeta+(0,1)|^{-n-1}(1-|\zeta|^2).$$

Remark. This lemma yields still another proof of the well known last part of Lemma 10 (for the case $n = 1$ see [34] and also [22]).

Proposition 4. *The transformation I sets up a topological isomorphism between the space of harmonic functions U in R_+^{n+1} satisfying*

$$\sup_{y>0} (1+y)^{-1}\|U(\cdot, y)\|_{p,\omega} = \|U\|[H_\omega^p(R_+^{n+1})]$$

and the space of harmonic functions V in B^{n+1} satisfying

(88) $$\sup_{0\leqslant\varrho<1} \|V(\varrho\,\cdot)\|_{p,\nu} = \|V\|[H_\nu^p(B^{n+1})] \quad \text{where} \quad \nu(\lambda) = \omega(\lambda)\,(1+\lambda)^{2n/p-n+1}$$

provided $(p, \omega)\in S^{*2}$ and $\alpha_1 < n/p'-1$ or $p = 1$, $(1, \omega)\in S_1^{*1}$, $\alpha_1 < -1$ or $p = \infty$, $(\infty, \omega)\in S_0^{*1}$, $\alpha_1 \leqslant n-1$. Furthermore under the same assumptions on p, ω, I maps the cone of non-negative subharmonic functions in R_+^{n+1} satisfying (39) *and* (40) *isomorphically onto the cone of non-negative subharmonic functions in B^{n+1} satisfying* (88), *i.e., the norm M_0+M_1 where M_0, M_1 are given by* (39), (40) *is equivalent to* (88).

Proof.

$$\int_{S^n} |f(\sigma)|^p\nu(\tan(\theta/2))^p\,d\sigma = \int_{R^n} |f(Ix)|^p\nu(|x|/2)^p(1+|x|^2/4)^{-n}dx$$
$$= \int_{R^n} |(I^{-1}f)(x)|^p\nu(|x|/2)^p(1+|x|^2/4)^{(n-1)p/n-n}dx.$$

For $\mu\in\mathscr{M}(S^n)$ such that $\mu(\{(0,-1)\}) = 0$ it follows from (87) that

$$\int_{S^n} \nu(\tan(\theta/2))\,\mu(d\sigma) = \int \nu(|x|/2)\,(1+|x|^2/4)^{-(n+1)/2}I^{-1}\mu(dx).$$

Given ω define ν by

$$\omega(\lambda) = \nu(\lambda/2)\,(1+\lambda^2/4)^{(n-1)/2-n/p}$$

then, e.g., $(p, \nu)\in T^{*2}$ if and only if for some α_1 in the definition of "$(p, \omega)\in S^{*2}$"

$$n-1-2n/p-\alpha_1 = -\beta > -n/p \quad \text{i.e.,} \quad \alpha_1 < n/p'-1$$

and

$$n-1-2n/p+\beta_1 = \alpha < n/p' \quad \text{i.e.} \quad \beta_1 < n/p+1.$$

(Also note that if $\beta_1 < n/p+1$ then $\delta = 0$ in Proposition 2 and (38)). The assertion now follows from Proposition 2 or the corollary to Proposition 3, respectively and Lemma 18.

It follows in particular that if $n/(n-1) < p < \infty$ and

$$\|U\|[H^p(R_+^{n+1})] = \sup_{y<0}\|U(\cdot, y)\|_p = \lim_{y\to 0}\|U(\cdot, y)\|_p \quad \text{(see [29])}$$

then

(89) $$\lim_{\varrho\to 1}\|IU(\varrho\,\cdot)\|_{p,\nu} = \|U\|[H^p(R_+^{n+1})]$$

(proof of precise equality is similar to the proof of (41), if $p = \infty$ then (89) certainly holds if $\lim_{|z|\to\infty} U(z) = 0$).

Added in proof: A somewhat different proof of the criterion for harmonic majorization in Proposition 3 has appeared earlier in: Ü. Kuran, *A criterion of harmonic majorization in half-spaces*, Bull. London Math. Soc. 3 (1971), pp. 21–22. For lemmas similar to those in Section 3 see: Ü. Kuran, *Harmonic majorizations in half balls and halfspaces*, Proc. London Math. Soc., 21 (1970), pp. 614–636.

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