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Irregularities of distribution, VII

by

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1. Introduction. We shall prove a result about the distribution of an arbitrary sequence of numbers in an interval.

Let U be the unit interval consisting of numbers ξ with $0 \leqslant \xi < 1$, and let $\omega = \{\xi_1, \xi_2, \ldots\}$ be a sequence of numbers in this interval. Given an a in U and a positive integer n, we write $Z(n, a)$ for the number of integers i with $1 \leqslant i \leqslant n$ and $0 \leqslant \xi_i < a$, and we put

$$D(n, a) = |Z(n, a) - na|.$$

The *discrepancy* $D(n)$ is defined by

$$D(n) = \sup_{a \in U} D(n, a).$$

The sequence ω is called *uniformly distributed* if $D(n) = o(n)$. In answer to a question of Van der Corput ([3], remark after Satz 6), Mrs. Van Aardenne-Ehrenfest [1] showed that $D(n)$ cannot remain bounded. Later [2] she proved that there are infinitely many integers n with $D(n) > c_1 \log\log n / \log\log\log n$ where $c_1 > 0$ is an absolute constant. This was improved by Roth [8] who showed that $D(n) > c_2 (\log n)^{1/2}$ for infinitely many n. In the present paper we shall show that $D(n) > c_3 \log n$ for infinitely many values of n.

THEOREM 1. *Suppose $N \geqslant 1$. There is an integer n with $1 \leqslant n \leqslant N$ and*

$$D(n) > 10^{-2} \log N. \tag{1}$$

A result of this type, with the right hand side of (1) replaced by $c_4 (\log N)^{1/2}$, had been shown by Roth [8]. The theorem implies that

$$D(n) > 10^{-2} \log n \tag{2}$$

for infinitely many n.

Now let θ be irrational and let $\omega(\theta)$ be the sequence $\{\theta\}, \{2\theta\}, \ldots$ where $\{\ \}$ denotes fractional parts. For sequences $\omega(\theta)$ an inequality of the type (2) has been known for some time: Ostrowski [7] and Hardy and Littlewood [5] had shown that for sequences $\omega(\theta)$ the function

$$S(n) = \sum_{i=1}^{n} (\{\xi_i\} - \tfrac{1}{2})$$

satisfies $|S(n)| > c_5 \log n$ for infinitely many n, and since $|S(n)| \leqslant D(n)$ by a result of Koksma [6], this gives $D(n) > c_5 \log n$.

Ostrowski ([7], page 95) showed that

$$D(n) \leqslant 36 A \log n$$

if $\omega = \omega(\theta)$ and if the partial denominators in the continued fraction of θ do not exceed A. Later Van der Corput ([4], Hilfssatz 4) constructed another sequence ω with $D(n) \leqslant c_6 \log n$. These results show that except for the value of the constant (namely 10^{-2}), the estimate (2) is best possible. No effort is made in the present paper to obtain a good value for this constant.

THEOREM 2. *Suppose* $P_1 = (\xi_1, \eta_1), \ldots, P_N = (\xi_N, \eta_N)$ *are points in the unit square* Q *defined by* $0 \leqslant \xi < 1$, $0 \leqslant \eta < 1$. *For every* (α, β) *in* Q *put* $E(\alpha, \beta) = |\nu(\alpha, \beta) - N\alpha\beta|$ *where* $\nu(\alpha, \beta)$ *is the number of points* P_i *in the rectangle* $0 \leqslant \xi < \alpha$, $0 \leqslant \eta < \beta$. *Then there is a point* (α_0, β_0) *in* Q *with*

$$E(\alpha_0, \beta_0) > (700)^{-1} \log N. \tag{3}$$

Roth [8] had proved this result with the right hand side of (3) replaced by $c_7 (\log N)^{1/2}$, and he showed that the right hand side may not be replaced by $c_8 \log N$ with arbitrarily small $c_8 > 0$. He also showed that Theorems 1 and 2 are essentially equivalent, and that the truth of Theorem 1 with some function $f(N)$ on the right hand side of (1) implies the truth of Theorem 2 with $\frac{1}{7} f(N)$ on the right hand side of (3). (Namely, this follows from the inequality $M^* \leqslant 7M$ in [8], § 5.) Hence it will suffice to prove Theorem 1. Our proof will be similar to but simpler than a proof of a related result given in the preceding paper [9] of this series.

It appears to be difficult to improve the known estimates [8] of the discrepancy of sequences in the unit cube of k-dimensional space where $k > 1$.

2. A proposition which implies Theorem 1. By $I, J, \ldots$ we shall denote intervals of the type $a < n \leqslant b$ where a, b are integers with $0 \leqslant a < b$. The number of integers in such an interval I is equal to its length $l(I) = b - a$. If n is a positive integer and α is in U, write

$$f(n, \alpha) = Z(n, \alpha) - n\alpha,$$

so that $D(n, \alpha) = |f(n, \alpha)|$. For arbitrary α write $f(n, \alpha) = f(n, \{\alpha\})$, so that $f(n, \alpha)$ is periodic in α with period 1. For intervals I and arbitrary α put

$$g^+(I, \alpha) = \max_{n \in I} f(n, \alpha), \qquad g^-(I, \alpha) = \min_{n \in I} f(n, \alpha)$$

and

$$h(I, \alpha) = g^+(I, \alpha) - g^-(I, \alpha).$$

PROPOSITION. *Suppose* t *is a positive integer and* β *is arbitrary. Let* I *be an interval with* $l(I) \geqslant 4^t$. *Then*

$$4^{-t} \sum_{j=1}^{4^t} h(I, \beta + j4^{-t}) \geqslant 2^{-5} t. \tag{4}$$

We are going to show that this proposition implies Theorem 1. Suppose at first that $N \geqslant 4^9$. There is an integer $t \geqslant 9$ with $4^t \leqslant N < 4^{t+1}$. Let I be the interval $0 < n \leqslant 4^t$. Then (4) will hold for every β. The truth of this inequality for any particular β shows that there is an α in U with $h(I, \alpha) \geqslant 2^{-5} t$. There are integers n_1, n_2 in I with $f(n_1, \alpha) - f(n_2, \alpha) \geqslant 2^{-5} t$. Hence either $f(n_1, \alpha)$ or $f(n_2, \alpha)$ is $\geqslant 2^{-6} t$ in absolute value, and there is an integer n with $0 < n \leqslant 4^t \leqslant N$ and $D(n, \alpha) \geqslant 2^{-6} t$, hence with $D(n) \geqslant 2^{-6} t$. Since $t \geqslant 9$, we have $t \geqslant (9/10)(t+1)$, whence

$$D(n) \geqslant 2^{-6}(9/10)(t+1) > 2^{-6}(9/10)\log N / \log 4 > 10^{-2} \log N.$$

To deal with the case when $1 \leqslant N < 4^9$, it will suffice to show that $D(1) \geqslant \frac{1}{2}$. But $f(1, \frac{1}{2})$ equals $\frac{1}{2}$ or $-\frac{1}{2}$ depending on whether ξ_1 lies in $0 \leqslant \xi_1 < \frac{1}{2}$ or in $\frac{1}{2} \leqslant \xi_1 < 1$, and this implies that $D(1) \geqslant D(1, \frac{1}{2}) = \frac{1}{2}$.

3. An inequality. We have to introduce more notation. Write

$$f(n, \alpha, \beta) = f(n, \beta) - f(n, \alpha),$$

$$g^+(I, \alpha, \beta) = \max_{n \in I} f(n, \alpha, \beta), \qquad g^-(I, \alpha, \beta) = \min_{n \in I} f(n, \alpha, \beta),$$

and for a pair of intervals J, J' put

$$h(J, J', \alpha, \beta) = \max\big(g^-(J, \alpha, \beta) - g^+(J', \alpha, \beta),\ g^-(J', \alpha, \beta) - g^+(J, \alpha, \beta)\big).$$

LEMMA. *Suppose* J, J' *are subintervals of some interval* I. *Then*

$$\begin{aligned} h(I, \alpha) + h(I, \beta) \\ \geqslant h(J, J', \alpha, \beta) + \tfrac{1}{2}\big(h(J, \alpha) + h(J, \beta) + h(J', \alpha) + h(J', \beta)\big). \end{aligned} \tag{5}$$

Proof. This lemma had already been proved in [9]. For completeness, we shall reproduce the proof. We may assume without loss of generality that

$$h(J, J', \alpha, \beta) = g^-(J, \alpha, \beta) - g^+(J', \alpha, \beta).$$

Then we have $f(n, \alpha, \beta) - f(n', \alpha, \beta) \geqslant h(J, J', \alpha, \beta)$, i.e.

$$f(n, \beta) - f(n, \alpha) - f(n', \beta) + f(n', \alpha) \geqslant h(J, J', \alpha, \beta) \tag{6}$$

for every $n \in J$ and every $n' \in J'$. Let $m_\alpha, n_\alpha, m_\beta, n_\beta$ be integers in J with $f(m_\alpha, \alpha) = g^+(J, \alpha)$, $f(n_\alpha, \alpha) = g^-(J, \alpha)$, $f(m_\beta, \beta) = g^+(J, \beta)$, $f(n_\beta, \beta) = g^-(J, \beta)$. Then

$$f(m_\alpha, \alpha) - f(n_\alpha, \alpha) = h(J, \alpha), \tag{7}$$

$$f(m_\beta, \beta) - f(n_\beta, \beta) = h(J, \beta). \tag{8}$$

Similarly, there are elements $m'_\alpha, n'_\alpha, m'_\beta, n'_\beta$ of J' with

$$f(m'_\alpha, \alpha) - f(n'_\alpha, \alpha) = h(J', \alpha), \tag{9}$$

$$f(m'_\beta, \beta) - f(n'_\beta, \beta) = h(J', \beta). \tag{10}$$

Applying (6) with $n = m_\alpha$, $n' = m'_\beta$ we obtain

$$f(m_\alpha, \beta) - f(m_\alpha, \alpha) - f(m'_\beta, \beta) + f(m'_\beta, \alpha) \geqslant h(J, J', \alpha, \beta).$$

Applying (6) with $n = n_\beta$, $n' = n'_\alpha$ we obtain

$$f(n_\beta, \beta) - f(n_\beta, \alpha) - f(n'_\alpha, \beta) + f(n'_\alpha, \alpha) \geqslant h(J, J', \alpha, \beta).$$

Adding these two inequalities and the four equations (7), (8), (9), (10), we get

$$\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 \geqslant 2h(J, J', \alpha, \beta) + h(J, \alpha) + h(J, \beta) + h(J', \alpha) + h(J', \beta),$$

where

$$\varphi_1 = f(m'_\alpha, \alpha) - f(n_\alpha, \alpha), \quad \varphi_2 = f(m'_\beta, \alpha) - f(n_\beta, \alpha),$$
$$\varphi_3 = f(m_\beta, \beta) - f(n'_\beta, \beta), \quad \varphi_4 = f(m_\alpha, \beta) - f(n'_\alpha, \beta).$$

Since $h(I, \alpha) \geqslant \varphi_1$, $h(I, \alpha) \geqslant \varphi_2$, $h(I, \beta) \geqslant \varphi_3$, $h(I, \beta) \geqslant \varphi_4$, the lemma follows.

4. Proof of the proposition. We shall proceed by induction on t. First we note that

$$f(n+1, \beta + \tfrac{1}{2}) - f(n, \beta + \tfrac{1}{2}) - \big(f(n+1, \beta) - f(n, \beta)\big)$$

is an integer minus $(n+1)(\beta + \frac{1}{2}) - n(\beta + \frac{1}{2}) - (n+1)\beta + n\beta$, hence is an integer minus $\frac{1}{2}$. Therefore every I with $l(I) \geqslant 2$ has $h(I, \beta + \frac{1}{2}) + h(I, \beta) \geqslant \frac{1}{2}$. This shows that

$$\sum_{j=1}^{4} h\big(I, \beta + (j/4)\big) = \big(h(I, \beta) + h(I, \beta + \tfrac{1}{2})\big) + \big(h(I, \beta + \tfrac{1}{4}) + h(I, \beta + \tfrac{3}{4})\big)$$
$$\geqslant \tfrac{1}{2} + \tfrac{1}{2} > 4 \cdot 2^{-5},$$

and the proposition is true for $t = 1$.

Now suppose the proposition is true for a particular t, and let I be an interval of length $l(I) \geqslant 4^{t+1}$. If I is given by $a < n \leqslant b$, say, let J be the interval $a < n \leqslant a + 4^t$ and J' the interval $a + 2 \cdot 4^t < n \leqslant a + 3 \cdot 4^t$. By our hypothesis, the inequality (4) is true for J and for J'.

If we extend the definition of $Z(n, \alpha)$ to every α by putting

$$Z(n, \alpha) = Z(n, \{\alpha\}) + n(\alpha - \{\alpha\}),$$

then $Z(n, \alpha)$ is always an integer, and we have $f(n, \alpha) = Z(n, \alpha) - n\alpha$. The function $Z(n, \alpha)$ is non-decreasing in n and in α, and the identity $Z(n, \alpha + 1) - Z(n, \alpha) = n$ holds. In particular we have

$$Z(a + 2 \cdot 4^t, \beta + 1) - Z(a + 2 \cdot 4^t, \beta) - \big(Z(a + 4^t, \beta + 1) - Z(a + 4^t, \beta)\big) = 4^t. \tag{11}$$

Write

$$\alpha_j = \beta + j 4^{-t-1} \quad (j = 0, 1, 2, \ldots)$$

and

$$z_j = Z(a + 2 \cdot 4^t, \alpha_j) - Z(a + 2 \cdot 4^t, \alpha_{j-1}) - \big(Z(a + 4^t, \alpha_j) - Z(a + 4^t, \alpha_{j-1})\big).$$

The numbers z_j are non-negative integers which have

$$\sum_{j=1}^{4^{t+1}} z_j = 4^t \tag{12}$$

by (11).

For every n in J and every n' in J' one has

$$Z(n', \alpha_j) - Z(n', \alpha_{j-1}) - \big(Z(n, \alpha_j) - Z(n, \alpha_{j-1})\big) \geqslant z_j \quad (j = 0, 1, 2, \ldots),$$

whence

$$f(n', \alpha_{j-1}, \alpha_j) - f(n, \alpha_{j-1}, \alpha_j)$$
$$\geqslant z_j - (n' - n)(\alpha_j - \alpha_{j-1}) \geqslant z_j - 3 \cdot 4^t 4^{-t-1} = z_j - (3/4).$$

This yields $h(J, J', \alpha_j, \alpha_{j-1}) \geqslant z_j - (3/4)$, and if z_j is positive then

$$h(J, J', \alpha_j, \alpha_{j-1}) \geqslant \tfrac{1}{4} z_j.$$

In conjunction with (5) this gives

$$h(I, \alpha_j) + h(I, \alpha_{j-1}) \tag{13}$$
$$\geqslant \tfrac{1}{4} z_j + \tfrac{1}{2}\big(h(J, \alpha_j) + h(J, \alpha_{j-1}) + h(J', \alpha_j) + h(J', \alpha_{j-1})\big).$$

Obviously this inequality is also true if $z_j = 0$, and hence it is true in general.

We divide the sum

$$\sum_{j=1}^{4^{t+1}} h(J, \alpha_j) \tag{14}$$

into four parts which correspond to the four residue classes of j modulo 4. Each of these four parts is a sum like the one on the left hand side of (4).

Now (4) is true by induction for our particular value of t, and hence each of the four parts has the lower bound $4^t 2^{-5} t$, and the sum (14) itself is at least $4 \cdot 4^t 2^{-5} t = 2^{-3} 4^t t$. The same lower bound holds if $h(J, a_j)$ in (14) is replaced by $h(J, a_{j-1})$, $h(J', a_j)$ or $h(J', a_{j-1})$. We now take the sum of (13) over $j = 1, 2, \ldots, 4^{t+1}$, and we obtain

$$2\sum_{j=1}^{4^{t+1}} h(I, a_j) \geqslant \tfrac{1}{4}\sum_{j=1}^{4^{t+1}} z_j + \tfrac{1}{2} 4 (2^{-3} 4^t t) \geqslant 4^{t-1} + 4^{t-1} t = 2 \cdot 4^{t+1} \big(2^{-5}(t+1)\big)$$

by (12). Dividing by $2 \cdot 4^{t+1}$ and recalling the definition of a_j we obtain (4) with t replaced by $t+1$.

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An application of Minkowski's theorem in the geometry of numbers

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In memory of Professor Wacław Sierpiński

The classical result in the geometry of numbers is given by Minkowski's

THEOREM 1. *An n-dimensional closed convex region symmetrical around the origin and of volume not less than 2^n, contains a point other than the origin of every lattice L in n variables of determinant one.*

Very few applications of this theorem are to be found in the usual literature. They are mostly concerned with sums of powers of linear forms or separate linear forms. As problems are rather scarce, I notice another application which may be of interest and which is given by

THEOREM 2. *Let L be a lattice in $2n$ variables $(x_1, \ldots, x_{2n})$ of determinant one. Then the region given by*

$$(1) \qquad |x_r| \leqslant a \ \ (r = 1, \ldots, 2n), \qquad \sum_{r=1}^{n} |x_{2r-1} - x_{2r}| \leqslant 2b, \ \ a > b,$$

contains a point other than the origin of L if

$$(2) \qquad b^{2n}\left(\frac{1}{n!} - \frac{n}{1\cdot(n+1)!} + \frac{n \cdot n-1}{2!(n+2)!} + \ldots + \frac{(-1)^n}{(2n)!}\right) \geqslant 2^{-n}.$$

The condition $a > b$ is imposed to exclude the case when the lattice L contains a sublattice of determinant one in x_1, x_2; for if $a < b$, a trivial solution may exist in which $x_3 = \ldots = x_{2n} = 0$.

We have to express the condition that the volume V of (1) is $\geqslant 2^{2n}$. Make the substitution

$$\sqrt{2}x_{2r-1} \to x_{2r-1} + x_{2r}, \qquad \sqrt{2}x_{2r} \to x_{2r-1} - x_{2r} \qquad (r = 1, 2, \ldots, n).$$