in which every sentence $[n]^0$ *is true and every sentence* $[n]$ *for* $n > 1$ *is false* (4).

Proof. Let A and B be arbitrary finite sets of positive integers and $\min B > 1$. We shall show that there exists a model M for $ZF' \cup \cup \{[n]^0\colon n \in A\} \cup \{\neg[n]\colon n \in B\}$. Let $p_1, \ldots, p_k$ be prime numbers such that every element of the set B is divisible by at least one of these numbers. From the assumption it follows that there exist prime numbers $q_1, \ldots, q_k$ such that $(p_i^{q_i}-1)/(p_i-1)$ are prime numbers greater than the numbers of the set A. By lemma 2 there exist groups $G_1, \ldots, G_k$ such that the index of every proper subgroup of the group G_i is divisible by at least one of the numbers of the set $C_i = \{p_i^{q_i}, (p_i^{q_i}-1)/(p_i-1)\}$. Let $G = \prod_{i=1}^{k} G_i$. Using lemma 1, we infer that the index of every proper finite subgroup of the group G is divisible by at least one of the numbers of the set $\bigcup_{i=1}^{k} C_i$. In virtue of theorem 2 there exists a model in which the propositions $[n]^0$ for $n \in A$ are true and the propositions $[n]$ for $n \in B$ are false. Using the compactness theorem we obtain the assertion of theorem 4.

(4) The famous conjecture on the existence of infinitely many Mersenne primes (i.e. the numbers 2^n-1) is a particular case of the conjecture stated in the assumption of the theorem.

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Reçu par la Rédaction le 27. 1. 1970

Some properties of convex metric spaces

by

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1. Introduction. A point x of a metric space (X, ϱ) is said to be a *frontier point* (see [6]) if there exists a point $y \in X$ such that for every point $z \in X\setminus\{x\}$ we have

$$\varrho(y, x) + \varrho(x, z) > \varrho(y, z)\,.$$

The aim of this paper is to give a topological characterization of a frontier point of a compact strongly convex (1) finite-dimensional metric space (X, ϱ) without ramifications (2) (denoted by $(X, \varrho) \in$ SCWR). Holsztyński and Kuperberg have proved (see [4]) that every frontier point of a space $(X, \varrho) \in$ SCWR is a labile point in X (3). It follows from [3] that the set of the frontier points of a space $(X, \varrho) \in$ SCWR is a boundary set (see [8] too). In the present note it is shown that this set is compact. I give some remarks concerning the set of the stable points (3) of the SCWR-spaces.

2. Property of a ball. Let $(X, \varrho) \in$ SCWR. Then there exists exactly one function $\lambda\colon X\times X\times I \to X$ where $I = \langle 0, 1\rangle$ such that

$$\varrho\big(x, \lambda(x, y, t)\big) = t\varrho(x, y) \quad \text{and} \quad \varrho\big(y, \lambda(x, y, t)\big) = (1-t)\,\varrho(x, y)\,.$$

It is not difficult to see that the function λ defined above is continuous. Let us write $|x, y| = \lambda\big((x, y)\times I\big)$. This means that $z \in |x, y|$ if and only if

$$\varrho(x, z) + \varrho(z, y) = \varrho(x, y)\,.$$

(1) A metric space (X, ϱ) is said to be *strongly convex* (see [1]) if for every two points $x, y \in X$ there exists exactly one point $z \in X$ such that $\varrho(x, z) = \varrho(y, z) = \frac{1}{2}\varrho(x, y)$.

(2) A metric space (X, ϱ) is said to be *without ramifications* if for all points $x, y, z, s \in X$ the conditions $\varrho(x, y) + \varrho(y, z) = \varrho(x, z)$, $\varrho(x, y) + \varrho(y, s) = \varrho(x, s)$, $\varrho(x, z) = \varrho(x, s)$ imply $z = s$ (see [6]).

(3) A point p of topological space X is said to be a *labile point* in X if for any neighbourhood U of p there exists a homotopy $h\colon X\times I \to X$ such that the following conditions hold: (i) $h(x, 0) = x$ for every $x \in X$, (ii) $h(x, t) = x$ for every $x \in X\setminus U$, $t \in I$, (iii) $h(x, t) \in U$ for every $x \in U$, $t \in I$, (iv) $h(x, 1) \neq p$ for every $x \in X$ (see [2]). A point of topological space X is said to be a *stable point* in X if is not a labile point.

Let $L(X)$ and $F(X)$ denote the sets of labile points and frontier points of X, respectively. We put $S(X) = X\setminus F(X)$. Since the set $L(X)$ is a boundary set (see [3]) and $L(X) \supset F(X)$ (see [4]), for every point $p \in S(X)$ there exist numbers $r_0 > 2r_1 > 0$ such that a closed ball K_i with centre p and radius r_i is contained in $S(X)$ $(i = 0, 1)$. Let $\dot{K}_i = K_i \cap \overline{X\setminus K_i}$ and $\mathring{K}_i = K_i\setminus \dot{K}_i$ $(i = 0, 1)$.

LEMMA 1. *For every point $a \in \mathring{K}_1$ there exists a homeomorphism $h\colon K_0 \to K_1$ such that $h(a) = a$ and $h(K_0) = K_1$.*

Proof. Let $r_1 > r_2 > \varrho(a, p)$ and K_2 be a closed ball with centre p and radius r_2 and let $\dot{K}_2 = K_2 \cap \overline{X\setminus K_2}$. Since X is without ramifications, for every point $x \in K_0\setminus\{p\}$ there exists exactly one point $\varphi_0(x)$ such that

$$\varrho(p, x) + \varrho\big(x, \varphi_0(x)\big) = \varrho\big(p, \varphi_0(x)\big) = r_0 .$$

Let $\varphi_i(x) = |p, \varphi_0(x)| \cap \dot{K}_i$ for $i = 1, 2$. It is easily seen that the function $\varphi_i\colon K_i\setminus\{p\} \to \dot{K}_i$ is continuous and $\varphi_i(x) = x$ for every $x \in \dot{K}_i$ $(i = 0, 1, 2)$.

Let $t_0 = (r_0 - r_1)/(r_0 - r_2)$. Then for every point $x \in \dot{K}_0$ we have

$$\lambda\big(x, \varphi_2(x), t_0\big) = \varphi_1(x) .$$

Putting

$$h(x) = \begin{cases} \lambda\big(x, \varphi_2(x), t_0\big) & \text{for} \quad x \in \overline{K_0\setminus K_2}, \\ x & \text{for} \quad x \in K_2 \end{cases}$$

we obtain the mapping $h\colon K_0 \to K_1$. It is not difficult to see that h is a homeomorphism. Since $a \in K_2$, we have $h(a) = a$.

3. r-points. A point p of a space X is said to be an *r-point* if each neighbourhood of p contains a neighbourhood U of p (is called a canonical neighbourhood) such that for each $q \in U$, $\operatorname{Fr} U = \overline{U} \cap \overline{X\setminus U}$ is a deformation retract of $U\setminus\{q\}$ (see [5]).

THEOREM 1. *$\mathring{K}_1$ is a canonical neighbourhood.*

Before giving the proof, we establish the following

LEMMA 2. *For every point $q \in \mathring{K}_1$, $\dot{K}_0$ is a deformation retract of $K_0\setminus\{q\}$.*

Proof. Let K_2 be a closed ball with centre q and radius r_2 where $\varrho(p, q) < r_2 < r_1$. We define a map $\varphi\colon K_2\setminus\{q\} \to \dot{K}_2$ by the formula

$$x \in |\varphi(x), q| .$$

It is worth noting that $\varphi(x) = x$ for every $x \in \dot{K}_2$. Let a map ξ: $K_0\setminus\{q\} \to \overline{K_0\setminus K_2}$ be such that

$$\xi(x) = \begin{cases} \varphi(x) & \text{for} \quad x \in K_2\setminus\{q\}, \\ x & \text{for} \quad x \in \overline{K_0\setminus K_2}. \end{cases}$$

Then a map $f\colon (K_0\setminus\{q\}) \times I \to \overline{K_0\setminus K_2}$ given by the formula

$$f(x, t) = \lambda\big(x, \xi(x), t\big)$$

is a deformation retraction of $K_0\setminus\{q\}$ onto $\overline{K_0\setminus K_2}$. We define a map $g\colon \overline{K_0\setminus K_2} \times I \to \dot{K}_0$ putting

$$g(x, t) = \xi\lambda\big(x, \varphi_0(x), t\big)$$

where φ_0 is the map defined in the proof of lemma 1. Since $|p, q| \cap \overline{K_0\setminus K_2} = 0$, we have $q \notin \lambda(\overline{K_0\setminus K_2} \times \overline{K_0\setminus K_2} \times I)$ and g is well-defined. We easily see that g is a deformation retraction of $\overline{K_0\setminus K_2}$ onto $\dot{K}_0$. Putting

$$h_q(x, t) = \begin{cases} f(x, 2t) & \text{for} \quad x \in K_0\setminus\{q\},\ 0 \leqslant t \leqslant \tfrac{1}{2}, \\ g(x, 2t-1) & \text{for} \quad x \in \overline{K_0\setminus K_2},\ \tfrac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

we obtain a deformation retraction h_q of $K_0\setminus\{q\}$ onto $\dot{K}_0$. This completes the proof of the lemma and now we are ready for the

Proof of theorem 1. Let $a \in \mathring{K}_1$. It follows from lemma 1 that there exists a homeomorphism $h\colon K_0 \to K_1$ such that $h(a) = a$ and $h(\dot{K}_0) = \dot{K}_1$. It follows from lemma 2 that $\dot{K}_0$ is a deformation retract of $K_0\setminus\{a\}$. Hence $\dot{K}_1 = h(\dot{K}_0)$ is deformation retract of $K_1\setminus\{a\} = h(K_0\setminus\{a\})$. This means that $\mathring{K}_1$ is a canonical neighbourhood.

4. Locally homogeneous spaces. A metric space (X, ϱ) is called *locally homogeneous* (see [7]) if it is connected and locally compact and if for every point $q \in X$ there exists a neighbourhood U of q such that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $a \in \overline{U}$, $b \in X$ and $\varrho(a, b) < \delta$ then there exists a map $h\colon \overline{U} \times I \to X$ which satisfies the following conditions:

(a) $h(x, 0) = x$ for every $x \in \overline{U}$,

(b) for t fixed, $h(x, t)$ is a homeomorphism,

(c) $h(a, 1) = b$,

(d) $\varrho\big(x, h(x, t)\big) < \varepsilon$ for every $(x, t) \in \overline{U} \times I$.

Now let us prove the following

THEOREM 2. *$\mathring{K}_1$ is locally homogeneous.*

Proof. It is plain that $\mathring{K}_1$ is a connected and locally compact space. Let $q \in \mathring{K}_1$ and let $U = \mathring{K}$ be a closed ball with centre q and radius $r = \frac{1}{4}\big(r_1 - \varrho(p, q)\big)$. Let $\varepsilon > 0$ and let $\delta = \min(\varepsilon, r)$, $a \in K$, $b \in \mathring{K}_1$ such that $\varrho(a, b) < \delta$. Since $F(X) \cap K_1 = 0$ and K_1 is a compact subset of X, there exists a point $c \in K_1$ such that $b \in |a, c|$ and $\varrho(q, c) = 3r$. It is plain that $|a, c| \subset K_1$. We define a map $f\colon K \to I$ putting $f(x) = \varrho(a, b)/\varrho(x, c)$. Then for every $x \in K$ we have

$$\varrho\Big(x, \lambda\big(x, c, f(x)\big)\Big) = \varrho(a, b) .$$

It is not difficult to see that f is continuous and $f(x) \neq 1$ for all $x \in K$. Now let a map $h\colon K \times I \to K_1$ be defined by the formula

$$h(x,t) = \lambda\big(x, c, tf(x)\big).$$

It is clear that conditions (a)–(c) hold. Moreover, we have

$$\varrho\big(x, h(x,t)\big) \leqslant \varrho\big(x, h(x,1)\big) = \varrho(a,b) < \delta \leqslant \varepsilon.$$

Thus it is shown that the condition (d) holds and the proof is finished.

5. Invariability of area. Kosiński has shown (see [5]) that

(5.1) If U is an n-dimensional canonical neighbourhood of p in a space X and V is a canonical neighbourhood of p such that the closure of V is contained in U, then the set $\operatorname{Fr} V = \overline{V} \cap \overline{X \setminus V}$ is the carrier of an $(n-1)$-dimensional essential cycle z which is homologous to zero in $\overline{V}$ and not homologous to zero in any proper subset of $\overline{V}$.

On the other hand, Montgomery has shown (see [7]) that

(5.2) If X is a locally homogeneous n-dimensional space and p a point of X, then there exists a neighbourhood U of p such that if A, B are compact subsets of U and B carries an essential $(n-1)$-cycle z and A is minimal with respect to the properties (a) $B \subset A$, (b) z is homologous to zero in A, then $A \setminus B$ is open in X.

We now prove the following

LEMMA 3. *If F and G are homeomorphic subsets of $\mathring{K}_1$ and G is open, then F is open.*

Proof. Let $\dim K_1 = n$ and let $h\colon G \to F$ be a homeomorphism onto F. Let $a \in F$ and let K be a closed ball with centre $h^{-1}(a)$, $K \subset G$ and $h(K) \subset U \subset \mathring{K}_1$ where U is from (5.2). Let $\dot{K} = K \cap \overline{X \setminus K}$. Since $G \subset \mathring{K}_1$, we infer by theorem 1 and (5.1) that $h(\dot{K})$ carries an $(n-1)$-dimensional essential cycle z which is homologous to zero in $h(K)$ and is not homologous to zero in any proper subset of $h(K)$. It follows from (5.2) that $h(K) \setminus h(\dot{K})$ is open in X; hence F is open in X.

6. Frontier points of X. Let $F_p(X)$ denote the set of all points $x \in X$ such that for every point $y \in X \setminus \{x\}$ we have

$$\varrho(p,x) + \varrho(x,y) > \varrho(p,y).$$

COROLLARY 1. *For every point $q \in F_p(X)$, the set $X \setminus \{q\}$ is contractible in itself.*

Proof. The mapping $h\colon (X \setminus \{q\}) \times I \to X \setminus \{q\}$ given by the formula

$$h(x,t) = \lambda(x,p,t)$$

is the desired homotopy.

LEMMA 4. *If $A \subset \mathring{K}_1$ and $a \in \operatorname{Int} A$, then the set $A \setminus \{a\}$ is not contractible in itself.*

Proof. Let K be an open ball with centre a such that $K \subset \operatorname{Int} A$. Since $K \subset S(X)$, we infer by theorem 1 that K is a canonical neighbourhood. For every point $x \in A \setminus \{a\}$ there exists exactly one point $f(x) \in \dot{K} = \overline{K} \cap \overline{X \setminus K}$ such that

(a) $x \in |f(x), a|$ for $x \in K \setminus \{a\}$,

(b) $f(x) \in |x, a|$ for $x \in \overline{A \setminus K}$.

The mapping $f\colon A \setminus \{a\} \to \dot{K}$ is a retraction of $A \setminus \{a\}$ onto $\dot{K}$. It follows from theorem 1 and (5.1) that $\dot{K}$ is not contractible in itself; hence $A \setminus \{a\}$ is not contractible in itself.

Let us prove the following

THEOREM 3. *$F_p(X)$ is homeomorphic with $\dot{K}_1$.*

Proof (4). Since X is without ramifications, for every point $x \in \dot{K}_1$ there exists exactly one point $y \in F_p(X)$ such that $x \in |p, y|$. Then $\dot{K}_1$ is a 1-1 image of $F_p(X)$. It remains to show that $F_p(X)$ is compact. Let $f\colon X \to \mathring{K}_1$ be a homeomorphism of X into $\mathring{K}_1$ given by the formula

$$f(x) = \lambda(x, p, t_0)$$

where $0 < t_0 < 1$ is such that $f(X) \subset \mathring{K}_1$. Let us write $B = f\big(F_p(X)\big)$ and let $x_0 \in B$. It follows from corollary 1 that the set $f(X) \setminus \{x_0\}$ is contractible in itself, and by lemma 4 we infer that $x_0 \in \operatorname{Fr} f(X) = \overline{f(X)} \cap \overline{X \setminus f(X)}$. Hence $B \subset \operatorname{Fr} f(X)$.

Now let $x_0 \notin B$ and let $q \neq x_0$ be such that $q \in B$ and $f^{-1}(x_0) \in |p, f^{-1}(q)|$. Then there exists a $0 < t_1 < 1$ such that $f^{-1}(x_0) = \lambda\big(p, f^{-1}(q), t_1\big)$. Putting

$$g(x) = f\big(\lambda(x, f^{-1}(q), t_1)\big) \quad \text{for} \quad x \in f(X),$$

we obtain a homeomorphism of $f(X)$ into $f(X)$ such that

$$g(p) = f\big(\lambda(p, f^{-1}(q), t_1)\big) = f\big(f^{-1}(x_0)\big) = x_0.$$

Since $p \in \operatorname{Int} f(X)$, we infer by lemma 3 that $x_0 \in \operatorname{Int} g\big(f(X)\big) \subset \operatorname{Int} f(X)$. This means that $B \supset \operatorname{Fr} f(X)$, we conclude that $B = \operatorname{Fr} f(X)$. Since $\operatorname{Fr} f(X)$ is compact, $F_p(X)$ is compact. This completes the proof.

Let us observe that K_1 is a cone over $\dot{K}_1$ with a vertex p (5); hence theorem 3 implies that

COROLLARY 2. *X is a cone over $F_p(X)$ with a vertex p.*

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(4) By the same method as D. Rolfsen [8].

(5) Let $I = [0, 1]$. The space obtained from the Cartesian product $X \times I$ by identifying the set $X \times \{0\}$ with one point is called a *cone over* X. The point corresponding to the set $X \times \{0\}$ in the identification space is called a *cone vertex*.

COROLLARY 3. *X is homeomorphic with K_1.*

Kosiński (see [5]) has shown that if U is a canonical neighbourhood in a finite-dimensional space X, then every point $x \in U$ is stable in U. Since $L(X) \supset F(X) \supset F_p(X)$, from theorem 3, corollary 2 and corollary 3 we obtain the following

THEOREM 4. *If $X \in$ SCWR and $\dim X = n$, then the set $F(X)$ of frontier points of X is compact, $(n-1)$-dimensional and identical with the set $L(X)$ of labile points of X.*

From theorem 2 and lemma 3 we infer

THEOREM 5. *If $X \in$ SCWR then the set $S(X)$ is identical with the set of stable points of X and is locally homogeneous. If F and G are homeomorphic subsets of $S(X)$ and F is open, then G is open.*

Since for every point $p \in S(X)$ we have $F_p(X) = F(X)$, we obtain

THEOREM 6. *If $X \in$ SCWR, then the set $S(X)$ of stable points of X is convex.*

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Reçu par la Rédaction le 24. 3. 1970

Coreflective subcategories in general topology

by

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§ 1. Introduction. There are several standard methods of representing a given topological space by something with more convenient properties, e.g.:

METHOD I. Determine its image under a functor into a more amenable category (e.g. nth homotopy, homology and cohomology functors).

METHOD II. Embed the space in another which possesses the desired characteristics (e.g. compactifications, realcompactifications, completions, and H-closed extensions).

METHOD III. Modify the topology of the space to obtain a new space with the desired properties (e.g. Arhangel'skiĭ's k-modification, Franklin's sequential modification, Gleason's locally connected refinement, Young's a-, c-, lc-, and r-modifications, and Katětov's semi-regular modification).

In order that the changes imposed be not too drastic, it is desirable that a method of type II or III actually determines a functor from the category of topological spaces into itself. "Nice" embeddings are often characterized by universal mapping properties which automatically yield reflective functors, whereas "nice" modifications of the topology usually yield coreflective functors. Interestingly enough, every coreflective functor from the category of topological spaces into itself can be obtained in this way.

In this paper, which is the second in a series, we concern ourselves with "nice" modifications of topologies, i.e. with coreflective subcategories of the category $\mathfrak{T}$ of topological spaces and continuous functions and the category $\mathfrak{H}$ of Hausdorff spaces and continuous functions. In the first paper, "Coreflective subcategories" [11], results were presented within the framework of general category theory. It will be shown here that many topological results are simple consequences of some of the general theorems of that paper. Most of the topological results are of a fairly recent nature:

(a) In 1946 Young [15] discovered that the topology of a given space can be modified "in a natural way" to obtain a locally connected space.