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REMARK ON FOURIER-STIELTJES TRANSFORMS OF CONTINUOUS MEASURES

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The following theorem pertains to a question (P 711) raised by S. Hartman and C. Ryll-Nardzewski ([1], p. 275).

THEOREM. *Let $P(B)$ be the set of probability measures on a perfect set B in $[0, 2\pi)$ and $P_c(B)$ the subset of continuous (or diffuse) measures. For a number $\delta \in (0,1)$ and $\mu \in P(B)$ put $E(\mu, \delta) = \{n: |\hat{\mu}(n)| > \delta\}$, where $\hat{\mu}$ denotes the Fourier-Stieltjes transform of μ. If $\{n_k\}$ is an increasing sequence of natural numbers and $r_k \to 0$ $(r_k > 0)$, then the set Z of measures in $P_c(B)$ such that*

$$|E(\mu, \delta) \cap [-n_k, n_k]| = O(r_k n_k)$$

is of the first category in $P_c(B)$ in its weak topology. (By $|A|$ we denote the number of elements of the set A.)

Proof. The set S of measures in $P(B)$ such that

$$|E(\mu, \delta) \cap [-n_k, n_k]| \leqslant r_k n_k \quad \text{for } k = 1, 2, 3, \ldots$$

is closed and nowhere dense. In fact, for each k, $|E(\mu, \delta) \cap [-n_k, n_k]|$ is a lower semi-continuous function of μ, so S is closed. Moreover, S contains no measure concentrated in a finite set, for $\hat{\mu}$ would be then almost periodic, $\hat{\mu}(0) > \delta$, whence $E(\mu, \delta)$ would have a positive lower density. The set $S \cap P_c(B)$ is nowhere dense in $P_c(B)$; this is plain by virtue of the density of $P_c(B)$ in $P(B)$. The set Z is evidently a countable union of sets of the type $S \cap P_c(B)$.

COROLLARY. *Since $P_c(B)$ is a dense G_δ-set in the weak topology of $P(B)$* (see [3], p. 55), *the set $P_c(B) \setminus Z$ is not empty. In particular, putting $n_k = k$ and $r_k = \dfrac{\log k}{k}$ we see that there are continuous measures μ such that $\hat{\mu}(n)$ is bounded away from zero on a set which is not a Sidon set* (compare [2], p. 132-133).

RÉFÉRENCES

[1] S. Hartman et C. Ryll-Nardzewski, *Quelques résultats et problèmes en algèbre des mesures continues*, ce volume, p. 271–277.
[2] J.-P. Kahane, *Séries de Fourier absolument convergentes*, Ergebnisse der Mathematik Bd. 50 (1970).
[3] K. P. Parthasarathy, *Probability measures in metric spaces*, New York-London 1967.

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