COROLLARY 3. *X is homeomorphic with K_1.*

Kosiński (see [5]) has shown that if U is a canonical neighbourhood in a finite-dimensional space X, then every point $x \in U$ is stable in U. Since $L(X) \supset F(X) \supset F_p(X)$, from theorem 3, corollary 2 and corollary 3 we obtain the following

THEOREM 4. *If $X \in$ SCWR and $\dim X = n$, then the set $F(X)$ of frontier points of X is compact, $(n-1)$-dimensional and identical with the set $L(X)$ of labile points of X.*

From theorem 2 and lemma 3 we infer

THEOREM 5. *If $X \in$ SCWR then the set $S(X)$ is identical with the set of stable points of X and is locally homogeneous. If F and G are homeomorphic subsets of $S(X)$ and F is open, then G is open.*

Since for every point $p \in S(X)$ we have $F_p(X) = F(X)$, we obtain

THEOREM 6. *If $X \in$ SCWR, then the set $S(X)$ of stable points of X is convex.*

References

[1] K. Borsuk, *On metrization of polytopes*, Fund. Math. 47 (1959), pp. 325–341.

[2] K. Borsuk and J. W. Jaworowski, *On labil and stabil points*, Fund. Math. 39 (1952), pp. 159–175.

[3] T. Ganea, *Homotopically stable points*, Com. Acad. R. P. Romine 9 (1959), pp. 1113–1116.

[4] W. Holsztyński and W. Kuperberg, *Star spaces and 2-dimensional contractible polytopes*, Bull. Acad. Polon. Sci. 13 (1965), pp. 561–564.

[5] A. Kosiński, *On manifolds and r-spaces*, Fund. Math. 42 (1955), pp. 111–124.

[6] A. Lelek and W. Nitka, *On convex metric spaces I*, Fund. Math. 49 (1961), pp. 183–204.

[7] D. Montgomery, *Locally homogeneous spaces*, Ann. of Math. 52 (1950), pp. 261–271.

[8] D. Rolfsen, *Characterizing the 3-cell by its metric*, Fund. Math. 68 (1970), pp. 215–223.

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Coreflective subcategories in general topology

by

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§ 1. Introduction. There are several standard methods of representing a given topological space by something with more convenient properties, e.g.:

METHOD I. Determine its image under a functor into a more amenable category (e.g. nth homotopy, homology and cohomology functors).

METHOD II. Embed the space in another which possesses the desired characteristics (e.g. compactifications, realcompactifications, completions, and H-closed extensions).

METHOD III. Modify the topology of the space to obtain a new space with the desired properties (e.g. Arhangel'skiĭ's k-modification, Franklin's sequential modification, Gleason's locally connected refinement, Young's a-, c-, lc-, and r-modifications, and Katětov's semi-regular modification).

In order that the changes imposed be not too drastic, it is desirable that a method of type II or III actually determines a functor from the category of topological spaces into itself. "Nice" embeddings are often characterized by universal mapping properties which automatically yield reflective functors, whereas "nice" modifications of the topology usually yield coreflective functors. Interestingly enough, every coreflective functor from the category of topological spaces into itself can be obtained in this way.

In this paper, which is the second in a series, we concern ourselves with "nice" modifications of topologies, i.e. with coreflective subcategories of the category $\mathfrak{T}$ of topological spaces and continuous functions and the category $\mathfrak{H}$ of Hausdorff spaces and continuous functions. In the first paper, "Coreflective subcategories" [11], results were presented within the framework of general category theory. It will be shown here that many topological results are simple consequences of some of the general theorems of that paper. Most of the topological results are of a fairly recent nature:

(a) In 1946 Young [15] discovered that the topology of a given space can be modified "in a natural way" to obtain a locally connected space.

(b) In 1963 Gleason [10] rediscovered this fact, proved essentially that the full subcategory of locally connected spaces is coreflective in $\mathfrak{T}$, and pointed out the general nature of his constructions.

(c) In the same year, Arhangel'skiĭ [1] proved essentially that the full subcategory of compactly generated spaces (i.e. k-spaces) is coreflective in $\mathfrak{H}$.

(d) In 1964 Freyd's *Abelian Categories* [8] appeared with the general definition of coreflective subcategories and the first general results concerning them (e.g. the adjoint functor theorem, preservation of colimits, and the relation to limits).

(e) In 1965 Kennison [12] characterized coreflective subcategories of $\mathfrak{T}$ as those which are closed under formation of disjoint topological unions and topological quotients. He also established that all (1) coreflection maps in $\mathfrak{T}$ are one-to-one and onto.

(f) More recently Franklin [6], [7] began an investigation of coreflective subcategories of $\mathfrak{T}$ via a "natural covering" technique, generalizing results for sequential spaces and compactly generated spaces.

(g) Several authors recently discovered independently that the category of compactly generated Hausdorff spaces is complete, cocomplete, and most suitable for the purposes of algebraic topology (cf., e.g. Steenrod [14]).

In § 3 below some general categorical concepts are interpreted for $\mathfrak{T}$ and $\mathfrak{H}$. § 4 provides several characterization theorems for coreflective subcategories of $\mathfrak{T}$ and $\mathfrak{H}$. § 5 contains additional properties of coreflective subcategories of $\mathfrak{T}$ and $\mathfrak{H}$, especially those pertaining to coverings, subobjects, limits and colimits. These results about coverings and the results of § 7 show that coreflections are usually of a more "local" character whereas reflections are usually of a "global" nature. In § 6 it is shown that for every subcategory $\mathfrak{A}$ of $\mathfrak{T}$ (resp. $\mathfrak{H}$) there exists a smallest coreflective subcategory of $\mathfrak{T}$ (resp. $\mathfrak{H}$) containing $\mathfrak{A}$. Various consequences of this "global" generation are obtained. § 7 deals with internal generation of coreflective subcategories. A space X is called $\mathfrak{A}$-generated provided that its topology is completely determined by its $\mathfrak{A}$-subspaces in the following sense: A subset of X is closed in X iff its intersection with each $\mathfrak{A}$-subspace B of X is closed in B. If $\mathfrak{A}$ is preserved by continuous functions, then the $\mathfrak{A}$-generated spaces constitute precisely the smallest coreflective subcategory containing $\mathfrak{A}$. This method of obtaining coreflective subcategories of $\mathfrak{T}$ or $\mathfrak{H}$ is simpler but essentially the same as Franklin's technique by means of "natural covers". However, whereas all coreflective subcategories of $\mathfrak{T}$ or $\mathfrak{H}$ can be obtained as the $\mathfrak{A}$-generated spaces for some $\mathfrak{A}$, not all can be obtained as the $\mathfrak{A}$-generated spaces for some $\mathfrak{A}$ which is preserved by continuous functions; e.g. the subcategory of locally connected spaces.

(1) Except for the degenerate coreflective subcategory consisting of the empty space.

§ 2. Preliminaries. All undefined categorical terminology will be that of [13]. All subcategories will be assumed to be both full and replete (2). Furthermore, every subcategory of topological spaces will be assumed to be non-degenerate (i.e. containing at least one non-empty space).

Recall that a subcategory $\mathfrak{A}$ of a category $\mathfrak{C}$ is said to be *coreflective in* $\mathfrak{C}$ provided that the inclusion functor has a coadjoint; i.e. for each object X in $\mathfrak{C}$ there exists an object $X_{\mathfrak{A}}$ in $\mathfrak{A}$ and a morphism $c_{\mathfrak{A}}\colon X_{\mathfrak{A}} \to X$ such that for each object A in $\mathfrak{A}$ and each morphism $f\colon A \to X$, there exists a unique morphism $f'\colon A \to X_{\mathfrak{A}}$ such that $f = c_{\mathfrak{A}} f'$. $c_{\mathfrak{A}}\colon X_{\mathfrak{A}} \to X$ is called the *coreflection* morphism from $X_{\mathfrak{A}}$ to X, and the functor associated with the coreflection is called a *coreflector*. If each coreflection morphism is a monomorphism (resp. epimorphism), then $\mathfrak{A}$ is said to be *mono-coreflective* (resp. *epi-coreflective*) in $\mathfrak{C}$.

In that which follows we will be concerned with subcategories of the category $\mathfrak{T}$ of all topological spaces and continuous functions (i.e. maps). The special symbols $\mathfrak{H}$, $\mathfrak{C}$, $\mathfrak{PC}$, $\mathfrak{K}$, and $\mathfrak{D}$ will be used to denote the subcategories consisting of, respectively, the Hausdorff, connected, pathwise connected, compact, and discrete spaces. $\mathfrak{P}$ will denote the category of all "paths", i.e. continuous images of the closed unit interval, $\mathfrak{S}$ will denote all convergent sequences (including the finite ones), and $\mathfrak{T}_0$ (resp. $\mathfrak{H}_0$) will denote all non-empty (Hausdorff) spaces. (Note that we thus let a symbol $\mathfrak{A}$ denote a topological property and at the same time the category of all spaces possessing that property.)

We will need to consider two types of "local properties". If $\mathfrak{A}$ is a topological property, then a space X will be said to be *locally* $\mathfrak{A}$ (denoted by: X is in $\mathfrak{A}_{\mathrm{L}}$) provided that for each point $x \in X$ and every neighborhood U of x, there is a neighborhood V of x such that $V \subset U$ and V is an $\mathfrak{A}$-subspace of X. E.g. $\mathfrak{C}_{\mathrm{L}}$ is the category of locally connected spaces.

A space X is called *weak locally* $\mathfrak{A}$ (denoted by: X is in $\mathfrak{A}_{\mathrm{WL}}$) provided that each point of X has some neighborhood which is in $\mathfrak{A}$. Thus, for example, $\mathfrak{K}_{\mathrm{WL}} \cap \mathfrak{H} = \mathfrak{K}_{\mathrm{L}} \cap \mathfrak{H}$.

A category $\mathfrak{A} \subset \mathfrak{T}$ will be called *closed-* (resp. *open-*) *hereditary* provided that for each X in $\mathfrak{A}$, each closed (resp. open) subset of X is in $\mathfrak{A}$. $\mathfrak{A} \subset \mathfrak{T}$ will be called *map-invariant* provided that for each A in $\mathfrak{A}$ and each continuous onto function $f\colon A \to B$, $f(A) = B$ must be in $\mathfrak{A}$. E.g. $\mathfrak{K}$ is both closed-hereditary and map-invariant.

(2) A subcategory $\mathfrak{A}$ of $\mathfrak{C}$ is said to be *replete*, provided that each object in $\mathfrak{C}$ which is isomorphic to an object in $\mathfrak{A}$ must be in $\mathfrak{A}$.

For the reader's convenience we now state the following results from [11] which will be used in the sequel.

THEOREM (I) 1 (Freyd). *Let I be a small category and $\mathfrak{A}$ be a coreflective subcategory of $\mathfrak{C}$. If a diagram D in $\mathfrak{A}$ over I has a colimit* (3) $\{D(i)\xrightarrow{q_i}L\}$ *in $\mathfrak{C}$, then L is in $\mathfrak{A}$.*

THEOREM (I) 2 (Freyd). *Let I be a small category and let $\mathfrak{A}$ be a coreflective subcategory of a category $\mathfrak{C}$, with coreflection morphisms $c_{\mathfrak{A}}\colon X_{\mathfrak{A}}\to X$. If a diagram D in $\mathfrak{A}$ over I has a limit $\{L\xrightarrow{q_i}D(i)\}$ in $\mathfrak{C}$, then it has the limit $\{L_{\mathfrak{A}}\xrightarrow{c_{\mathfrak{A}}}L\xrightarrow{q_i}D(i)\}$ in $\mathfrak{A}$.*

PROPOSITION (I) 2. *In any category, if $e\colon X\to Y$ is an equalizer for morphisms $f, g\colon Y\to W$, then e is an extremal monomorphism* (4).

THEOREM (I) 3. *Let $\mathfrak{C}$ be a category which:*

(i) *has products,*

(ii) *has the epi-mono factorization property,*

(iii) *is either locally small or colocally small; Then $\mathfrak{C}$ has the unique extremal epi-mono factorization property* (5).

THEOREM (I) 5. *If $\mathfrak{A}$ is a coreflective subcategory of a constant-generated* (6) *category $\mathfrak{C}$, then $\mathfrak{A}$ is both mono-coreflective and epi-coreflective in $\mathfrak{C}$.*

THEOREM (I) 6. *If $\mathfrak{C}$ is a category which*

(i) *is locally small,*

(ii) *has coproducts,*

(iii) *has the extremal epi-mono factorization property;*

(3) For the definitions of diagram, limit and colimit, cf. [11].

(4) An *extremal monomorphism* is a monomorphism with the property that each of its factorizations for which the right factor is an epimorphism is such that the right factor must be an isomorphism. *Extremal epimorphism* is defined dually. *Extremal sub* (resp. *quotient*) *objects* are sub (resp. quotient) objects where the morphisms involved are extremal.

(5) A category is said to have the *extremal epi-mono factorization property* provided that each of its morphisms, f, has a factorization $f = gh$, where h is an extremal epimorphism and g is a monomorphism. The factorization is called *unique* provided that whenever $f = g'h'$ is also an extremal epi-mono factorization, then there exists an isomorphism u such that $uh' = h$ and $gu = g'$. The property is called *strong* provided that the composition of extremal epimorphisms is an extremal epimorphism. The dual notion is (*strong*) (*unique*) *epi-extremal mono-factorization property*.

(6) A morphism $f\colon X\to Y$ is called *constant* provided that for each object Z and each pair of morphisms $g, h\colon Z\to X$, it follows that $fg = fh$. A category $\mathfrak{C}$ is said to be *constant-generated* provided that for each pair of objects (X, Y) in $\mathfrak{C}$: (i) the set of morphisms from X to Y is non-empty; (ii) if $f, g\colon X\to Y$ are distinct morphisms, then there exists an object Z and a constant morphism $k\colon Z\to X$ such that gk and fk are distinct.

and if $\mathfrak{A}$ is a subcategory of $\mathfrak{C}$, then the following statements are equivalent:

(1) *$\mathfrak{A}$ is mono-coreflective in $\mathfrak{C}$.*

(2) *$\mathfrak{A}$ is closed under the formation of coproducts and extremal quotient objects.*

PROPOSITION (I) 7. *If $\mathfrak{C}$ is a category which*

(i) *is locally small,*

(ii) *has coproducts,*

(iii) *has the extremal epi-mono factorization property, and if $\mathfrak{A}$ is a subcategory of $\mathfrak{C}$, then there exists a smallest mono-coreflective subcategory, $\mathfrak{C}(\mathfrak{A})$ of $\mathfrak{C}$ containing $\mathfrak{A}$. Furthermore, if $\mathfrak{C}$ has the strong unique extremal epi-mono factorization property, then the objects of $\mathfrak{C}(\mathfrak{A})$ are exactly all extremal quotient objects of coproducts of objects in $\mathfrak{A}$.*

THEOREM (I) 9. *If $\mathfrak{C}$ is a category which*

(i) *is locally small,*

(ii) *has coproducts,*

(iii) *has the strong unique extremal epi-mono factorization property, and if $\mathfrak{A}$ is any subcategory of $\mathfrak{C}$, then each monomorphism in $\mathfrak{C}$ which is $\mathfrak{A}$-liftable* (7) *is also $\mathfrak{C}(\mathfrak{A})$-liftable.*

§ 3. Categories of topological spaces. In this section we consider the particular categories $\mathfrak{T}$ and $\mathfrak{H}$ and establish them as categories which satisfy the hypotheses of general theorems in [11].

The following proposition is well known and easily verified:

PROPOSITION 1. *In the category $\mathfrak{T}$ (resp. $\mathfrak{H}$):*

(i) *isomorphism means homeomorphism;*

(ii) *monomorphism means one-to-one map;*

(iii) *epimorphism means onto map (resp. dense map* (8)*);*

(iv) *retraction means topological retraction map;*

(v) *constant morphism means constant function;*

(vi) *product means topological product;*

(vii) *coproduct means disjoint topological union* (9).

(7) If $\mathfrak{A}$ is a subcategory of $\mathfrak{C}$, then a morphism $f\colon X\to Y$ in $\mathfrak{C}$ is said to be *$\mathfrak{A}$-liftable* provided that for each object A in $\mathfrak{A}$ and each morphism $g\colon A\to Y$, there exists a unique morphism $g'\colon A\to X$ such that $g = fg'$.

(8) A map $f\colon X\to Y$ is said to be *dense* provided that $f(X)^{-} = Y$.

(9) We will denote the disjoint topological union of spaces $\{X_i,\ i\in I\}$ by: $\overset{+}{\bigcup}\{X_i,\ i\in I\}$, and the injection maps by: $u_i\colon X_i\to\overset{+}{\bigcup}\{X_i,\ i\in I\}$.

In [11] the importance of the notions "extremal monomorphism" and "extremal epimorphism" in the general theories of reflections and coreflections was established. We now interpret them topologically.

THEOREM 1. *In the category* $\mathfrak{T}$ (*resp.* $\mathfrak{H}$) *the following notions are equivalent*:

(i) *equalizer*,

(ii) *extremal monomorphism*,

(iii) *topological embedding* (*resp. closed embedding* ([10])).

Proof. [(i) ⇒ (ii)]. Proposition (I)2.

[(ii) ⇒ (iii)]. Let $f\colon X \to Y$ be an extremal monomorphism. Let $Z = f(X)$ (resp. $Z = f(X)^-$) with the subspace topology, let $f'\colon X \to Z$ be defined by $f'(x) = f(x)$, and let $i\colon Z \to Y$ be the inclusion map. f' is onto (resp. dense) so that it is an epimorphism. Thus by the definition of extremal monomorphism, f' is a homeomorphism; hence f is an embedding (resp. closed embedding).

[(iii) ⇒ (i)]. Let $f\colon X \to Y$ be an embedding (resp. closed embedding), let $Y_1 = Y_2 = Y$, and let $Z = Y_1 \overset{+}{\bigcup} Y_2$ with injections $u_1, u_2\colon Y \to Z$. Define an equivalence relation R on Z as follows:

$$\text{for each } y \in f(X)\,, \quad u_1(y) R u_2(y) \text{ and } u_2(y) R u_1(y)\,;$$

$$\text{for each } z \in Z\,, \quad zRz\,.$$

Let $W = Z/R$ with the quotient topology and let $g\colon Z \to W$ be the canonical map. Clearly $(gu_1)f = (gu_2)f$. Now suppose that $h\colon S \to Y$ is a map such that $(gu_1)h = (gu_2)h$. Thus $h(S) \subset f(X)$ so that since f is an embedding $f^{-1}h\colon S \to X$ is continuous and is the unique map whose composition with f is h. Consequently f equalizes gu_1 and gu_2.

THEOREM 2. *In either category* $\mathfrak{T}$ *or* $\mathfrak{H}$, *the following notions are equivalent*:

(i) *coequalizer*,

(ii) *extremal epimorphism*,

(iii) *topological quotient map* ([11]).

Proof. [(i) ⇒ (ii)]. The dual of proposition (I)2.

[(ii) ⇒ (iii)]. Let $f\colon X \to Y$ be an extremal epimorphism, let $Z = f(X)$ with the finest topology such that f is continuous, let $f'\colon X \to Z$ be defined by $f'(x) = f(x)$ for each x and let $i\colon Z \to Y$ be the inclusion. Clearly f' is a topological quotient map, i is a monomorphism, and $f = if'$. Thus by the definition of extremal epimorphism, i is an isomorphism. Thus $f = f'$ is a topological quotient map.

[(iii) ⇒ (i)]. Suppose that $f\colon X \to Y$ is a topological quotient map. For each $y \in Y$, let $A_y = f^{-1}(y) \times f^{-1}(y)$ with projection functions $p_{y,1}$ and $p_{y,2}$. Let $W = \bigcup \{A_y |\ y \in Y\}$ equipped with the discrete topology. Define $\alpha, \beta\colon W \to X$ as follows: $\alpha | A_y = p_{y,1}$ and $\beta | A_y = p_{y,2}$. Now for each $y \in Y$ and each $(c, d) \in A_y$,

$$f\alpha(c, d) = f(c) = y = f(d) = f\beta(c, d)\,.$$

Therefore $f\alpha = f\beta$. Suppose that $g\colon Y \to Z$ is a map such that $g\alpha = g\beta$. Then for $c, d \in f^{-1}(y)$, $g(c) = g\alpha(c, d) = g\beta(c, d) = g(d)$. Thus for each $y \in Y$, $g|f^{-1}(y)$ is a constant function. Now define $h\colon Y \to Z$ by $h(y) = g[f^{-1}(y)]$. h is well defined and $hf = g$. Since g is continuous and f is a topological quotient, h is continuous. Since f is an epimorphism, h is unique. Thus f coequalizes α and β.

THEOREM 3. *Let* $\mathfrak{C}$ *represent either* $\mathfrak{T}$ *or* $\mathfrak{H}$ *and let* $\mathfrak{C}_0$ *represent either* $\mathfrak{T}_0$ *or* $\mathfrak{H}_0$. *Then*:

(i) $\mathfrak{C}$ *has products*;

(i)* $\mathfrak{C}$ *has coproducts*;

(ii) $\mathfrak{C}$ *has equalizers*;

(ii)* $\mathfrak{C}$ *has coequalizers*;

(iii) $\mathfrak{C}$ *is complete*;

(iii)* $\mathfrak{C}$ *is cocomplete*;

(iv) $\mathfrak{C}$ *is locally small*;

(iv)* $\mathfrak{C}$ *is colocally small*;

(v) $\mathfrak{C}$ *has the strong unique epi-extremal mono factorization property*;

(v)* $\mathfrak{C}$ *has the strong unique extremal epi-mono factorization property*;

(vi) $\mathfrak{C}_0$ *is constant generated*;

(vii) $\mathfrak{H}$ *is epi-reflective in* $\mathfrak{T}$.

Proof. [(i) and (i)*]. These clearly hold since the topological product and disjoint topological union of any set of (Hausdorff) spaces are again (Hausdorff) spaces.

[(ii) and (ii)*]. If $f, g\colon X \to Y$ are maps, then the inclusion of $\{x \in X |\ f(x) = g(x)\}$ into X is clearly their equalizer and the natural map $h\colon Y \to Y/R$ (where aRb iff $\forall$ morphism k such that $kf = kg$, $k(a) = k(b)$) is their coequalizer.

[(iii) and (iii)*]. These follow from (i), (i)*, (ii), (ii)* and the theorem on p. 77 of [8] and its dual.

[(iv)]. Given any space, it is clear that there exists at most a set of spaces on each subset of its underlying set and hence at most a set of one-to-one maps from these spaces into it.

([10]) I.e. an embedding $f\colon X \to Y$ such that $f(X) = f(X)^-$.

([11]) I.e. a map which is onto, and which is such that the topology on its range is the finest (i.e. strongest) one which makes it continuous.

[(iv)*]. Let X be in $\mathfrak{C}$, let S be the underlying set for X, let M be some set with cardinality $2^{2^{|S|}}$, and let P be all pairs (f, Y) where Y is a space on M and f is a map from X to Y. Clearly P is a set and contains a system of representatives for the class of all categorical quotients of X (since if D is a dense subset of a Hausdorff space on a set T, then $|T| \leqslant 2^{2^{|D|}}$).

[(v) and (v)*]. Clearly $\mathfrak{C}$ has the epi-mono factorization property (every map can be factored into the same map restricted to its image together with the inclusion map). Thus by (i), (i)*, (iv) and theorem (I)3 and its dual, $\mathfrak{C}$ has both the unique epi-extremal mono factorization property and the unique extremal epi-mono factorization property. Since the composition of (closed) embeddings is a (closed) embedding and since the composition of topological quotient maps is a topological quotient map, we have by theorems 1 and 2 that both factorization properties are strong.

[(vi)]. Obviously true since if $f, g\colon X \to Y$ and $f \neq g$ then $f(x) \neq g(x)$ for some $x \in X$. The discrete space on $\{x\}$ is in $\mathfrak{C}$ and the inclusion i is a constant map such that $fi \neq gi$.

[(vii)]. This follows from the dual of theorem (I)6 since $\mathfrak{H}$ is hereditary and closed under products. Cf. also [12].

§ 4. Characterizations of coreflective subcategories of $\mathfrak{T}$ and $\mathfrak{H}$.

Remark. Recall that throughout the paper, each subcategory of $\mathfrak{T}$ (resp. $\mathfrak{H}$) will be assumed to contain at least one non-empty space.

THEOREM 4 (Kennison). *If $\mathfrak{A}$ is coreflective in $\mathfrak{T}$ (resp. $\mathfrak{H}$), then each coreflection map $c_{\mathfrak{A}}$ is both one-to-one and onto.*

Proof. By theorem 3 (vi) and theorem (I)5 every coreflective subcategory of $\mathfrak{T}_0$ (resp. $\mathfrak{H}_0$) is both mono-coreflective and epi-coreflective in $\mathfrak{T}_0$ (resp. $\mathfrak{H}_0$). Since we only deal with non-trivial subcategories and since $\mathfrak{A}$ is coreflective in $\mathfrak{T}$ if and only if $\mathfrak{A} \cap \mathfrak{T}_0$ is coreflective in $\mathfrak{T}_0$, we have that each coreflection morphism is both one-to-one and onto (resp. one-to-one and dense). If some coreflection map $c_{\mathfrak{A}}\colon X_{\mathfrak{A}} \to X$ is not onto, then for any A in $\mathfrak{A}$ and any $x \in X \setminus c_{\mathfrak{A}}(X_{\mathfrak{A}})$, the constant map from A to x cannot be factored through $X_{\mathfrak{A}}$.

Remark. For the next two theorems (and occasionally elsewhere) we denote a topological space by a pair (X, T), where X is the underlying set and T is the topology on X.

THEOREM 5. *A subcategory $\mathfrak{A}$ of $\mathfrak{T}$ (resp. $\mathfrak{H}$) is coreflective in $\mathfrak{T}$ (resp. $\mathfrak{H}$) if and only if for each space (X, T) in $\mathfrak{T}$, there exists a topology $T^{\mathfrak{A}}$ on X with the following properties:*

(i) *$T^{\mathfrak{A}}$ is finer than T (i.e. $T^{\mathfrak{A}} \supset T$);*

(ii) *$(X, T^{\mathfrak{A}})$ is in $\mathfrak{A}$;*

(iii) *$T^{\mathfrak{A}}$ is the coarsest (i.e. weakest) topology on X satisfying (i) and (ii);*

(iv) *for each space (Y, S) and each continuous function $f\colon (X, T) \to (Y, S)$, the same set function $f\colon (X, T^{\mathfrak{A}}) \to (Y, S^{\mathfrak{A}})$ is continuous.*

Proof. Necessity. By theorem 4 each coreflection morphism is one-to-one and onto. Thus (up to isomorphism in the category of sets) the domain of each correflection map has the same underlying set as its codomain. Since $\mathfrak{A}$ is replete, we may therefore consider each coreflection map to be the identity function on the underlying sets $c_{\mathfrak{A}}\colon (X, T^{\mathfrak{A}}) \to (X, T)$, where $(X, T^{\mathfrak{A}}) \in \mathfrak{A}$. Since $c_{\mathfrak{A}}$ is continuous, $T^{\mathfrak{A}} \supset T$. Thus (i) and (ii) are established. If $(X, T') \in \mathfrak{A}$ and $T' \supset T$, then the identity function $f\colon (X, T') \to (X, T)$ is continuous, so that by the definition of coreflection there exists a map $f'\colon (X, T') \to (X, T^{\mathfrak{A}})$ such that $c_{\mathfrak{A}} f' = f$. Since f and $c_{\mathfrak{A}}$ are identity functions, f' must be the identity function, so that $T' \supset T^{\mathfrak{A}}$. Thus (iii) holds. Now suppose that $f\colon (X, T) \to (Y, S)$ is continuous. By the above, the coreflections can be represented as $c_{\mathfrak{A}}\colon (X, T^{\mathfrak{A}}) \to (X, T)$ and $c'_{\mathfrak{A}}\colon (Y, S^{\mathfrak{A}}) \to (Y, S)$. Since $fc_{\mathfrak{A}}$ has domain in $\mathfrak{A}$, there exists a unique morphism $\hat{f}\colon (X, T^{\mathfrak{A}}) \to (Y, S^{\mathfrak{A}})$ such that $c'_{\mathfrak{A}} \hat{f} = f c_{\mathfrak{A}}$. Since $c_{\mathfrak{A}}$ and $c'_{\mathfrak{A}}$ are identities, $\hat{f}$ must be the same set function as f, so that (iv) holds.

Sufficiency. For each space (X, T), let $c_{\mathfrak{A}}$ be the identity map $c_{\mathfrak{A}}\colon (X, T^{\mathfrak{A}}) \to (X, T)$. Clearly by (i), (ii), and (iii) if $(X, T) \in \mathfrak{A}$, then $T^{\mathfrak{A}} = T$. Now suppose that $(A, S) \in \mathfrak{A}$ and $f\colon (A, S) \to (X, T)$ is continuous. By (iv), the same set function $\hat{f}\colon (A, S^{\mathfrak{A}}) \to (X, T^{\mathfrak{A}})$ is continuous. But $S^{\mathfrak{A}} = S$. Thus

$$\begin{tikzcd}
(X, T^{\mathfrak{A}}) \arrow[rr, "c_{\mathfrak{A}}"] & & (X, T) \\
& (A, S) \arrow[ul, "\hat{f}"] \arrow[ur, "f"'] &
\end{tikzcd}$$

commutes. And since $c_{\mathfrak{A}}$ is one-to-one, $\hat{f}$ is unique. Thus $\mathfrak{A}$ is coreflective in $\mathfrak{T}$.

Remark. A characterization similar to that of theorem 5 has been essentially obtained by Gleason [10]. He actually dealt only with locally connected spaces, but his methods can be used to obtain the following:

THEOREM 6 (Gleason). *$\mathfrak{A}$ is a coreflective subcategory of $\mathfrak{T}$ if and only if for each set X there exists a function γ_X from the lattice of all spaces on X into itself such that:*

(i) *γ_X is increasing;*

(ii) *the fixed points of γ_X are precisely the $\mathfrak{A}$-spaces on X;*

(iii) *if $f\colon (Y, U) \to (Z, V)$ is continuous, then the same set function $f\colon \gamma_Y(Y, U) \to \gamma_Z(Z, V)$ is continuous.*

LEMMA 1 (Gleason). *Let L be a complete lattice, let $x \to x'$ be a function from L into itself which is increasing and order-preserving, and let F be the* *set of fixed points of this function. Then there exists a unique increasing, order-preserving function $x \to x^*$ which retracts L onto F.*

Proof of theorem 6. Necessity. Let γ_X be the function which assigns to each space (X, T), the coreflection space $(X, T^{\mathfrak{A}})$ of theorem 5. Parts (i), (ii), (iii) and (iv) of theorem 5 show that (i), (ii) and (iii) above hold.

Sufficiency. Suppose that for each set X there is a γ_X satisfying (i), (ii), and (iii). If S and T are topologies on X such that $S \supset T$, then the identity function $i\colon (X, S) \to (X, T)$ is continuous, so that by (iii) $i\colon \gamma_X(X, S) \to \gamma_X(X, T)$ is continuous. Hence the topology of $\gamma_X(X, S)$ is finer than that of $\gamma_X(X, T)$ so that γ_X is order-preserving.

Since the lattice $\mathfrak{L}_X$ of all topologies on X is complete, by lemma 1 we have the existence of a unique increasing order-preserving function $T \to T^*$ which retracts $\mathfrak{L}_X$ onto the set of all $\mathfrak{A}$-topologies on X.

We claim that this is a coreflection functor. For each space (X, S), let $c_{\mathfrak{A}}\colon (X, S^*) \to (X, S)$ be the identity function on X. Because of order-preservation $c_{\mathfrak{A}}$ is continuous. Now suppose that (A, U) is in $\mathfrak{A}$ and $f\colon (A, U) \to (X, S)$ is continuous. Let T be the quotient topology on X induced by f, and let $\hat{f}\colon (A, U) \to (X, T)$ be the function f. By (iii) $\hat{f}\colon \gamma_A(A, U) \to \gamma_X(X, T)$ is continuous, but by (ii) $\gamma_A(A, U) = (A, U)$ and so by the choice of T, $\gamma_X(X, T) = (X, T)$. Hence by (ii), (X, T) is in $\mathfrak{A}$, so that $T = T^*$. By order-preservation and the fact that $T \supset S$, we have $T = T^* \supset S^*$. Thus the identity function $i\colon (X, T) \to (X, S^*)$ is continuous and the diagram

$$\begin{tikzcd}
(X, T) \arrow[r, "i"] & (X, S^*) \arrow[r, "c_{\mathfrak{A}}"] & (X, S) \\
& (A, U) \arrow[ul, "\hat{f}"] \arrow[ur, "f"'] &
\end{tikzcd}$$

commutes. Uniqueness of $i\hat{f}$ follows since $c_{\mathfrak{A}}$ is one-to-one. Consequently, $c_{\mathfrak{A}}$ is a coreflection morphism and $\mathfrak{A}$ is coreflective in $\mathfrak{T}$.

Remark. The following theorem, whose validity for $\mathfrak{T}$ is due originally to Kennison [12], gives our third and most useful characterization of coreflective subcategories of $\mathfrak{T}$ and $\mathfrak{H}$.

THEOREM 7 (Kennison). *A subcategory of $\mathfrak{T}$ (resp. $\mathfrak{H}$) is coreflective in $\mathfrak{T}$ (resp. $\mathfrak{H}$) if and only if it is invariant under the formation of disjoint topological unions and topological quotient spaces in $\mathfrak{T}$ (resp. $\mathfrak{H}$).*

Proof. By theorem 3, $\mathfrak{T}$ (resp. $\mathfrak{H}$) satisfies the hypotheses of theorem (I)6. By theorem 4, the coreflective subcategories of $\mathfrak{T}$ (resp. $\mathfrak{H}$) are

precisely the mono-coreflective subcategories of $\mathfrak{T}$. Thus applying theorem (I)6, we have that the coreflective subcategories of $\mathfrak{T}$ (resp. $\mathfrak{H}$) are precisely those that are closed under the formation of coproducts and extremal quotient objects in $\mathfrak{T}$ (resp. $\mathfrak{H}$). But by proposition 1 and theorem 2, these are the disjoint topological unions and topological quotient spaces in $\mathfrak{T}$ (resp. $\mathfrak{H}$).

COROLLARY. *Every coreflective subcategory of $\mathfrak{T}$ is closed under the formation of adjunction spaces.*

§ 5. Further properties of coreflective subcategories of $\mathfrak{T}$ and $\mathfrak{H}$.

Remark. Throughout this section, we will have the standing assumption that $\mathfrak{A}$ is a non-trivial coreflective subcategory of $\mathfrak{T}$ (resp. $\mathfrak{H}$).

A. Relationship to coverings. The results below partially answer the question as to when a cover of a space X by members of the coreflective subcategory $\mathfrak{A}$ tells us that X itself is in $\mathfrak{A}$. A similar problem, namely: What types of covers of topological spaces determine coreflective subcategories of $\mathfrak{T}$? has been investigated by Franklin [6] and [7].

LEMMA 2. *Let $\{U_i \mid i \in I\}$ be a cover of a space X in $\mathfrak{T}$ (resp. $\mathfrak{H}$) such that (as a subspace) each U_i is in $\mathfrak{A}$. If the map $f\colon \overset{+}{\bigcup}\{U_i \mid i \in I\} \to X$ induced by the embeddings $U_i \to X$ is a topological quotient map, then X is in $\mathfrak{A}$.*

Proof. Immediate from theorem 7.

THEOREM 8. *Let $\mathfrak{V}$ be a cover of a space X in $\mathfrak{T}$ (resp. $\mathfrak{H}$) such that (as a subspace) each member of $\mathfrak{V}$ is in $\mathfrak{A}$. If X has the property that each of its points is in the interior of some member of $\mathfrak{V}$, then X is in $\mathfrak{A}$.*

Proof. Let $Y = \overset{+}{\bigcup}\mathfrak{V}$. For each $V \in \mathfrak{V}$ let $u_V\colon V \to Y$ be the injection and $i_V\colon V \to X$ be the inclusion. By lemma 2, it suffices to show that the map $f\colon Y \to X$ induced by the inclusions is a topological quotient map.

Suppose that $G \subset X$ and $f^{-1}(G)$ is open. Let $q \in G$. By hypothesis there is some $W \in \mathfrak{V}$ such that $q \in W^0$. Clearly $u_W i_W^{-1}(W^0) \cap f^{-1}(G) = f^{-1}(W^0 \cap G)$ is open in Y, so that since $f|u_W(W)$ is a homeomorphism onto its image, $W^0 \cap G = f\big(f^{-1}(W^0 \cap G)\big)$ is open in W; hence open in W^0, so open in X. Thus since $q \in W^0 \cap G \subset G$, G must be open. Consequently f is a topological quotient map.

COROLLARY 1. *If $\mathfrak{B}$ is coreflective in $\mathfrak{T}$, then $\mathfrak{B} = \mathfrak{B}_{WL}$.*

Remark. The above corollary does not give a characterization of the coreflective subcategories of $\mathfrak{T}$ since if $\mathfrak{B}$ is the category of all T_1 spaces, $\mathfrak{B} = \mathfrak{B}_{WL}$, but $\mathfrak{B}$ is not coreflective in $\mathfrak{T}$.

COROLLARY 2. *If X is a member of $\mathfrak{T}$ (resp. $\mathfrak{H}$) which has an open cover by members of $\mathfrak{A}$, then X is in $\mathfrak{A}$.*

THEOREM 9. *If X is in $\mathfrak{T}$ (resp. $\mathfrak{H}$) and if $\mathfrak{F}$ is a locally finite cover of X by means of closed subspaces each of which is in $\mathfrak{A}$, then X is in $\mathfrak{A}$.*

Proof. Let $Y = \dot{\bigcup}\mathfrak{F}$ and for each $F \in \mathfrak{F}$ let $u_F\colon F \to Y$ be the injection and $i_F\colon F \to X$ be the inclusion map. By proposition 2, we need only show that the map $f\colon Y \to X$ induced by the inclusions is a topological quotient map.

Let G be closed in Y. Then for each $F \in \mathfrak{F}$, $G \cap u_F(F)$ is closed in $u_F(F)$ so that since i_F is the inclusion onto a closed set, $f(G \cap u_F(F))$ is closed in X. Suppose that $p \in X \setminus f(G)$. Then since $\mathfrak{F}$ is locally finite, there exists an open neighborhood U of p such that U meets only finitely many (say $F_1, F_2, \ldots, F_n$) of the members of $\mathfrak{F}$. Let $H = \bigcup \{f(G \cap u_{F_j}(F_j)) \mid j = 1, 2, \ldots, n\}$. Then H is closed and $H \cap U = f(G) \cap U$. Thus $f(G) \cap U$ is closed in U, so that $p \notin f(G)^-$. Consequently f is a closed map, so that it is a topological quotient map.

Corollary. *If X is in $\mathfrak{T}$ (resp. $\mathfrak{H}$) and if $\mathfrak{F}$ is a finite closed cover of X by spaces in $\mathfrak{A}$, then X is in $\mathfrak{A}$.*

B. Hereditary properties.

Proposition 3. *If X is a retract of any space in $\mathfrak{A}$, then X is in $\mathfrak{A}$.*

Proof. Clearly, every retraction in $\mathfrak{T}$ (resp. $\mathfrak{H}$) is a topological quotient map in $\mathfrak{T}$ (resp. $\mathfrak{H}$). Apply theorem 7.

Corollary. *Every clopen subspace of a member of $\mathfrak{A}$ is also in $\mathfrak{A}$.*

Proof. Clearly by the definition of coreflectivity the empty space is in $\mathfrak{A}$, and every non-empty clopen subspace is a retract.

Remark. Coreflective subcategories of $\mathfrak{T}$ and $\mathfrak{H}$ are, in general, neither closed-hereditary nor open-hereditary: e.g. $\mathfrak{C}_L$ is coreflective in $\mathfrak{T}$ (since it is closed under formation of disjoint topological unions and topological quotients) but is not closed hereditary whereas $\mathfrak{K}_G$ is coreflective in $\mathfrak{T}$ (cf. § 7) but is not open hereditary. (See also propositions 8, 9, and 10 and their corollaries.)

C. Completeness properties.

Theorem 10. *Every coreflective subcategory of $\mathfrak{T}$ (resp. $\mathfrak{H}$) is cocomplete and is a cocomplete subcategory of $\mathfrak{T}$ (resp. $\mathfrak{H}$).*

Proof. Immediate from the fact that $\mathfrak{T}$ (resp. $\mathfrak{H}$) is cocomplete (theorem 3 (iii)*) and theorem (I)1.

Corollary 1. *As a category:*

(i) *$\mathfrak{A}$ has coproducts, and the coproduct is precisely the disjoint topological union.*

(ii) *$\mathfrak{A}$ has coequalizers, and the coequalizers are precisely the topological quotient maps in $\mathfrak{A}$.*

Proof. Theorem 2, theorem 3 (iii), and theorem 10.

Corollary 2. *Every coreflective subcategory of $\mathfrak{T}$ or $\mathfrak{H}$ is closed under the formation of topological direct limit spaces.*

Theorem 11. *Every coreflective subcategory $\mathfrak{A}$ of $\mathfrak{T}$ (resp. $\mathfrak{H}$) is complete and if $c_{\mathfrak{A}}\colon X_{\mathfrak{A}} \to X$ are the coreflection morphisms and if D is a diagram in $\mathfrak{A}$ with limit $\{L \xrightarrow{q_i} D(i)\}$ in $\mathfrak{T}$ (resp. $\mathfrak{H}$), then $\{L_{\mathfrak{A}} \xrightarrow{c_{\mathfrak{A}}} L \xrightarrow{q_i} D(i)\}$ is the limit of D in $\mathfrak{A}$.*

Proof. Immediate from theorem 3(iii) and theorem (I)2.

Corollary. *If $\mathfrak{A}$ is coreflective in $\mathfrak{T}$ or $\mathfrak{H}$, then $\mathfrak{A}$ has categorical products, inverse limits, and equalizers in its own right.*

Remark. Thus, for example, even though the topological product of compactly generated Hausdorff spaces (i.e. k-spaces) need not be compactly generated, this is not a serious defect since the category of compactly generated Hausdorff spaces is coreflective in $\mathfrak{H}$ (cf. § 7) and thus has products. By theorem 11 this product is obtained by first finding the topological product and then applying the coreflection functor. Steenrod [14] has recently pointed out this and other desirable properties of the category of compactly generated Hausdorff spaces. Also cf. Brown [3] and Gabriel [9].

§ 6. Global generation of coreflective subcategories of $\mathfrak{T}$ and $\mathfrak{H}$. In this section we use a general result from [11] to determine a method of global generation of coreflective subcategories of $\mathfrak{T}$ and determine the exact connection between the coreflective subcategories of $\mathfrak{T}$ and those of $\mathfrak{H}$.

Theorem 12. *If $\mathfrak{A}$ is a subcategory of $\mathfrak{T}$ (resp. $\mathfrak{H}$) then there exists a smallest coreflective subcategory $\mathfrak{T}(\mathfrak{A})$ of $\mathfrak{T}$ (resp. $\mathfrak{H}(\mathfrak{A})$ of $\mathfrak{H}$) containing $\mathfrak{A}$. Furthermore, $\mathfrak{T}(\mathfrak{A})$ (resp. $\mathfrak{H}(\mathfrak{A})$) is precisely all topological quotients in $\mathfrak{T}$ (resp. $\mathfrak{H}$) of disjoint topological unions of members of $\mathfrak{A}$.*

Proof. By theorem 3, $\mathfrak{T}$ (resp. $\mathfrak{H}$) is locally small, has coproducts and has the strong unique extremal epi-mono factorization property. By theorem 4, the coreflective subcategories of $\mathfrak{T}$ (resp. $\mathfrak{H}$) are precisely the mono-coreflective subcategories. The result thus follows from proposition (I)7 together with facts that in $\mathfrak{T}$ (resp. $\mathfrak{H}$) the coproducts are precisely the disjoint topological unions and the extremal epimorphisms are precisely the topological quotient maps in $\mathfrak{T}$ (resp. $\mathfrak{H}$) (theorem 2).

Corollary 1. *If $\mathfrak{A}$ is a subcategory of $\mathfrak{H}$, then $\mathfrak{H}(\mathfrak{A}) = \mathfrak{T}(\mathfrak{A}) \cap \mathfrak{H}$.*

Corollary 2. *$\mathfrak{A}$ is coreflective in $\mathfrak{T}$ (resp. $\mathfrak{H}$) if and only if $\mathfrak{A} = \mathfrak{T}(\mathfrak{A})$ (resp. $\mathfrak{A} = \mathfrak{H}(\mathfrak{A})$).*

Lemma 3. *If $\mathfrak{A}$ is coreflective in $\mathfrak{T}$ and if $\mathfrak{C}$ is a subcategory of $\mathfrak{T}$ with the property that (X, T) in $\mathfrak{C}$ and $T' \supset T$ implies that (X, T') is in $\mathfrak{C}$; then $\mathfrak{A} \cap \mathfrak{C}$ is coreflective in $\mathfrak{C}$.*

Proof. Since by theorem 4 each coreflection map $c_{\mathfrak{A}}\colon X_{\mathfrak{A}} \to X$ is one-to-one and onto, $X_{\mathfrak{A}}$ is homeomorphic to a space which has the same underlying set as X and a stronger topology. Thus if X is in $\mathfrak{C}$, $X_{\mathfrak{A}}$ is in $\mathfrak{A} \cap \mathfrak{C}$.

THEOREM 13. *A subcategory $\mathfrak{A}$ of $\mathfrak{T}$ is coreflective in $\mathfrak{H}$ if and only if there exists a coreflective subcategory $\mathfrak{B}$ of $\mathfrak{T}$ such that $\mathfrak{A} = \mathfrak{B} \cap \mathfrak{H}$.*

Proof. The sufficiency follows from lemma 3; the necessity from corollaries 1 and 2 of theorem 12.

Remark. We now translate the statements of theorem (I)9 and its corollary for the particular category $\mathfrak{T}$.

THEOREM 14. *If $\mathfrak{A}$ is subcategory of $\mathfrak{T}$ (resp. $\mathfrak{H}$) and if f is a one-to-one $\mathfrak{A}$-liftable map in $\mathfrak{T}$ (resp. $\mathfrak{H}$), then f is $\mathfrak{T}(\mathfrak{A})$-liftable (resp. $\mathfrak{H}(\mathfrak{A})$-liftable).*

Proof. By theorem 3, $\mathfrak{T}$ (resp. $\mathfrak{H}$) satisfies the hypotheses of theorem (I)9.

COROLLARY. *Let $\mathfrak{A}$ be a subcategory of $\mathfrak{T}$ (resp. $\mathfrak{H}$), let Y be an object in $\mathfrak{T}(\mathfrak{A})$ and let $f\colon Y \to X$ be an $\mathfrak{A}$-liftable one-to-one map in $\mathfrak{T}$ (resp. $\mathfrak{H}$). Then f is the coreflection morphism for $\mathfrak{T}(\mathfrak{A})$ (resp. $\mathfrak{H}(\mathfrak{A})$).*

Remark. We now list some examples of subclasses of $\mathfrak{T}$ and the smallest coreflective subcategory of $\mathfrak{T}$ generated by each. If $\mathfrak{A}$ is a single space, we say that $\mathfrak{T}(\mathfrak{A})$ is *simply-generated*. The first five examples in the table are simply-generated. The sequential spaces of example 4 have been extensively investigated recently, cf. [2], [4], [5], [7]. Additions to the table will be made in § 7. (In the table $W(\alpha)$ is used to denote the space of all ordinal numbers less than or equal to α, with the order topology.)

$\mathfrak{A}$	$\mathfrak{T}(\mathfrak{A})$
1. a single space consisting of a singleton	all discrete spaces
2. a single space consisting of two points only one of which is isolated	all spaces with the property that the intersection of open sets is open
3. a single indiscrete space consisting of two points	all locally indiscrete spaces (= all spaces with the property that every open set is closed)
4. the space $W(\omega)$ where ω is the first limit ordinal	all sequential spaces
5. the space $W(\alpha)$ where α is a limit ordinal	all α-sequential spaces
6. all the spaces $W(\alpha)$ where α is a limit ordinal	all order-sequential spaces
7. $\mathfrak{C} \cap \mathfrak{C}_L$ — all connected locally connected spaces	$\mathfrak{C}_L$
8. $\mathfrak{PC} \cap (\mathfrak{PC})_L$ — all pathwise connected locally pathwise connected spaces	$(\mathfrak{PC})_L$

§ 7. Local generation of coreflective subcategories. Here, in contrast to § 6, we examine and determine coreflective subcategories of $\mathfrak{T}$ by means of internal characteristics of the spaces.

DEFINITION. Let $\mathfrak{A}$ be a subcategory of $\mathfrak{T}$. A space X is called *$\mathfrak{A}$-generated* provided that each $C \subset X$ is closed if and only if for each $\mathfrak{A}$-subspace A of X, $A \cap C$ is closed in A. (Thus the $\mathfrak{A}$-spaces completely determine the topology on X.)

Notation. We denote the category of $\mathfrak{A}$-generated spaces by $\mathfrak{A}_G$.

Remark. Two particular examples of $\mathfrak{A}$-generated spaces are especially well known: If $\mathfrak{A} = \mathfrak{K}$ (the compact spaces) then $\mathfrak{K}_G$ is the category of all compactly generated spaces (cf. [1], [3], [14]) sometimes referred to as k-spaces ([12]). If $\mathfrak{A} = \mathfrak{S}$ (the category of all convergent sequences) then $\mathfrak{S}_G$ is the category of all sequential spaces (cf. [2], [4], [5], [6], [7]).

LEMMA 4. *If $\mathfrak{A} \subset \mathfrak{B}$, then $\mathfrak{A} \subset \mathfrak{A}_G \subset \mathfrak{B}_G$.*

PROPOSITION 4. *For every subcategory $\mathfrak{A}$ of $\mathfrak{T}$,*

$$\mathfrak{A}_G = (\mathfrak{A}_G)_{WL} = (\mathfrak{A}_{WL})_G$$

(i.e. the $\mathfrak{A}$-generated spaces are precisely the weak locally ($\mathfrak{A}$-generated) spaces, and these are precisely the (weak locally $\mathfrak{A}$)-generated spaces).

Proof. Clearly by the definition $\mathfrak{A} \subset \mathfrak{A}_{WL}$, for every $\mathfrak{A} \subset \mathfrak{T}$. Thus $\mathfrak{A}_G \subset (\mathfrak{A}_G)_{WL}$ and by lemma 4, $\mathfrak{A}_G \subset (\mathfrak{A}_{WL})_G$. To show that $(\mathfrak{A}_G)_{WL} \subset \mathfrak{A}_G$, suppose that X is in $(\mathfrak{A}_G)_{WL}$ and C is a subspace of X with the property that for every $\mathfrak{A}$-subspace, A of X, $A \cap C$ is closed in A. Let $p \in C^-$. By our supposition there exists a neighborhood N of p such that N is in $\mathfrak{A}_G$. Consequently $N \cap C$ is closed in N so that since $p \in N \cap C^-$, p must be in C. Thus C is closed, so that X is $\mathfrak{A}$-generated.

To show that $(\mathfrak{A}_{WL})_G \subset \mathfrak{A}_G$, suppose that X is in $(\mathfrak{A}_{WL})_G$ and C is a subspace of X such that for every $\mathfrak{A}$-subspace A of X, $A \cap C$ is closed in A. Let B be any $\mathfrak{A}_{WL}$-subspace of X. We wish to show that $B \cap C$ is closed in B. Let $p \in B \cap C^-$. Since B is in $\mathfrak{A}_{WL}$, there is a B-neighborhood N of p such that N is in $\mathfrak{A}$. Thus by the property assumed for C, $N \cap C^- = N \cap C$. Hence $p \in N \cap C \subset B \cap C$, so that $B \cap C$ is closed in B. Consequently, since X is in $(\mathfrak{A}_{WL})_G$, C is closed; so that X must be in $\mathfrak{A}_G$.

Notation. For any category $\mathfrak{A} \subset \mathfrak{T}$, $\Sigma(\mathfrak{A})$ denotes all disjoint topological unions of members of $\mathfrak{A}$.

([12]) Many authors restrict the definition of compactly generated spaces to Hausdorff spaces. In our context, such a restriction would be artificial. Note, however, (theorem 13 and theorem 15) that the Hausdorff compactly generated spaces do constitute a coreflective subcategory of $\mathfrak{H}$, and that a Hausdorff space is compactly generated iff it is (compact Hausdorff)-generated.

PROPOSITION 5. *For every subcategory* $\mathfrak{A}$ *of* $\mathfrak{T}$,

$$\mathfrak{A} \subset \Sigma(\mathfrak{A}) \subset \mathfrak{A}_{\mathrm{WL}} \subset \mathfrak{A}_{\mathrm{G}} \subset \mathfrak{T}(\mathfrak{A}) .$$

Proof. $\mathfrak{A} \subset \Sigma(\mathfrak{A}) \subset \mathfrak{A}_{\mathrm{WL}}$ follows immediately from the definitions. From lemma 4, $\mathfrak{A} \subset \mathfrak{A}_{\mathrm{G}}$, so that by the above $\mathfrak{A}_{\mathrm{WL}} \subset (\mathfrak{A}_{\mathrm{G}})_{\mathrm{WL}}$. Hence by proposition 4, $\mathfrak{A}_{\mathrm{WL}} \subset \mathfrak{A}_{\mathrm{G}}$.

To show that $\mathfrak{A}_{\mathrm{G}} \subset \mathfrak{T}(\mathfrak{A})$, suppose that X is an $\mathfrak{A}$-generated space, $\mathfrak{A}'$ is the set of all $\mathfrak{A}$-subspaces of X, and W is the collection of all points in X which are contained in some $\mathfrak{A}$-subspace. Clearly by the definition of $\mathfrak{A}_{\mathrm{G}}$, $X-W$ is either empty or discrete and $X = W \overset{+}{\bigcup} (X-W)$. Since each discrete space is in $\mathfrak{T}(\mathfrak{A})$ for every $\mathfrak{A}$, by theorem 7 it suffices to show that W is in $\mathfrak{T}(\mathfrak{A})$. Let $i\colon \overset{+}{\bigcup} \mathfrak{A}' \to W$ be the map induced by the inclusions $A \to W$ for each A in $\mathfrak{A}'$. Suppose that $C \subset W$ and $i^{-1}(C)$ is closed. Then for each A in $\mathfrak{A}'$, $i^{-1}(C) \cap A$ is closed in A. Thus since X is in $\mathfrak{A}_{\mathrm{G}}$, C is closed in X and hence is closed in W. Consequently, i is a topological quotient map, so that by theorem 7, W is in $\mathfrak{T}(\mathfrak{A})$.

COROLLARY 1. $\mathfrak{A}$ *is coreflective in* $\mathfrak{T}$ *if and only if*

$$\mathfrak{A} = \Sigma(\mathfrak{A}) = \mathfrak{A}_{\mathrm{WL}} = \mathfrak{A}_{\mathrm{G}} = \mathfrak{T}(\mathfrak{A}) .$$

COROLLARY 2. *Every coreflective subcategory* $\mathfrak{A}$ *of* $\mathfrak{T}$ *is locally generated; in particular* $\mathfrak{A} = \mathfrak{A}_{\mathrm{G}}$.

COROLLARY 3. *For every subcategory* $\mathfrak{A}$ *of* $\mathfrak{T}$,

$$\mathfrak{T}(\mathfrak{A}) = \mathfrak{T}\big(\Sigma(\mathfrak{A})\big) = \mathfrak{T}(\mathfrak{A}_{\mathrm{WL}}) = \mathfrak{T}(\mathfrak{A}_{\mathrm{G}}) .$$

Proof. $\mathfrak{T}(\mathfrak{A}) \subset \mathfrak{T}\big(\Sigma(\mathfrak{A})\big) \subset \mathfrak{T}(\mathfrak{A}_{\mathrm{WL}}) \subset \mathfrak{T}(\mathfrak{A}_{\mathrm{G}}) \subset \mathfrak{T}\big(\mathfrak{T}(\mathfrak{A})\big) = \mathfrak{T}(\mathfrak{A})$.

Remark. All of the inclusions in the above proposition can be strict. In theorems 15 and 17 below we exhibit conditions on $\mathfrak{A}$ which force some of the inclusions to become equalities.

THEOREM 15. *If* $\mathfrak{A}$ *is a map-invariant subcategory of* $\mathfrak{T}$, *then* $\mathfrak{A}_{\mathrm{G}} = \mathfrak{T}(\mathfrak{A})$, *i.e.* $\mathfrak{A}_{\mathrm{G}}$ *is the smallest coreflective subcategory of* $\mathfrak{T}$ *which contains* $\mathfrak{A}$.

Proof. By proposition 5, we need only show that $\mathfrak{T}(\mathfrak{A}) \subset \mathfrak{A}_{\mathrm{G}}$. Let X be in $\mathfrak{T}(\mathfrak{A})$. By theorem 12 there exists a collection $\mathfrak{B} \subset \mathfrak{A}$ and a topological quotient map $f\colon \overset{+}{\bigcup} \mathfrak{B} \to X$. Suppose that $C \subset X$ has the property that for every $\mathfrak{A}$-subspace A of X, $C \cap A$ is closed in A. Since $\mathfrak{A}$ is map-invariant, we have that for every $B \in \mathfrak{B}$, $C \cap f(B)$ is closed in $f(B)$. Thus for each B, $f^{-1}(C) \cap B$ is closed in $\overset{+}{\bigcup} \mathfrak{B}$, so that $f^{-1}(C)$ is closed. Since f is a topological quotient map, C is closed. Consequently X is in $\mathfrak{A}_{\mathrm{G}}$.

COROLLARY. *The category of* (*Hausdorff*) *compactly generated spaces and the category of* (*Hausdorff*) *sequential spaces are each coreflective in* $\mathfrak{T}$ (*resp.* $\mathfrak{H}$).

Remark. If $\mathfrak{A}$ is a map-invariant subcategory of $\mathfrak{T}$, then the family of all subspaces of a given space X which belong to $\mathfrak{A}$ forms a "natural cover" of X in the sense of Franklin [6]. Furthermore, all natural covers can be obtained in this way. However, not every coreflective subcategory of $\mathfrak{T}$ can be obtained by this method as the next theorem shows:

THEOREM 16. *There exists no map-invariant subcategory* $\mathfrak{A}$ *of* $\mathfrak{T}$ *such that* $\mathfrak{A}_{\mathrm{G}} = \mathfrak{C}_{\mathrm{L}}$ *or* $\mathfrak{A}_{\mathrm{G}} = (\mathfrak{P}\mathfrak{C})_{\mathrm{L}}$.

LEMMA 5. *Every infinite Hausdorff space possesses an infinite collection of non-empty pairwise disjoint open sets.*

PROPOSITION 6. *If* $\mathfrak{A}$ *is a map-invariant subcategory of* $\mathfrak{T}$ *and if* $\mathfrak{A}$ *contains an infinite Hausdorff space* (X, T), *then* $\mathfrak{A}$ *contains a space which is not locally connected.*

Proof. By lemma 5, there exists a countably infinite pairwise disjoint family $\{U_i\}$ of non-empty members of T. Pick $p \in U_1$. Define a new topology T' by setting $A \in T'$ if and only if $A \in T$ and whenever $p \in A$, then there exists some n such that for every $m \geqslant n$, $U_m \subset A$. Clearly (X, T') is not locally connected, and the identity function on X is continuous from (X, T) to (X, T'), so that (X, T') is in $\mathfrak{A}$.

Proof of Theorem 16. Suppose that $\mathfrak{A}$ is map-invariant and $\mathfrak{A}_{\mathrm{G}} = \mathfrak{C}_{\mathrm{L}}$ or $\mathfrak{A}_{\mathrm{G}} = (\mathfrak{P}\mathfrak{C})_{\mathrm{L}}$. Let R be the real line and let S be the sequence $\{1/n \mid n = 1, 2, \ldots\}$. Clearly R is locally pathwise connected, so that by the definition of generation, there exists an $\mathfrak{A}$-subspace A of R such that $A \cap S$ is not closed in A. Thus A is infinite, so that by proposition 6, $\mathfrak{A}$ contains a space which is not in $\mathfrak{C}_{\mathrm{L}}$, which is a contradiction.

Remark. We next obtain a general theorem, special cases of which yield specific characterizations for the smallest coreflective subcategory of $\mathfrak{T}$ containing all connected spaces (resp. all pathwise connected spaces).

DEFINITION. A collection $\mathfrak{C}$ of subsets of some set is said to be:

(i) *centered* if and only if $\bigcap \mathfrak{C} \neq \emptyset$.

(ii) *chained* if and only if for any $A, B \in \mathfrak{C}$ there exists a finite subfamily $C_1, C_2, \ldots, C_n$ of $\mathfrak{C}$ such that $C_1 = A$, $C_n = B$, and $C_i \cap C_{i+1} \neq \emptyset$ $i = 1, \ldots, n-1$.

DEFINITION. If $\mathfrak{A}$ is a subcategory of $\mathfrak{T}$, then an $\mathfrak{A}$-*component* of a space X is a maximal subspace of X belonging to $\mathfrak{A}$.

N.B. $\mathfrak{A}$-components of a space need not exist; e.g. the real line has no $\mathfrak{K}$-components.

PROPOSITION 7. *Let* $\mathfrak{A}$ *be a subcategory of* $\mathfrak{T}$ *which contains every singleton space. Then the following are equivalent:*

(i) *For every* X *in* $\mathfrak{T}$, *the set of all* $\mathfrak{A}$-*components of* X *forms a disjoint cover of* X.

(ii) *For every X in $\mathfrak{T}$, the property that the collection of all non-empty $\mathfrak{A}$-subspaces of X is chained, implies that X is in $\mathfrak{A}$.*

(iii) *For every X in $\mathfrak{T}$, the union of every centered collection of $\mathfrak{A}$-subspaces of X is in $\mathfrak{A}$.*

Proof. (i)$\to$(ii). Suppose that (i) holds for some space X and that the collection of non-empty $\mathfrak{A}$-subspaces of X is chained. Let C be an $\mathfrak{A}$-component of X and $y \in X$. Then since singletons are in $\mathfrak{A}$, there exist $\mathfrak{A}$-subspaces $A_1, A_2, \ldots, A_n$ of X such that $A_1 = C$, $A_n = \{y\}$ and $A_i \cap A_{i+1} \neq \emptyset$, $i = 1, \ldots, n-1$. For each i let C_i be the $\mathfrak{A}$-component of X which contains A_i. Since the $\mathfrak{A}$-components are disjoint, $C = C_1 = \ldots = C_n \supset \{y\}$. Therefore $C = X$, so that X is in $\mathfrak{A}$.

(ii)$\to$(iii). Every centered collection is obviously chained.

(iii)$\to$(i). Let X be any space, let $y \in X$, and let $\mathfrak{F}$ be the collection of all $\mathfrak{A}$-subspaces of X which contain y. By (iii) $\bigcup\mathfrak{F}$ is in $\mathfrak{A}$, and by the construction it is maximal. Thus the $\mathfrak{A}$-components of X cover X. If A and B are $\mathfrak{A}$-components which are not disjoint, then by (iii) $A \cup B$ is in $\mathfrak{A}$ so that by the maximality $A = B$.

Definition. $\mathfrak{A}$ is said to be a *component subcategory* of $\mathfrak{T}$ if and only if it contains all singleton spaces and satisfies the equivalent conditions of proposition 7.

Theorem 17. *If $\mathfrak{A}$ is a map-invariant component subcategory of $\mathfrak{T}$, then*

$$\Sigma(\mathfrak{A}) = \mathfrak{A}_{\mathrm{WL}} = \mathfrak{A}_{\mathrm{G}} = \mathfrak{T}(\mathfrak{A})\,.$$

Proof. By proposition 5, it suffices to show that $\mathfrak{T}(\mathfrak{A}) \subset \Sigma(\mathfrak{A})$.

Let X be in $\mathfrak{T}(\mathfrak{A})$. By theorem 12 there exists a collection $\mathfrak{B} \subset \mathfrak{A}$ and a topological quotient map $f\colon \overset{+}{\bigcup}\mathfrak{B} \to X$. Let $\mathfrak{C}$ be the collection of $\mathfrak{A}$-components of X, and let $C \in \mathfrak{C}$. If $B \in \mathfrak{B}$ is such that $B \cap f^{-1}(C) \neq \emptyset$, then $f(B) \cap C \neq \emptyset$, so that since $\mathfrak{A}$ is map-invariant and C is maximal we have by proposition 7(iii) that $f(B) \cup C = C$. Thus $B \subset f^{-1}(C)$. Consequently

$$f^{-1}(C) = \bigcup\{B \in \mathfrak{B} \mid B \cap f^{-1}(C) = \emptyset\}$$

which is a clopen subset of $\overset{+}{\bigcup}\mathfrak{B}$. Since f is a topological quotient map, C must be clopen in X. Thus $X = \overset{+}{\bigcup}\mathfrak{C}$, so that X is in $\Sigma(\mathfrak{A})$.

Corollary 1. $\Sigma(\mathfrak{C}) = \mathfrak{C}_{\mathrm{WL}} = \mathfrak{C}_{\mathrm{G}} = \mathfrak{T}(\mathfrak{C})$.

Corollary 2. $\Sigma(\mathfrak{P}\mathfrak{C}) = (\mathfrak{P}\mathfrak{C})_{\mathrm{WL}} = (\mathfrak{P}\mathfrak{C})_{\mathrm{G}} = \mathfrak{T}(\mathfrak{P}\mathfrak{C})$.

Remark. Note that if in theorem 17, $\mathfrak{A}$ is the class of all singleton spaces (resp. all indiscrete spaces) we obtain $\mathfrak{T}(\mathfrak{A})$ as the class of all discrete spaces (resp. all locally indiscrete spaces) (cf. § 6). It should also be observed that for any map-invariant component subcategory $\mathfrak{A}$ of $\mathfrak{T}$, the $\mathfrak{T}(\mathfrak{A})$-coreflection of a space X is obtained by forming coarsest (i.e. weakest) topology on X which contains the given topology and at the same time all $\mathfrak{A}$-components of X.

Using information gained so far, we now expand the table of § 6.

$\mathfrak{A}$	$\mathfrak{T}(\mathfrak{A})$
9. $\mathfrak{K}$ (all compact spaces)	$\mathfrak{K}_{\mathrm{G}}$ (all compactly generated spaces)
10. $\mathfrak{S}$ (all convergent sequences including finite ones)	$\mathfrak{S}_{\mathrm{G}}$ (all sequential spaces)
11. $\mathfrak{C}$ (all connected spaces)	$\Sigma(\mathfrak{C})$ (all disjoint topological unions of connected spaces)
12. $\mathfrak{P}\mathfrak{C}$ (all pathwise connected spaces)	$\Sigma(\mathfrak{P}\mathfrak{C})$
13. $\mathfrak{P}$ (all paths, i.e. continuous images of $[0,1]$	$\mathfrak{P}_{\mathrm{G}}$ (all path generated spaces)
14. $\mathfrak{C} \cap \mathfrak{K}$ (all continua)	$(\mathfrak{C} \cap \mathfrak{K})_{\mathrm{G}} \subsetneqq \mathfrak{C}_{\mathrm{G}} \cap \mathfrak{K}_{\mathrm{G}}$

We conclude with some results relating hereditary properties and local generation.

Proposition 8. *If $\mathfrak{A}$ is a closed-hereditary subcategory of $\mathfrak{T}$, then $\mathfrak{A}_{\mathrm{G}}$ is closed-hereditary.*

Proof. Let X be in $\mathfrak{A}_{\mathrm{G}}$, B be a closed subspace of X, and $C \subset B$ such that for every $\mathfrak{A}$-subspace, A, of B, $C \cap A$ is closed in A. Now let A' be any $\mathfrak{A}$-subspace of X. Since $A' \cap B$ is an $\mathfrak{A}$-subspace of B, $C \cap A' = C \cap (A' \cap B)$ is closed in $A' \cap B$ which is closed in A'. Thus C is closed in B.

Corollary (Arhangel'skiĭ [1], Franklin [5]). *Every closed subspace of a compactly generated (resp. sequential) space is compactly generated (resp. sequential).*

Proposition 9. *If $\mathfrak{A}$ is a closed-hereditary subcategory of $\mathfrak{T}$, then the class of all regular $\mathfrak{A}$-generated spaces is open-hereditary.*

Proof. Let X be regular and $\mathfrak{A}$-generated and let U be open in X. If $p \in U$ then there exists a closed neighborhood N of p such that $N \subset U$. By proposition 8, N is $\mathfrak{A}$-generated. Thus U is weak locally ($\mathfrak{A}$-generated), so that by proposition 4 it is $\mathfrak{A}$-generated.

Corollary (Arhangel'skiĭ [1]). *Every open subset of a regular compactly generated space is compactly generated.*

Proposition 10. *If $\mathfrak{A}_{\mathrm{G}} = \mathfrak{T}(\mathfrak{A})$ (resp. $\mathfrak{H}(\mathfrak{A})$) and if every open subspace of every $\mathfrak{A}$-space is in $\mathfrak{T}(\mathfrak{A})$ (resp. $\mathfrak{H}(\mathfrak{A})$), then $\mathfrak{T}(\mathfrak{A})$ (resp. $\mathfrak{H}(\mathfrak{A})$) is open-hereditary.*

Proof. Let X be in $\mathfrak{T}(\mathfrak{A})$. Let V be open in X and let $i\colon \overset{+}{\bigcup}\{A \in \mathfrak{A} \mid A \subset X\} \to X$ be the map induced by the inclusions $A \to X$. By hypothesis each $i^{-1}(V) \cap A$ is in $\mathfrak{T}(\mathfrak{A})$ (resp. $\mathfrak{H}(\mathfrak{A})$), so that by theorem 12, $i^{-1}(V)$ is in $\mathfrak{T}(\mathfrak{A})$ (resp. $\mathfrak{H}(\mathfrak{A})$). Since X is in $\mathfrak{A}_G$, $i|\ i^{-1}(V)\colon i^{-1}(V) \to V$ is a topological quotient map. Thus by theorem 12, V is in $\mathfrak{T}(\mathfrak{A})$ (resp. $\mathfrak{H}(\mathfrak{A})$).

COROLLARY (Franklin [6]). *Each of the properties "Hausdorff compactly generated" and "sequential" is open-hereditary.*

Proof. Clearly by theorem 13, $\mathfrak{H}(\mathfrak{K} \cap \mathfrak{H}) = \mathfrak{T}(\mathfrak{K}) \cap \mathfrak{H}$. Also by the corollary to proposition 9 every open subset of a compact Hausdorff space is compactly generated. Similarly, every open subset of every convergent sequence is a convergent sequence (possibly finite) and so is sequential.

References

[1] A. Arhangel'skiĭ, *Bicompact sets and the topology of spaces*, Dokl. Akad. Nauk, SSSR, 150 (9) (1963).
[2] C. E. Aull, *Sequences in topological spaces*, Notices Amer. Math. Soc. 12 (1965), p. 222.
[3] R. Brown, *Ten topologies for $X \times Y$*, Quant. J. Math., Oxford, 14 (2) (1963), pp. 303–319.
[4] R. M. Dudley, *On sequential convergence*, Trans. Amer. Math. Soc. 112 (1964), pp. 483–504.
[5] S. P. Franklin, *Spaces in which sequences suffice*, Fund. Math. 57 (1965), pp. 107–116.
[6] — *Natural covers*, Compositio Math. 21 (1969), pp. 253–261.
[7] — *The categories of k-spaces and sequential spaces*, class notes, Mathematics Department, Carnegie-Mellon University.
[8] P. Freyd, *Abelian Categories*, New York 1964.
[9] P. Gabriel, *Réalisation géomètrique des ensembles semi-simpliciaux* (rédigé d'après un exposé de C. Godbillon), Séminaire Homotopique, Strasbourg 1963/64.
[10] A. M. Gleason, *Universal locally connected refinements*, Illinois J. Math. 7 (1963), pp. 521–531.
[11] H. Herrlich and G. E. Strecker, *Coreflective subcategories*, Trans. Amer. Math. Soc. 157 (1971), pp. 205–226.
[12] J. F. Kennison, *Reflective functors in general topology and elsewhere*, Trans. Amer. Math. Soc. 118 (1965), pp. 303–315.
[13] B. Mitchell, *Theory of Categories*, New York 1965.
[14] N. E. Steenrod, *A convenient category of topological spaces*, Michigan Math. J. 14 (1967), pp. 133–153.
[15] G. S. Young, *The introduction of local connectivity by change of topology*, Amer. J. Math. 68 (1946), pp. 479–494.

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Svenonius sentences and Lindström's theory on preservation theorems

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Introduction

Let us call the infinitary sentence $q \bigwedge \Theta$ a *Svenonius sentence* or *S-sentence* if q is a prefix of length $\leqslant \omega$ and Θ is a countable set of finitary first order formulas appropriate for a fixed countable similarity type ξ. It is immediate from the semantics of infinitary formulas (see [5], [6], [7] and § 0 below) that any class K of structures of type ξ, definable by an S-sentence, is a PC_Δ class. Svenonius [12] showed a partial converse of this fact.

Let us call K a $PC_{\bar{\delta}}$ class if K is the class of the reducts to ξ of the countable models of a countable set of ordinary sentences. Svenonius' theorem says that the $PC_{\bar{\delta}}$ classes are exactly the classes of countable models of S-sentences (1). Svenonius [12] also showed that Craig's interpolation theorem [1] is an easy consequence of this theorem. Considered from this point of view, (the proof of) Svenonius' theorem yields perhaps the most elementary model-theoretical proof of the interpolation theorem, or more particularly, it demonstrates that the ideas of Henkin [4], if properly applied, are sufficient for proving the interpolation theorem.

Knowing the close connection between certain preservation theorems and interpolation theorems, it is natural to ask whether there exist analogs involving S-sentences of known preservation theorems such that the original theorems are consequences of the new ones. This paper gives a positive answer to this question.

Call an S-sentence $q \bigwedge \Theta$ positive if every element of Θ is positive in the usual sense. Our Corollary 2.4 (a) says that K is a $PC_{\bar{\delta}}$ class closed under homomorphisms iff K is the class of countable models of a positive S-sentence. We also show that Lyndon's well-known preservation theorem [9] is an easy consequence of this result.

(1) See Theorem 2 in [12]. In [12] a different terminology is used and S-sentences are mentioned only in passing.