On shape

by

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1. Introduction. In [2] K. Borsuk introduced a relation of equivalence between compact metrizable spaces which he later called *having the same* (*topological*) *shape* (¹), and he proved that (i) any two compacta that belong to the same homotopy type necessarily have the same shape, and (ii) two compact absolute neighborhood retracts belong to the same homotopy type if and only if they have the same shape.

My first objective is to extend this concept to arbitrary metrizable spaces. This is done in § 3. The method used is based on the same idea that allowed the notions of absolute retract and absolute neighborhood retract, originally defined for compact metrizable spaces, to be extended first to arbitrary separable metrizable spaces and later to arbitrary metrizable spaces. (Cf. [1], p. 209.) The extension is such that the properties (i) and (ii) are preserved (Theorems 3.3 and 3.4). The proof that for compacta the extended concept is identical with Borsuk's original concept occupies § 4.

Over the past few years I have been investigating the relationship between fundamental group and covering spaces (Cf. [9]), and when I learned recently from Borsuk about this new concept I realized that it was exactly what was needed to complete the theory that I had been developing.

The problem was the following: The fundamental theorem of covering space theory asserts that the (connected) d-fold covering spaces of a connected, locally connected and semi-locally 1-connected space X are in bi-unique correspondence with the homotopy (²) classes of (transitive) representations of $\pi(X)$ in the symmetric group Σ_d of

(¹) In [2] the relation is called *being of the same fundamental type*. In a lecture in March 1968 at Princeton, Borsuk used instead of *fundamental type*, the term *topological shape*. I have taken the liberty of adopting this latter name, *topological shape*, for my generalization of the fundamental type, retaining the term *fundamental type* for the original concept.

(²) Homotopy of homomorphies is defined in § 6.

degree d. The question is: what happens to this correspondence when X does not have these local connectivity properties? It is not difficult (3) to remove the condition of semi-local 1-connectivity by the device of topologizing the group $\pi(X)$, but the removal of the condition of local connectivity eluded me until I made the startling discovery that for this fundamental correspondence to hold for spaces X that are not locally connected, $\pi(X)$ should not be a group at all. In fact what $\pi(X)$ must be is a group shape $\Pi(X)$, a concept analogous to that of topological shape.

What underlies the two concepts is, of course, a concept of shape in an abstract category. This is developed in § 2. The basic concept is that of similarity type of inverse systems (see (2.12) and above), which should be regarded as a concept more fundamental than that of inverse limit.

Since there have been a number of proposed generalizations of the covering space concept to the case of spaces that are not locally connected, I call my generalized covering spaces *overlays* (4). The definition is given in § 5. The rest of § 5 is devoted to the proof of what I call *the extension theorem*, which is not only the main part of the fundamental theorem of overlay theory (proved in § 6), but utilizes a lemma (5) (5.5) that could be of wide applicability (e.g. to the theory of homology or homotopy groups).

Finally, in § 7, I note how these ideas can be used to give a systematic foundation to local theories in topology. An earlier version appeared in [6].

I am especially indebted to R. J. Knill, who directed my attention to the Kuratowski–Wojdysławski theorem that makes § 3 possible.

2. Inverse systems. Let us consider an arbitrary category $\mathcal{E}$ of objects $U, V, W, \ldots$ and morphisms $u, v, w, \ldots, f, g, h, \ldots$ and let $\sim$ be a symmetric, reflexive, transitive and compositive relation between morphisms, called *similarity*. Of course one such relation is that of equality, but for the purposes of this paper this is not the most useful interpretation of $\sim$. (Cf. § 2 et seq.) The basic constructions of this paragraph take place in $\mathcal{E}$ and depend on $\sim$, i.e. they take place in $(\mathcal{E}, \sim)$. Two objects A, B of $\mathcal{E}$ are *of the same* (*similarity*) *type* (written $A \sim B$) when there are morphisms $f\colon A \to B$ and $g\colon B \to A$ such that $gf \sim 1_A$ and $fg \sim 1_B$.

Let us call a set of morphisms *concurrent* when they all have the same domain and the same range. A (*right similarity*) *equalizer* of a set $u_1, \ldots, u_n$ of concurrent morphisms is a morphism u for which $u_1 u \sim \ldots \sim u_n u$, a state of affairs that is, of course, only possible when the range of u is the common domain of $u_1, \ldots, u_n$.

In a category $\boldsymbol{U}$ (i.e. in a subcategory $\boldsymbol{U}$ of $\mathcal{E}$) an object U is a *predecessor* of an object U' (and U' is a *successor* of U) when there is in $\boldsymbol{U}$ a morphism whose domain is U and whose range is U'. Generalizing somewhat the usual definition, let us call a category $\boldsymbol{U}$ an *inverse system* when

(2.1) *any two objects of $\boldsymbol{U}$ have in $\boldsymbol{U}$ a common predecessor,*

and

(2.2) *any two concurrent morphisms of $\boldsymbol{U}$ have in $\boldsymbol{U}$ an equalizer.*

It is an easy exercise to show that in an inverse system

(2.1)′ *any finite number of objects have a common predecessor,*

and

(2.2)′ *any finite number of concurrent morphisms have an equalizer,*

from which it follows that

(2.3) *if $u_i\colon U_i \to U$ $(i = 1, \ldots, n)$ are morphisms belonging to an inverse system $\boldsymbol{U}$ that have the same range U then there are in $\boldsymbol{U}$ morphisms $u_i'\colon U' \to U_i$ $(i = 1, \ldots, n)$ having the same domain U', such that $u_1 u_1' \sim \ldots \sim u_n u_n'$.*

By a *mutation* $f\colon \boldsymbol{U} \to \boldsymbol{V}$ from an inverse system $\boldsymbol{U}$ to an inverse system $\boldsymbol{V}$ I shall mean any collection of morphisms $f\colon U \to V$ from objects U of $\boldsymbol{U}$ to objects V of $\boldsymbol{V}$ which is such that

(2.4) *for morphisms $u \in \boldsymbol{U}$, $f \in \boldsymbol{f}$, $v \in \boldsymbol{V}$, the morphism vfu belongs to $\boldsymbol{f}$ whenever it is defined,*

(2.5) *every object of $\boldsymbol{V}$ is the range of a least one of the morphisms belonging to the collection $\boldsymbol{f}$,*

(2.6) *any two concurrent morphisms belonging to the collection $\boldsymbol{f}$ have an equalizer in $\boldsymbol{U}$.*

In other words a collection of morphisms $f\colon \boldsymbol{U} \to \boldsymbol{V}$ is a mutation when and only when the constituent objects of $\boldsymbol{U}$ and $\boldsymbol{V}$ together with

(3) I did this in lectures at the University of Mexico in the summer of 1951. It has since been discovered independently by others, and appears for example in [18] on p. 82.

(4) All the reasonable generalizations agree for spaces that are connected, locally connected and semi-locally 1-connected. A good general reference for this classical theory is [15], Ch. 5. The difference between an overlay and a coordinate bundle with discrete group is only a technical one, and this, together with theorem 6.1 of the present paper shows that "overlay" is the only correct generalization of "covering space". It should be noted that the generalized covering spaces given for example in [12], p. 104, [11], p. 247, [18], p. 62, and [10], p. 17, are not overlays. The term overlay, which is just a literal translation of the original term Ueberlagerung, is an attempt to avoid the terminological confusion between covering space and covering (cover) by open sets.

(5) It has been pointed out to me that lemma 5.5 had already been proved by Kuratowski, although under slightly more stringent conditions. See [4], p. 240, lemma 13.1.

the constituent morphisms of $\boldsymbol{U}$, $\boldsymbol{V}$ and $\boldsymbol{f}$ constitute an inverse system. Notice that it follows from (2.3) that

(2.7) *if f_i: $U_i \to V$ $(i = 1, \ldots, n)$ are constituent morphisms of a mutation $\boldsymbol{f}$ that have the same range V then there are in $\boldsymbol{U}$ morphisms u_i': $U' \to U_i$ $(i = 1, \ldots, n)$ having the same domain U', such that $f_1 u_1' \sim \ldots \sim f_n u_n'$.*

The *composition* $\boldsymbol{gf}$ of mutations $\boldsymbol{f}$ and $\boldsymbol{g}$ is defined whenever the range of $\boldsymbol{f}$ is the domain of $\boldsymbol{g}$; it is the collection of all the morphisms gf, $f \in \boldsymbol{f}$, $g \in \boldsymbol{g}$, that are defined. It is not difficult to show that $\boldsymbol{gf}$ is a mutation; the verification of (2.4) and (2.5) is trivial, but the verification of (2.6) involves a little more because $g_1 f_1$ and $g_2 f_2$ can be concurrent even when neither f_1 and f_2 nor g_1 and g_2 are concurrent.

Given $U \in \boldsymbol{U}$, $V_1, V_2 \in \boldsymbol{V}$, $W \in \boldsymbol{W}$, $f_1, f_2 \in \boldsymbol{f}$, $g_1, g_2 \in \boldsymbol{g}$, with f_i: $U \to V_i$, g_i: $V_i \to W$ $(i = 1, 2)$, use (2.7) to construct $V \in \boldsymbol{V}$, v_i: $V \to V_i$ $(i = 1, 2)$ satisfying $g_1 v_1 \sim g_2 v_2$. [Cf. diagram 1.]

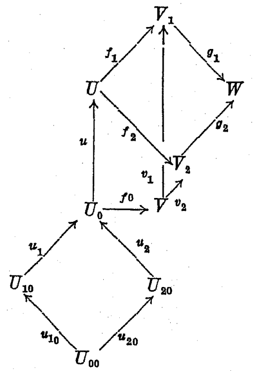

Diagram 1

By (2.5), (2.1) and (2.4) there must be an object U_0 of $\boldsymbol{U}$ and morphisms f_0: $U_0 \to V$ in $\boldsymbol{f}$ and u: $U_0 \to U$ in $\boldsymbol{U}$. By (2.2) the concurrent morphisms $f_i u$ and $v_i f_0$ must have an equalizer u_i: $U_{i0} \to U_0$ in $\boldsymbol{U}$ $(i = 1, 2)$. Finally use (2.3) to construct $U_{00} \in \boldsymbol{U}$ and u_{i0}: $U_{00} \to U_{i0}$ $(i = 1, 2)$ satisfying $u_1 u_{10} \sim u_2 u_{20}$. Then $g_1 f_1 u u_1 u_{10} \sim g_1 v_1 f_0 u_1 u_{10} \sim g_2 v_2 f_0 u_2 u_{20} \sim g_2 f_2 u u_2 u_{20} \sim g_2 f_2 u u_1 u_{10}$, which shows that $u u_1 u_{10}$ is an equalizer of $g_1 f_1$ and $g_2 f_2$.

The collection $\boldsymbol{u}$ of morphisms that belong to an inverse system $\boldsymbol{U}$ is a mutation from $\boldsymbol{U}$ to itself. It acts as an identity, for $\boldsymbol{fu} = \boldsymbol{f}$ and $\boldsymbol{ug} = \boldsymbol{g}$ whenever these compositions are defined.

Two mutations $\boldsymbol{f}, \boldsymbol{g}$: $\boldsymbol{U} \to \boldsymbol{V}$ will be called *similar* (written $\boldsymbol{f} \sim \boldsymbol{g}$) if

(2.8) *concurrent morphisms $f \in \boldsymbol{f}$ and $g \in \boldsymbol{g}$ always have an equalizer in $\boldsymbol{U}$.*

This means just that *the constituent morphisms of $\boldsymbol{f}$ and $\boldsymbol{g}$ together form a mutation*, which, in turn, means that *the constituent objects of $\boldsymbol{U}$ and $\boldsymbol{V}$, together with the constituent morphisms of $\boldsymbol{U}$, $\boldsymbol{V}$, $\boldsymbol{f}$ and $\boldsymbol{g}$, together form an inverse system.*

Similarity of mutations is obviously symmetric and reflexive. For a reason explained above the verifications of transitivity and compositivity are a little more involved.

Given $\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}$: $\boldsymbol{U} \to \boldsymbol{V}$ with $\boldsymbol{f} \sim \boldsymbol{g}$ and $\boldsymbol{g} \sim$, to prove transitivity we have to consider concurrent morphisms f, h: $U \to V$. [Cf. diagram 2.] By (2.5), (2.1) and (2.4) there is an object $U_0 \in \boldsymbol{U}$ and morphisms g: $U_0 \to V$ in $\boldsymbol{g}$ and u: $U_0 \to U$ in $\boldsymbol{U}$.

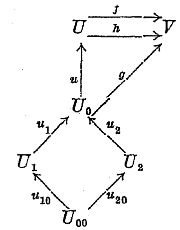

Diagram 2

Let u_1: $U_1 \to U_0$ be an equalizer of the concurrent morphisms $fu \in \boldsymbol{f}$ and g, and let u_2: $U_2 \to U_0$ be an equalizer of g and $hu \in \boldsymbol{h}$. By (2.3) there is an object $U_{00} \in \boldsymbol{U}$ and morphisms u_{i0}: $U_{00} \to U_i$ $(i = 1, 2)$ in $\boldsymbol{U}$ that satisfy $u_1 u_{10} \sim u_2 u_{20}$. Since $fuu_1 u_{10} \sim gu_1 u_{10} \sim gu_2 u_{20} \sim huu_2 u_{20} \sim huu_1 u_{10}$ we see that $uu_1 u_{10}$ is an equalizer of f and h. This completes the proof of transitivity. To prove compositivity, we are given $\boldsymbol{f}, \boldsymbol{g}$: $\boldsymbol{U} \to \boldsymbol{V}$ and $\boldsymbol{h}, \boldsymbol{k}$: $\boldsymbol{V} \to \boldsymbol{W}$ with $\boldsymbol{f} \sim \boldsymbol{k}$ and $\boldsymbol{g} \sim \boldsymbol{h}$, and we have to show that $\boldsymbol{hf} \sim \boldsymbol{kg}$. Thus we consider f_0: $U \to V_1$, g_0: $U \to V_2$, h_0: $V_1 \to W$ and k_0: $V_2 \to W$, and have to show that $h_0 f_0$ and $k_0 g_0$ have an equalizer in $\boldsymbol{U}$. It was observed above that $\boldsymbol{f} \sim \boldsymbol{g}$ means that the collection $\boldsymbol{f}, \boldsymbol{g}$ is a mutation, similarly $\boldsymbol{h}, \boldsymbol{k}$ is a mutation. The composition of these two mutations is the mutation whose constituents are just those morphisms hf, hg, kf, kg that are defined. Hence, by (2.6), the morphisms $h_0 f_0$ and $k_0 g_0$ that we are considering must have in $\boldsymbol{U}$ an equalizer.

We come now to the main definition. Two inverse systems $\boldsymbol{U}$ and $\boldsymbol{V}$ will be said to be *of the same* (*similarity*) *type* (written $\boldsymbol{U} \sim \boldsymbol{V}$) when there exist mutations $\boldsymbol{f}$: $\boldsymbol{U} \to \boldsymbol{V}$ and $\boldsymbol{g}$: $\boldsymbol{V} \to \boldsymbol{U}$ such that $\boldsymbol{g} \cdot \boldsymbol{f} \sim \boldsymbol{u}$ and $\boldsymbol{f} \cdot \boldsymbol{g} \sim \boldsymbol{v}$.

The preceding may be summarized as follows: given a category $\mathcal{E}$ and in it a relation $\sim$, we have constructed a new category $\bar{\mathcal{E}}$ and in it a relation also denoted by $\sim$. The objects of $\bar{\mathcal{E}}$ are the inverse systems of $\mathcal{E}$ and the morphisms of $\bar{\mathcal{E}}$ are the mutations of the inverse systems of $\mathcal{E}$. Similarity in $\bar{\mathcal{E}}$ means similarity of mutations, and similarity type in $\bar{\mathcal{E}}$ means similarity type of inverse systems. Now each object U of $\mathcal{E}$ may itself be considered as an inverse system of a rudimentary sort (one in which there is only the one object U and the one morphism 1_U: $U \to U$), and each morphism f: $U \to V$ of $\mathcal{E}$ may be considered as a mutation of a rudimentary sort (one that has only one constituent morphism). This means that there is naturally defined a biunique functor from $\mathcal{E}$ onto a certain subcategory of $\bar{\mathcal{E}}$. Thus $\bar{\mathcal{E}}$ may be regarded as a natural extension of $\mathcal{E}$. Moreover two rudimentary mutations f, g: $U \to V$ are similar in $\bar{\mathcal{E}}$ if and only if as morphisms they are similar in $\mathcal{E}$, and two rudimentary inverse systems U, V are of the same similarity type in $\bar{\mathcal{E}}$ if and only if as objects they are of the same similarity type in $\mathcal{E}$. Thus similarity and similarity type in $\bar{\mathcal{E}}$ may be regarded as natural extensions of similarity and similarity type in $\mathcal{E}$.

Two concepts that are usually associated with that of "inverse system" are "coinitial system" and "inverse limit". (In fact historically the first two of these concepts were developed out of the third one, — the inverse limit concept.) Let us now examine the concepts of coinitial system and inverse limit in the context of our more general definition of inverse system.

Generalizing somewhat the usual definition, let us call in inverse system $\boldsymbol{U}$ *coinitial* in an inverse system $\boldsymbol{W}$ when

(2.9) *each object of* $\boldsymbol{U}$ *is an object of* $\boldsymbol{W}$, *and each morphism of* $\boldsymbol{U}$ *is a morphism of* $\boldsymbol{W}$, i.e. $\boldsymbol{U}$ *is subcategory of* $\boldsymbol{W}$,

(2.10) *each object of* $\boldsymbol{W}$ *is preceded in* $\boldsymbol{W}$ *by at least one object of* $\boldsymbol{U}$,

(2.11) *any two concurrent morphisms of* $\boldsymbol{W}$ *whose common domain belongs to* $\boldsymbol{U}$ *have an equalizer in* $\boldsymbol{U}$.

This concept can be used to characterize similarity types of inverse systems:

(2.12) THEOREM. *Two disjoint inverse systems* $\boldsymbol{U}$ *and* $\boldsymbol{V}$ *of* $\mathcal{E}$ *are of the same type if and only if there is in* $\mathcal{E}$ *an inverse system* $\boldsymbol{W}$ *in which both* $\boldsymbol{U}$ *and* $\boldsymbol{V}$ *are coinitial.*

Proof. If $\boldsymbol{U}$ is coinitial in $\boldsymbol{W}$, let mutations $\boldsymbol{f}$: $\boldsymbol{U}\to\boldsymbol{W}$ and $\boldsymbol{g}$: $\boldsymbol{W}\to\boldsymbol{U}$ be defined as follows: $\boldsymbol{f}$ consists of all those morphisms w of $\boldsymbol{W}$ whose domains belong to $\boldsymbol{U}$; $\boldsymbol{g}$ consists of all those morphisms w of $\boldsymbol{W}$ whose ranges belong to $\boldsymbol{U}$. That $\boldsymbol{g}$ is a mutation is a consequence of the fact that $\boldsymbol{W}$ is an inverse system; that $\boldsymbol{f}$ is a mutation is a consequence of (2.10) and (2.11). That $\boldsymbol{fg}\sim\boldsymbol{w}$ is a consequence of the fact that $\boldsymbol{W}$ is an inverse system, that $\boldsymbol{gf}\sim\boldsymbol{u}$ is a consequence of (2.11).

If, conversely, $\boldsymbol{U}$ and $\boldsymbol{V}$ are of the same type, so that there exist mutations $\boldsymbol{f}$: $\boldsymbol{U}\to\boldsymbol{V}$ and $\boldsymbol{g}$: $\boldsymbol{V}\to\boldsymbol{U}$ such that $\boldsymbol{gf}\sim\boldsymbol{u}$ and $\boldsymbol{fg}\sim\boldsymbol{v}$, then an inverse system $\boldsymbol{W}$ in which both $\boldsymbol{U}$ and $\boldsymbol{V}$ are coinitial is the on whose objects are the objects U of $\boldsymbol{U}$ and V of $\boldsymbol{V}$, and whose morphisms are the morphisms u of $\boldsymbol{U}$, v of $\boldsymbol{V}$, f of $\boldsymbol{f}$, g of $\boldsymbol{g}$ and their compositions gf, fg, fgf, gfg,... That $\boldsymbol{W}$ is an inverse system, and that $\boldsymbol{U}$ and $\boldsymbol{V}$ are actually coinitial in $\boldsymbol{W}$ is due to the fact that $\boldsymbol{U}$ and $\boldsymbol{V}$ are inverse systems, that $\boldsymbol{f}$ and $\boldsymbol{g}$ are mutations, and that $\boldsymbol{gf}\sim\boldsymbol{u}$ and $\boldsymbol{fg}\sim\boldsymbol{v}$. ∎

Again, generalizing somewhat the usual definition, an object U of $\mathcal{E}$ will be called an *inverse limit* of an inverse system $\boldsymbol{W}$ when there is a mutation $\boldsymbol{f}$: $U\to\boldsymbol{W}$ such that

(2.13) *for each mutation* $\boldsymbol{g}$: $V\to\boldsymbol{W}$ *from an object* V *of* $\mathcal{E}$ *there is a morphism* h: $V\to U$ *in* $\mathcal{E}$ *for which* $\boldsymbol{f}h\sim\boldsymbol{g}$, and

(2.14) *if* h: $U\to U$ *is any morphism from* U *to itself for which* $\boldsymbol{f}h\sim\boldsymbol{f}$ *then* $h\sim 1_U$.

Let us note first that

(2.15) *if* U *is an inverse limit of* $\boldsymbol{W}$ *then an object* V *is also an inverse limit of* $\boldsymbol{W}$ *if and only if* V *is of the same type as* U.

Proof. If U and V are both inverse limits of $\boldsymbol{W}$, with associated mutations $\boldsymbol{f}$: $U\to\boldsymbol{W}$ and $\boldsymbol{g}$: $V\to\boldsymbol{W}$, then by (2.13) there are morphisms h: $U\to V$ and k: $V\to U$ for which $\boldsymbol{f}k\sim\boldsymbol{g}$ and $\boldsymbol{g}h\sim\boldsymbol{f}$. Hence $\boldsymbol{f}kh\sim\boldsymbol{f}$ and $\boldsymbol{g}hk\sim\boldsymbol{g}$, whence, by (2.14), $kh\sim 1_U$ and $hk\sim 1_V$. Thus U and V are of the same type.

Conversely, if U is an inverse limit of $\boldsymbol{W}$, with mutation $\boldsymbol{f}$: $U\to\boldsymbol{W}$, and V is of the same type as U, so that there are morphisms h: $U\to V$ and k: $V\to U$ for which $kh\sim 1_U$ and $hk\sim 1_V$, define $\boldsymbol{g}=\boldsymbol{f}k$. With this definition of $\boldsymbol{g}$ the conditions (2.13) and (2.14) are satisfied: if $\boldsymbol{l}$: $X\to\boldsymbol{V}$ is a mutation then, since U is an inverse limit of W, there must be a morphism m: $X\to U$ for which $\boldsymbol{f}m\sim\boldsymbol{l}$, and consequently the morphism hm: $X\to V$ is such that $\boldsymbol{g}hm=\boldsymbol{f}khm\sim\boldsymbol{f}m\sim\boldsymbol{l}$; if n: $V\to V$ is a morphism for which $\boldsymbol{g}n\sim\boldsymbol{g}$ then, since knh is a morphism from $\boldsymbol{U}$ to $\boldsymbol{U}$ for which $\boldsymbol{f}knh=\boldsymbol{g}nh\sim\boldsymbol{g}h=\boldsymbol{f}kh\sim\boldsymbol{f}$, by (2.14) $knh\sim 1_U$, whence $n\sim hk\sim 1_V$. Thus V is an inverse limit of $\boldsymbol{W}$. ∎

Secondly let us note that

(2.16) *inverse systems that are of the same type have the same inverse limits.*

Proof. Let $\boldsymbol{V}$ and $\boldsymbol{W}$ be inverse systems of the same type, so that there are mutations $\boldsymbol{h}$: $\boldsymbol{V}\to\boldsymbol{W}$ and $\boldsymbol{k}$: $\boldsymbol{W}\to\boldsymbol{V}$ for which $\boldsymbol{kh}\sim\boldsymbol{v}$ and $\boldsymbol{hk}\sim\boldsymbol{w}$, and let U be an inverse limit of $\boldsymbol{V}$ with associated mutation $\boldsymbol{f}$: $U\to\boldsymbol{V}$. Let us define $\boldsymbol{g}=\boldsymbol{hf}$. With this definition of $\boldsymbol{g}$ the conditions (2.13) and (2.14) are satisfied: if $\boldsymbol{l}$: $X\to\boldsymbol{W}$ is a mutation, then, applying (2.13) to the mutation $\boldsymbol{kl}$: $X\to\boldsymbol{V}$, there must be a morphism m: $X\to U$ for which $\boldsymbol{f}m\sim\boldsymbol{kl}$, and hence $\boldsymbol{g}m=\boldsymbol{hf}m\sim\boldsymbol{hkl}\sim\boldsymbol{l}$; if n: $U\to U$ is a morphism for which $\boldsymbol{g}n\sim\boldsymbol{g}$, then $\boldsymbol{hf}n\sim\boldsymbol{hf}$, and hence $\boldsymbol{f}n\sim\boldsymbol{khf}n\sim\boldsymbol{khf}\sim\boldsymbol{f}$, so that $n\sim 1_V$ by (2.14). Thus U is an inverse limit of $\boldsymbol{W}$. ∎

What (2.15) and (2.16) mean is that inverse limit associates to each type of inverse system a type of object. Of course in general the inverse limit need not always exist, although existence is guaranteed in some well known categories (e.g. in the category of topological spaces and continuous maps, and in the category of groups and homomorphies, with $\sim$ interpreted as equality).

If $\sim_1$ and $\sim_2$ are two similarity relations in $\mathcal{E}$ such that $f\sim_2 g$ whenever $f\sim_1 g$ then (a) $A\sim_2 B$ whenever $A\sim_1 B$. (b) each $(\sim_1)$-equalizer is a $(\sim_2)$-equalizer, (c) each $(\sim_1)$-inverse system is a $(\sim_2)$-inverse system, (d) each $(\sim_1)$-mutation is a $(\sim_2)$-mutation, (e) two such mutations are $(\sim_2)$-similar whenever they are $(\sim_1)$-similar, and (f) two $(\sim_1)$-inverse systems are of the same $(\sim_2)$-type whenever they are of the same $(\sim_1)$-type. Nevertheless even if an $(\sim_1)$-inverse system has inverse limits in both $(\mathcal{E}, \sim_1)$ and $(\mathcal{E}, \sim_2)$ they need not be even of the same $(\sim_2)$-type. An example will be given in § 7.

3. Topological shape. In this section the category $\mathcal{E}$ is specialized to the category $\mathfrak{R}$ of *absolute neighborhood retracts* and the *continuous mappings* between them and the relation $\sim$ is interpreted to mean *homotopy between continuous mappings* and denoted by $\simeq$. What a *topological shape* is just a homotopy type of inverse systems of the category $\mathfrak{R}$.

Consider a metrizable space X imbedded as a closed [6] set in an absolute neighborhood retract P. According to Hanner's first theorem ([1] IV, 10.1, p. 96) every neighborhood [7] of X in P is itself an absolute neighborhood retract. Hence the set that consists of all the neighborhoods U of X in P together with the indigenous inclusions is an inverse system of the category $\mathfrak{R}$. This system will be called the *complete neighborhood system* $\boldsymbol{U}(X,P)$ of X in P. By a *neighborhood system* of X in P I shall mean any coinitial subsystem of $\boldsymbol{U}(X,P)$.

If metrizable spaces X and Y are imbedded as closed sets in respective absolute neighborhood retracts P and Q then, according to a well-known theorem ([1] IV, 4.2 (ii), p. 88), any continuous mapping f from X to Y can be extended to a mapping $\bar{f}$ into Q of some neighborhood of X. Any such extension $\bar{f}$ determines uniquely a mutation $\boldsymbol{f}$ from the complete neighborhood system $\boldsymbol{U}(X,P)$ of X in P to the complete neighborhood system $\boldsymbol{V}(Y,Q)$ of Y in Q; the constituents of $\boldsymbol{f}$ are the mappings into neighborhoods V of Y in Q of neighborhoods U of X in $\bar{f}^{-1}(V)$ obtained by restricting $\bar{f}$ to U. Such a mutation $\boldsymbol{f}$ will be called an *extension* of the mapping f. Of course not every mutation from $\boldsymbol{U}(X,P)$ to $\boldsymbol{V}(Y,P)$ is an extension.

(3.1) Theorem. *Suppose that f and g are continuous mappings of a metrizable space X into a metrizable space Y, and let P and Q be absolute neighborhood retracts in which X and Y are respectively closed. If f and g are homotopic then any mutations $\boldsymbol{f}$ and $\boldsymbol{g}$: $\boldsymbol{U}(X,P)\to\boldsymbol{V}(Y,Q)$ that extend f and g respectively are homotopic.*

Proof. Let $h\colon X\times[0,1]\to Y$ be a homotopy between f and g, and let U_0 be a neighborhood of X in P such that f and g have respective extensions $\bar{f},\bar{g}\colon U_0\to Q$. Define $k\colon U_0\times[0]\cup X\times[0,1]\cup U_0\times[1]\to Q$ by the formulae

$$\begin{aligned} k(p,0)&=\bar{f}(u) && \text{for } p\in U_0,\\ k(p,t)&=h(u,t) && \text{for } p\in X,\ t\in[0,1],\\ k(p,1)&=\bar{g}(u) && \text{for } p\in U_0. \end{aligned}$$

(6) The requirement that X be closed in P is, in some cases, extremely inconvenient, as is shown by example 4 of § 6. Fortunately it has recently been shown by D. M. Hyman [13] that this condition can be removed.

(7) By *neighborhood* I shall always mean *open neighborhood*.

Since Q is an absolute neighborhood retract, and $U_0\times[0]\cup X\times\times[0,1]\cup U_0\times[1]$ is closed in $U_0\times[0,1]$, this can be extended to a mapping $\bar{k}$ into Q of a neighborhood N of $U_0\times[0]\cup X\times[0,1]\cup U_0\times[1]$. Since $[0,1]$ is compact there is a neighborhood U_1 of X in U_0 such that $U_1\times[0,1]\subset N$. Then $\bar{h}=\bar{k}|U_1\times[0,1]$ is a homotopy in Q between $\bar{f}|U_1$ and $\bar{g}|U_1$. It follows that $\boldsymbol{f}\simeq\boldsymbol{g}$. ■

(3.2) Theorem. *If metrizable spaces X and Y belong to the same homotopy type then, for any absolute neighborhood retracts P and Q in which X and Y are respectively closed, the complete neighborhood systems $\boldsymbol{U}(X,P)$ and $\boldsymbol{V}(Y,Q)$ are of the same homotopy type.*

Proof. Let $f\colon X\to Y$ and $g\colon Y\to X$ be continuous mappings for which $gf\simeq 1_X$ and $fg\simeq 1_Y$, and let $\boldsymbol{f}\colon\boldsymbol{U}(X,P)\to\boldsymbol{V}(Y,Q)$ and $\boldsymbol{g}\colon\boldsymbol{V}(Y,Q)\to\boldsymbol{U}(X,P)$ be mutations that extend f and g respectively. Then the mutation $\boldsymbol{gf}\colon\boldsymbol{U}(X,P)\to\boldsymbol{U}(X,P)$ and the identity mutation $\boldsymbol{u}\colon\boldsymbol{U}(X,P)\to\boldsymbol{U}(X,P)$ extend the mappings $gf\colon X\to X$ and $1_X\colon X\to X$ respectively. Consequently, according to the previous theorem, $\boldsymbol{gf}\simeq\boldsymbol{u}$. Similarly $\boldsymbol{fg}\simeq\boldsymbol{v}$. Thus $\boldsymbol{U}(X,P)$ and $\boldsymbol{V}(Y,Q)$ are seen to be of the same homotopy type. ■

Let us call the homotopy type to which $\boldsymbol{U}(X,P)$ belongs, which has just been shown to depend neither on the absolute neighborhood retract P nor on the manner in which X is imbedded as a closed set in P, the (topological) *shape*, $\operatorname{Sh}X$, *of* X. The rest of the statement of (3.2) may now be phrased as follows:

(3.3) Corollary. *Spaces that belong to the same homotopy type have the same shape.*

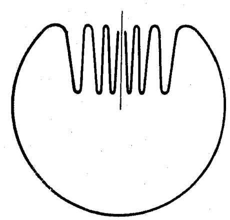

Diagram 3

The converse of (3.3) is generally false. For example the well-known 1-dimensional space shown in diagram 3 has the same shape as the circle, but they are not of the same homotopy type. However the converse does hold when the spaces involved are required to be absolute neighborhood retracts:

(3.4) Theorem. *If absolute neighborhood retracts X and Y have the same shape then they must be of the same homotopy type.*

Proof. Since X and Y have the same shape then, for any absolute neighborhood retracts P and Q in which X and Y are respectively closed, there are mutations $\boldsymbol{f}$: $\boldsymbol{U}(X, P) \to \boldsymbol{V}(Y, Q)$ and $\boldsymbol{g}$: $\boldsymbol{V}(Y, Q) \to \boldsymbol{U}(X, P)$ such that $\boldsymbol{gf} \simeq \boldsymbol{u}$ and $\boldsymbol{fg} \simeq \boldsymbol{v}$. Since X and Y are themselves absolute neighborhood retracts we may choose $P = X$ and $Q = Y$. But $\boldsymbol{f}$: $\boldsymbol{U}(X, X) \to \boldsymbol{V}(Y, Y)$ is just the mapping f: $X \to Y$, $\boldsymbol{g}$: $\boldsymbol{V}(Y, Y) \to \boldsymbol{U}(X, X)$ is just the mapping g: $Y \to X$, and the homotopies $\boldsymbol{gf} \simeq \boldsymbol{u}$ and $\boldsymbol{fg} \simeq \boldsymbol{v}$ are just the homotopies $gf \simeq 1_X$ and $fg \simeq 1_Y$. ■

If we define the *shape* of a mapping f of a metrizable space X into a metrizable space Y to be the class of all mutations $\boldsymbol{f}$: $\boldsymbol{U}(X, P) \to \boldsymbol{V}(Y, Q)$ that are homotopic to extensions of f, P and Q ranging over all absolute neighborhood retracts in which X and Y are respectively closed, then (3.1) shows that

(3.5) COROLLARY. *Mappings that are homotopic have the same shape.*

The converse of (3.5) is also generally false; it is not difficult to construct mappings of the space shown in diagram 3 onto itself that have the same shape but are not homotopic. However, again, the converse does hold when the spaces involved are required to be absolute neighborhood retracts:

(3.6) THEOREM. *If X and Y are absolute neighborhood retracts then mappings f, g: $X \to Y$ that have the same shape must be homotopic.*

The proof of (3.6) is analogous to that of (3.4).

4. The shape of a compactum. In [2] compacta X and Y are imbedded in the Hilbert cube (³) H, and a *fundamental sequence* $\boldsymbol{F}$ from X to Y is defined to be a sequence of maps F_k: $H \to H$, $k = 1, 2, \ldots$, such that for each neighborhood V of Y there is at least one neighborhood U of X for which

(4.1) $F_k|U \simeq F_{k+1}|U$ in V for almost all k.

Two fundamental sequences $\boldsymbol{F}$ and $\boldsymbol{G}$ are called *homotopic* (notation $\boldsymbol{F} \simeq \boldsymbol{G}$) if for each neighborhood V of Y there is at least one neighborhood U of X for which

(4.2) $F_k|U \simeq G_k|U$ in V for almost all k.

The *identity fundamental sequence* $\boldsymbol{I}$ is the sequence $I_1, I_2, \ldots$, of identity maps I_k: $H \to H$. The *composition* $\boldsymbol{GF}$ of two fundamental sequences $\boldsymbol{F}$ and $\boldsymbol{G}$ is the sequence $G_1F_1, G_2F_2, \ldots$ Two compacta X and Y are then described as *fundamentally equivalent*, or as belonging to the same fundamental type (¹), if there exist fundamental sequences $\boldsymbol{F}$ from X to Y and $\boldsymbol{G}$ from Y to X such that $\boldsymbol{GF} \simeq \boldsymbol{I}$ and $\boldsymbol{FG} \simeq \boldsymbol{I}$; it is shown in [2] that this does not depend on the manner of imbedding of X and Y in H.

(³) In order to fit my approach better, I have made some slight and unimportant modifications of Borsuk's definitions.

The object of this section is to show that the fundamental type of a compactum is precesely its shape.

(4.3) THEOREM. *Compacta X and Y belong to the same fundamental type if and only if they have the same shape.*

Before giving the proof of this theorem I would like to interpret it in categorical terms.

Let us consider the category $\mathfrak{B}$ whose objects are the compact subsets X of H, and whose morphisms are the fundamental sequences. In $\mathfrak{B}$ homotopy between sequences is a symmetric, reflexive, transitive and compositive relation; the quotient category $\mathfrak{B}/\simeq$ is just Borsuk's fundamental category ([2], p. 233).

Along with $\mathfrak{B}$ let us consider another category, — the subcategory $\mathfrak{K}$ of $\bar{\mathfrak{K}}$ that consists of the complete neighborhood systems of compacta X in H together with the mutations that subsist between them. In $\mathfrak{K}$ homotopy between mutations is, of course, a symmetric, reflexive, transitive and compositive relation. For the proof of (4.3) it is sufficient to construct a biunique functor from the category $\mathfrak{B}$ onto the category $\mathfrak{K}$ which is such that the homotopic fundamental sequences correspond to homotopic mutations and conversely. Of course this amounts to constructing a biunique functor from the quotient category $\mathfrak{B}/\simeq$ onto the quotient category $\mathfrak{K}/\simeq$.

Since X and H are compacta, the complete neighborhood systems $\boldsymbol{U}(X, H)$ now to be considered always have cofinal subsystems that are just sequences.

To each object X of the category $\mathfrak{B}/\simeq$ let us associate the object $\boldsymbol{U}(X, H)$ of the category $\mathfrak{K}/\simeq$. Clearly this establishes a biunique correspondence between the objects of these two categories.

Consider an arbitrary fundamental sequence $\boldsymbol{F}$ from X to Y. Let $\boldsymbol{f}$ be the collection of all those maps f: $U \to V$, $U \in \boldsymbol{U}(X, H)$, $V \in \boldsymbol{V}(Y, H)$, that are such that

(4.5) $f \simeq F_k|U$ in V for almost all k.

Using (4.1) it is easy to verify that the collection $\boldsymbol{f}$ satisfies conditions (2.4), (2.5) and (2.6), and is therefore a mutation.

The proof of (4.3) is now reduced to the proofs of the following two propositions:

(4.6) *If $\boldsymbol{F}$ and $\boldsymbol{G}$ are fundamental sequences from X to Y, and $\boldsymbol{f}$ and $\boldsymbol{g}$: $\boldsymbol{U}(X, H) \to \boldsymbol{V}(Y, H)$ are the corresponding mutations, then $\boldsymbol{F} \simeq \boldsymbol{G}$ if and only if $\boldsymbol{f} \simeq \boldsymbol{g}$.*

(4.7) *Given any mutation $\boldsymbol{f}$: $\boldsymbol{U}(X, H) \to \boldsymbol{V}(Y, H)$ there is a fundamental sequence $\boldsymbol{F}^*$ from X to Y whose corresponding mutation $\boldsymbol{f}^*$ is homotopic to $\boldsymbol{f}$.*

Proof of (4.6). Suppose that $f \in \boldsymbol{f}$ and $g \in \boldsymbol{g}$ are concurrent, i.e. that f and g are maps from $U \in \boldsymbol{U}(X, H)$ to $V \in \boldsymbol{V}(Y, H)$. If $\boldsymbol{F} \simeq \boldsymbol{G}$ then by (4.2) there is a neighborhood U' of X for which $F_k|U' \simeq G_k|U'$ in V for almost all k. Of course U' can be chosen so that $U' \subset U$, and then the inclusion $u\colon U' \subset U$ will be seen to be an equalizer of f and g.

Conversely if $\boldsymbol{f} \simeq \boldsymbol{g}$ the maps $f, g\colon U \to V$ must have an equalizer in $\boldsymbol{U}(X, H)$, and this means that there must be a neighborhood U' of X in U for which $f|U' \simeq g|U'$. It follows from the definition of $\boldsymbol{f}$ and $\boldsymbol{g}$ that $F_k|U' \simeq G_k|U'$ for almost all k.

Proof of (4.7). Let $V_1 \supset V_2 \supset \ldots$ be a neighborhood sequence of Y in H, and, for each positive integer j, let $f_j'\colon U_j' \to V_j$ be a constituent map of $\boldsymbol{f}$. Now it is easy to construct seriatim a sequence $U_1, U_2, \ldots$ of neighborhoods of X in H which is such that $U_1 = U_1'$, $\overline{U}_{j+1} \subset U_j$ for $j = 1, 2, \ldots$, and, for each positive integer j, the inclusion $U_{j+1} \subset U_{j+1}'$ is an equalizer of the concurrent maps $f_j'|U_{j+1}'\colon U_{j+1}' \to V_j$ and $f_{j+1}'\colon U_{j+1}' \to V_j$. Denote $f_j'|U_j$ by f_j. Thus we have a sequence of neighborhoods U_j, $j = 1, 2, \ldots$, of X in H and a sequence of maps $f_j\colon U_j \to V_j$, $j = 1, 2, \ldots$, such that, for each positive integer j,

$$\overline{U}_{j+1} \subset U_j \quad \text{and} \quad f_{j+1}|\overline{U}_{j+2} \simeq f_j|\overline{U}_{j+2} \quad \text{in} \quad V_j\,.$$

Now, fixing j temporarily, we shall construct, for each positive integer $k \leqslant j$, a map $f_j^k\colon \overline{U}_{k+2} \to V_k$ which is such that

$$f_j^k|\overline{U}_{i+2} \simeq f_i|\overline{U}_{i+2} \quad \text{in} \quad V_i \quad \text{for each integer } i = k, k+1, \ldots, j\,.$$

For $k = j$, this is easy; simply define $f_j^j = f_{j+1}|\overline{U}_{j+2}$. Suppose that $k < j$. Then f_j^{k+1} has been defined, and

$$f_j^{k+1}|\overline{U}_{i+3} \simeq f_{i+1}|\overline{U}_{i+3} \quad \text{in} \quad V_{i+1} \quad \text{for } i = k, k+1, \ldots, j-1\,.$$

Since V_k is an absolute neighborhood retract the homotopy between $f_j^{k+1}|\overline{U}_{k+3}$ and $f_{k+1}|\overline{U}_{k+3}$ can be extended, according to a well-known theorem ([1] IV, 8.1, p. 94), to a homotopy between some extension of $f_j^{k+1}|\overline{U}_{k+3}$ and the map $f_{k+1}|\overline{U}_{k+2}$. (Cf. diagram 4.) Let us denote this extension by f_j^k. Then the map $f_j^k\colon \overline{U}_{k+2} \to V_k$ is such that

$$f_j^k = f_j^k|\overline{U}_{k+2} \simeq f_{k+1}|\overline{U}_{k+2} \simeq f_k|\overline{U}_{k+2} \quad \text{in} \quad V_k$$

and

$$f_j^k|\overline{U}_{i+2} = f_j^{k+1}|\overline{U}_{i+2} \simeq f_i|\overline{U}_{i+2} \quad \text{in} \quad V_i \quad \text{for each } i = k+1, \ldots, j\,.$$

$$\begin{array}{lll}
f_1|\overline{U}_3 \simeq_{V_1} f_2|\overline{U}_3 & & \simeq_{V_1} f_j^1 \\
\quad f_2|\overline{U}_4 \simeq_{V_2} f_3|\overline{U}_4 & & \simeq_{V_2} f_j^2 \\
\qquad \vdots & & \quad \vdots \\
\qquad f_{j-1}|\overline{U}_{j+1} \simeq_{V_{j-1}} f_j|\overline{U}_{j+1} & & \simeq_{V_{j-1}} f_j^{j-1} \\
\qquad\qquad f_j|\overline{U}_{j+2} \simeq_{V_j} f_{j+1}|\overline{U}_{j+2} = f_j^j & &
\end{array}$$

Diagram 4

In particular, for $k = 1$, we find a map $f_j^1\colon \overline{U}_3 \to V_1$ which is such that $f_j^1|\overline{U}_{i+2} \simeq f_i|\overline{U}_{i+2}$ in V_i for each positive integer $i \leqslant j$. Let F_j^* be any mapping of H into itself that extends f_j^1. Then F^* has the following property

$$F_j^*|\overline{U}_{i+2} \simeq f_i|\overline{U}_{i+2} \quad \text{in} \quad V_i \quad \text{for each positive integer } i \leqslant j\,.$$

If V is any neighborhood of Y in H, and i is an index for which $V_i \subset V$, then

$$F_j^*|U_{i+2} \simeq f_i|U_{i+2} \quad \text{in} \quad V \quad \text{for each } j \geqslant i\,,$$

and hence

$$F_j^*|U_{i+2} \simeq F_{j+1}^*|U_{i+2} \simeq \ldots \quad \text{in} \quad V \quad \text{for each } j \geqslant i\,.$$

This shows that the sequence of maps $F_j^*\colon H \to H$, $j = 1, 2, \ldots$, is a fundamental sequence $\boldsymbol{F}^*$.

Now let us consider the mutation $\boldsymbol{f}^*$ that corresponds to $\boldsymbol{F}^*$. Let $f^*\colon U \to V$ be a constituent map of $\boldsymbol{f}^*$, and let $f\colon U \to V$ be a constituent map of $\boldsymbol{f}$ that is concurrent with f^*. By (4.5),

$$f^* \simeq F_k^*|U \quad \text{in} \quad V \quad \text{for almost all } k\,.$$

Let i be an index for which $V_i \subset V$. If U' is a neighborhood of $\boldsymbol{X}$ in $U \cap U_{i+2}$, it follows that

$$f^*|U' \simeq f_i|U' \quad \text{in} \quad V.$$

Finally, since $f|U'\colon U' \to V$ and $f_i|U'\colon U' \to V$ are concurrent, they have an equalizer $U'' \subset U'$, and thus

$$f^*|U'' \simeq f|U'' \quad \text{in} \quad V.$$

This shows that $U'' \subset U$ is an equalizer of f^* and f. Thus $\boldsymbol{f}^* \simeq \boldsymbol{f}$. ∎

5. Overlays. Let us consider separable metrizable spaces X and $\widetilde{X}$ and a continuous mapping e of $\widetilde{X}$ into X. A collection $\widetilde{\boldsymbol{M}} = \{\widetilde{M}_i^a\}$ of subsets of $\widetilde{X}$ will be said to *lie evenly over* a collection $\boldsymbol{M} = \{M_i\}$ of subsets of X when

(5.1) $e^{-1}(M_i) = \bigcup\limits_{a} \widetilde{M}_i^a$ for each index i; each $\widetilde{M}_i^a$ is open in $e^{-1}(M_i)$; each set $\widetilde{M}_i^a$ is mapped by $e_i^a = e|\widetilde{M}_i^a$ topologically onto M_i; and if $M_i \cap M_j \neq \emptyset$ then each set $\widetilde{M}_i^a$ meets exactly one of the sets $\widetilde{M}_j^\beta$ (in particular $M_i^a \cap M_i^\beta = \emptyset$ whenever $a \neq \beta$).

We may also say that $\boldsymbol{M}$ is *evenly overlaid by* $\widetilde{\boldsymbol{M}}$. A map $e\colon \widetilde{X} \to X$ will be called an *overlay* if $\widetilde{X}$ has an open cover $\widetilde{\boldsymbol{M}}$ that lies evenly over some open cover $\boldsymbol{M}$ of X.

Since $\widetilde{X}$ satisfies the second countability axiom the collection $\{\widetilde{M}_i^a\}$ is, for each index i, either finite or countably infinite. Let d_i denote the number of sets $\widetilde{M}_i^\alpha$. Thus α ranges over the index set $I(d_i)=\{1,2,\dots,d_i\}$ (or $\{\dots -1,0,1,\dots\}$ if $d_i=\infty$). In general d_i need not be the same as d_j, but of course they must be the same whenever M_i and M_j intersect. In such a case $I(d_i)=I(d_j)$ and a permutation ω_{ij} of $I(d)$ is determined as follows: $\omega_{ij}=\left(\dots\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\dots\right)$, where $\widetilde{M}_i^\alpha\cap\widetilde{M}_j^\beta\neq\emptyset$. Note that ω_{ii} is always just the identity permutation, that $\omega_{ji}=\omega_{ij}^{-1}$, and that $\omega_{ij}\omega_{jk}=\omega_{ik}$ whenever $M_i\cap M_j\cap M_k\neq\emptyset$. When all the numbers d_i are the same we may speak of a *d-fold overlay* or a *d-sheeted overlay*; the number d may then be called the *number of sheets*.

If $r\colon \widetilde{U}\to U$ is an overlay and $X\subset U$ then $e\colon \widetilde{X}\to X$, where $\widetilde{X}=r^{-1}(X)$ and $e=r|\widetilde{X}$, is also an overlay. In this situation we may describe e as a *restriction* of r, or r as an *extension* of e.

(5.2) EXTENSION THEOREM. *If X is a subset of a separable metrizable space P and $e\colon\widetilde{X}\to X$ is an overlay, then some neighborhood U of X in P has an overlay $r\colon\widetilde{U}\to U$ that is an extension of e.*

Before proving this theorem we must first prove some point set theoretical lemmas.

(5.3) LEMMA. *If X is any subset of a completely normal space P, and if M_1 and M_2 are disjoint and open in X, then there exist disjoint open subsets W_1 and W_2 of P such that $X\cap W_1=M_1$ and $X\cap W_2=M_2$.*

Proof. Since P is completely normal, $P'=P-\overline{M}_1\cap\overline{M}_2$ is normal. The sets $K_1=\overline{M}_1-\overline{M}_2$ and $K_2=\overline{M}_2-\overline{M}_1$ are disjoint and closed in P'; hence there exist disjoint open sets N_1 and N_2 of P' such that $K_1\subset N_1$ and $K_2\subset N_2$. Thus $M_1\subset K_1\subset N_1$ and $M_2\subset K_2\subset N_2$. Let G_1 and G_2 be any open sets of P which are such that $X\cap G_1=M_1$ and $X\cap G_2=N_2$, and define $W_1=N_1\cap G_1$ and $W_2=N_2\cap G_2$. Clearly W_1 and W_2 are disjoint and open in P, and $X\cap W_1=M_1$ and $X\cap W_2=M_2$. ∎

(5.4) LEMMA. *If X is any subset of a completely normal space P, and M is open in X, then there exists an open set W of P which is such that $X\cap W=M$ and $X\cap\overline{W}=X\cap\overline{M}$.*

Proof. Let $M'=X-\overline{M}$. Then M and M' are disjoint and open in X. Hence by the preceding lemma there exist disjoint open sets W and W' of P which are such that $X\cap W=M$ and $X\cap W'=M'$. Then $X\cap\overline{M}\subset X\cap\overline{W}\subset X\cap(P-W')=X-M'=X\cap\overline{M}$. ∎

If $M_1, M_2, \dots$ is an open cover of a space X and $A_1, A_2, \dots$ is a cover of X by sets that are open in some superspace P of X we may say that $\{A\}$ *extends* $\{M\}$ if $X\cap A_i=M_i$ for every i. If $\{M_i\}$ and $\{A_i\}$ are locally finite we may consider their nerves, which are locally finite dimensional complexes. I shall say that these nerves are naturally isomorphic if, for any finite sequence of indices $i_1, i_2, \dots, i_n$, the set $M_{i_1}\cap M_{i_2}\cap\dots\cap M_{i_n}$ is vacuous if and only if the set $A_{i_1}\cap A_{i_2}\cap\dots\cap A_{i_n}$ is vacuous.

(5.5) LEMMA. ([5]) *If $M_1, M_2, \dots$ is locally finite open cover of a space X, and P is a completely normal superspace of X, then $\{M_i\}$ can be extended to a cover $\{A_i\}$ of X by open sets of P which is such that its nerve is naturally isomorphic to the nerve of $\{M_i\}$.*

Proof. By the preceding lemma there exist in P open sets $W_1, W_2, \dots$ such that $X\cap W_i=M_i$ and $X\cap\overline{W}_i=X\cap\overline{M}_i$ for every i. For each index i let us define

$$A_i=W_i-\bigcup\overline{W}_\alpha-\bigcup\overline{W}_\alpha\cap\overline{W}_\beta-\bigcup\overline{W}_\alpha\cap\overline{W}_\beta\cap\overline{W}_\gamma\dots,$$

where in the first union α ranges over those indices for which $\alpha<i$ and $M_\alpha\cap M_i=\emptyset$, in the second union α and β range over those indices for which $\alpha<\beta<i$ and $M_\alpha\cap M_\beta\cap M_i=\emptyset$, in the third union α, β, γ range over those indices for which $\alpha<\beta<\gamma<i$ and $M_\alpha\cap M_\beta\cap M_\gamma\cap M_i=\emptyset$, etc. Each union is finite, and there cannot be more than $i-1$ unions; hence A_i must be open in P. Furthermore

$$X\cap A_i=M_i-\bigcup X\cap\overline{M}_\alpha-\bigcup X\cap\overline{M}_\alpha\cap\overline{M}_\beta-\bigcup X\cap\overline{M}_\alpha\cap\overline{M}_\beta\cap\overline{M}_\gamma=M_i.$$

Now if $\alpha<i$ and $M_\alpha\cap M_i=\emptyset$ then

$$A_\alpha\cap A_i\subset W_\alpha\cap(W_i-\overline{W}_\alpha)=\emptyset,$$

if $\alpha<\beta<i$ and $M_\alpha\cap M_\beta\cap M_i=\emptyset$ then

$$A_\alpha\cap A_\beta\cap A_i\subset W_\alpha\cap W_\beta\cap(W_i-\overline{W}_\alpha\cap\overline{W}_\beta)=\emptyset,$$

etc. ∎

Proof of the extension theorem. Since $e\colon\widetilde{X}\to X$ is an overlay there is an open cover $\boldsymbol{M}$ of X over which there lies evenly an open cover $\widetilde{\boldsymbol{M}}$ of $\widetilde{X}$. Since X is a separable metrizable space it may be assumed that $\boldsymbol{M}$ is countable and locally finite. Let us write $\boldsymbol{M}$ as a sequence $M_1, M_2, \dots$

By (5.5) $\boldsymbol{M}$ extends to a cover $\boldsymbol{A}=\{A_1, A_2, \dots\}$ of X by sets that are open in P whose nerve is naturally isomorphic to the nerve of $\boldsymbol{M}$. Let $U=\bigcup A_i$; this is a neighborhood of X in P.

For each positive integer i and index $\alpha\in I(d_i)$ let $\hat{A}_i^\alpha$ be a topological space homeomorphic to A_i, and let $\hat{r}_i^\alpha$ be a homeomorphism of $\hat{A}_i^\alpha$ onto A_i. In the conjunction $\hat{U}=\bigcup_{i\alpha}\hat{A}_i^\alpha$ of all the spaces $\hat{A}_i^\alpha$, $\alpha\in I(d_i)$, $i=1,2,\dots$, let us introduce the following equivalence relation $\approx$:

(5.6) *if $\hat{p}'\in\hat{A}_i^\alpha$ and $\hat{p}''\in\hat{A}_j^\beta$ then $\hat{p}'\approx\hat{p}''$ if and only if $\hat{r}_i^\alpha(\hat{p}')=\hat{r}_j^\beta(\hat{p}'')$ and $\omega_{ij}=\left(\dots\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\dots\right)$.*

It is easy to see that the relation $\approx$ is symmetric and reflexive. Its transitivity is a consequence of the fact that the 2-dimensional skeleton of the nerve of $\boldsymbol{A}$ is naturally isomorphic to the 2-dimensional skeleton of the nerve of $\boldsymbol{M}$. Hence the quotient space $\widetilde{U}=\hat{U}/\approx$ is defined.

Let us denote by q the quotient mapping of $\hat{U}$ onto $\widetilde{U}$, and by q_i^a the restriction $q|\hat{A}_i^a$ of q to $\hat{A}_i^a$. Obviously q_i^a maps $\hat{A}_i^a$ homeomorphically onto a subset $\widetilde{A}_i^a$ of $\widetilde{U}$, and, since $q^{-1}(\widetilde{A}_i^a) = \bigcup_{j\beta} (\hat{r}_j^\beta)^{-1}(A_i)$, it is clear that $\widetilde{A}_i^a$ is open in $\widetilde{U}$. Since each $\hat{A}_i^a$ is a regular space in which the second axiom of countability is satisfied, the same is true in the countable union $\widetilde{U} = U_{ia}\widetilde{A}_i^a$; thus $\widetilde{U}$ is a separable metrizable space.

For any point $\widetilde{p}$ of $\widetilde{U}$ there is at least one point $\hat{p}$ of $\hat{U}$ for which $q(\hat{p}) = \widetilde{p}$, and if $\hat{p}'$ and $\hat{p}''$ are two such points then $\hat{p}' \approx \hat{p}''$. Hence if we define $r(\widetilde{p})$ to be the image under an appropriate $\hat{r}_i^a$ of a point of $q^{-1}(\widetilde{p})$ the definition is unambiguous. Obviously $rq_i^a = \hat{r}_i^a$ for every i, a. Since the function r, defined on $\widetilde{U}$ with values in U, by its very definition maps each $\widetilde{A}_i^a$ topologically onto A_i, it is easily verified that r is an overlay. It remains only to show that r is an extension of e.

Consider a point $\widetilde{x}$ of $\widetilde{X}$. It belongs to at least one of the sets $\widetilde{M}_i^a$. Therefore the point $x = e(\widetilde{x})$ must belong to M_i and hence to A_i. Let $\hat{p}_i^a$ denote the point of $\hat{A}_i^a$ that is mapped by $\hat{r}_i^a$ into x, and let $\widetilde{p} = q(\hat{p}_i^a)$. Let us define $f(\widetilde{x})$ to be the point $\widetilde{p}$. This is an unambiguous definition for if $\widetilde{x} \in \widetilde{M}_i^a \cap \widetilde{M}_j^\beta$ then $x \in M_i \cap M_j$ and $\hat{p}_i^a \approx \hat{p}_j^\beta$. Since $f|\widetilde{M}_i^a$ is obviously a homeomorphism of $\widetilde{M}_i^a$ onto a subset of $\widetilde{A}_i^a$, and $\boldsymbol{M}$ is locally finite, it is easy to show that f is a homeomorphism. Then, finally we note that

$$r\big(f(\widetilde{x})\big) = r(\widetilde{p}) = r\big(q_i^a(\hat{p}_i^a)\big) = \hat{r}_i^a(\hat{p}_i^a) = x = e\widetilde{x}. \quad \blacksquare$$

6. The fundamental tropes. A mapping $f\colon U \to V$ of a pathwise connected space U into a pathwise connected space V induces, for each point x of U, a homomorphy $f_*\colon \pi(U, x) \to \pi\big(V, f(x)\big)$, but if $y \neq f(x)$ no single homomorphy $f_*\colon \pi(U, x) \to \pi(V, y)$ can be attached in a natural way. What we can do is to choose a path p in V from y to $f(x)$ and associate to each x-based loop a in U the y-based loop $p \cdot f(a) \cdot p^{-1}$; the resulting homomorphy $f_*\colon \pi(U, x) \to \pi(V, y)$ depends on the homotopy class of the path p, but any two such homomorphies differ only by an inner automorphism of $\pi(V, y)$.

With this observation as motivation let us define two homomorphies $\varphi, \psi\colon \Gamma \to \Delta$ of a group Γ into a group Δ to be *homotopic* (written $\varphi \simeq \psi$) when there is an inner automorphism θ of Δ for which $\psi = \theta\varphi$. The relation $\simeq$ of homotopy is symmetric, reflexive, transitive and compositive, and consequently in the category $\mathfrak{J}$ of groups and homomorphies it may be regarded as a legitimate interpretation of the similarity relation $\sim$ of § 2.

An object of the category $\overline{\mathfrak{J}}$, i.e. an inverse system of groups in $(\mathfrak{J}, \simeq)$, will be called a *trope*. Tropes and mutations of tropes generalize the concepts groups and homomorphies of groups, because that is what they reduce to when they are rudimentary. In [19] some of the most basic

concepts of group theory are generalized to trope theory. Homotopies of mutations and homotopy types of tropes are defined according to the scheme elaborated in § 3. A homotopy type of tropes may be called a *group shape*.

We are going to use the concept of representation of a trope $\boldsymbol{\Gamma}$ in a group Σ; this is nothing else than a mutation from $\boldsymbol{\Gamma}$ to the rudimentary trope Σ. Two representations of $\boldsymbol{\Gamma}$ in Σ are *homotopic* when they are homotopic as mutations from $\boldsymbol{\Gamma}$ to the rudimentary trope Σ. A representation of $\boldsymbol{\Gamma}$ in Σ_d, the symmetric group of degree d, is *transitive* when each of its constituent representations $\Gamma \to \Sigma_d$ is transitive. Any representation that is homotopic to a transitive representation is itself transitive. If $\boldsymbol{\Gamma}$ and $\boldsymbol{\Delta}$ belong to the same group shape, so that there exist mutations $\boldsymbol{\varphi}\colon \boldsymbol{\Gamma} \to \boldsymbol{\Delta}$ and $\boldsymbol{\psi}\colon \boldsymbol{\Delta} \to \boldsymbol{\Gamma}$ for which $\boldsymbol{\psi}\boldsymbol{\varphi} \simeq \boldsymbol{\gamma}$ and $\boldsymbol{\varphi}\boldsymbol{\psi} \simeq \boldsymbol{\delta}$, then $\boldsymbol{\psi}$ transforms each homotopy class of representations of $\boldsymbol{\Gamma}$ in Σ into a homotopy class of representations of $\boldsymbol{\Delta}$ in Σ, and $\boldsymbol{\varphi}$ transforms it back again. Thus the homotopy classes of representations of $\boldsymbol{\Gamma}$ in Σ are in biunique correspondence with the homotopy classes of representations of $\boldsymbol{\Delta}$ in Σ.

In order to avoid the inconvenience of dealing with spaces that have more than one component, the category $\mathfrak{K}$ of § 3 will now be replaced by the category $\mathfrak{K}^c$ of *connected* absolute neighborhood retracts (and the continuous mappings that subsist between them). This is purely a technical convenience; in the preceding paragraphs $\mathfrak{K}$ could have been replaced by its subcategory $\mathfrak{K}^c$ without any significant alteration in the theory. For example, in the category $\mathfrak{K}^c$ the complete neighborhood system of a closed connected subset X of an absolute neighborhood retract P is the collection $\boldsymbol{U}^c(X, P)$ of all the connected (open) neighborhoods U of X in P (together with the indigenous inclusions). If we choose in each U a base point o, and corresponding to each inclusion $U_1 \subset U_2$ a path l_{12} in U_2 from o_2 to o_1, the fundamental groups $\pi(U, o)$ together with the homomorphies $\pi(U_1, o_1) \to \pi(U_2, o_2)$ induced by the paths l form a trope $\overline{\pi}\big(\boldsymbol{U}^c(X, P), \boldsymbol{o}, \boldsymbol{l}\big)$. (Of course it would be possible to choose all the points o to be the same point x of X, but in the long run there is no special advantage in doing this.) The homotopy type to which this trope belongs does not depend on the choice of $\boldsymbol{o}$ or $\boldsymbol{l}$, nor, according to (3.2), on the choice of the absolute neighborhood retract P or the manner in which X is imbedded in P as a closed set; in fact it depends only on the topological shape of X. I shall call it the *fundamental group shape of* X and denote it by $\Pi(X)$. The various tropes $\overline{\pi}\big(\boldsymbol{U}^c(X, P), \boldsymbol{o}, \boldsymbol{l}\big)$ may be called the *fundamental tropes* of X and denoted by $\overline{\pi}(X)$. When X is a connected absolute neighborhood retract we may choose $P = X$ as in § 3, and observe that the corresponding fundamental tropes $\overline{\pi}(X)$, being rudimentary, are then just the fundamental groups of X.

It will be observed that in the theory of the fundamental tropes the fundamental groups of connected spaces that are not absolute neighborhood retracts play no role. This is not an unreasonable development, for it is well known that the fundamental groups of a space that is not adequately connected locally tend to misrepresent the gross structure of that space. (For spaces that are locally connected in dimension 0 the utility of its fundamental groups can be recaptured by topologizing them, but for spaces that are not locally connected in dimension 0 the situation is hopeless.)

(6.1) THE FUNDAMENTAL THEOREM OF OVERLAY THEORY. (9) *The d-fold overlays of a connected separable metrizable space X are in biunique correspondence with the homotopy classes of representations of an arbitrary one of its fundamental tropes $\overline{\pi}(X)$ in the symmetric group Σ_d of degree d.*

Proof. Let X be imbedded as a closed set in an absolute neighborhood retract P. For $\overline{\pi}(X)$ we choose $\overline{\pi}(\boldsymbol{U}^c(X,P))$, using a point x of X as base point for all the groups $\pi(U)$. Consider a representation $\boldsymbol{\omega}$ in Σ_d of $\overline{\pi}(X)$. According to the fundamental theorem of covering space theory each constituent representation $\omega\colon \pi(U)\to\Sigma_d$ determines a covering space $r_\omega\colon \widetilde{U}_\omega\to U$ and thereby an overlay $e_\omega\colon \widetilde{X}_\omega\to X$, where $\widetilde{X}_\omega = r_\omega^{-1}(X)$ and $e_\omega = r_\omega|\widetilde{X}_\omega$. If u_* denotes the homomorphy of $\pi(U_i)$ into $\pi(U_j)$ induced by the inclusion $u\colon U_i\subset U_j$, then to each constituent $\omega_j\colon \pi(U_j)\to\Sigma_d$ is associated the constituent $\omega_i = \omega_j u_*\colon \pi(U_i)\to\Sigma_d$, and $\widetilde{U}_{\omega i}$ may be identified with $r_{\omega j}^{-1}(U_i)$, and $r_{\omega i}$ with $r_{\omega j}|\widetilde{U}_{\omega i}$. If $\omega_1\colon \pi(U_1)\to\Sigma_d$ and $\omega_2\colon \pi(U_2)\to\Sigma_d$ are any two constituents of $\boldsymbol{\omega}$ then, according to (2.7), in $U_1\cap U_2$ there must be a connected neighborhood U_0 of X such that, denoting by u^1 and u^2 the respective inclusions $U_0\subset U_1$ and $U_0\subset U_2$, the representations $\omega_1u^1_*$ and $\omega_2u^2_*$ are homotopic. It follows that there is a homeomorphism f of $r_{\omega 1}^{-1}(U_0)$ onto $r_{\omega 2}^{-1}(U_0)$ such that $r_{\omega 2}f|r_{\omega 1}^{-1}(U_0) = r_{\omega 1}|r_{\omega 1}^{-1}(U_0)$, and consequently $f(\widetilde{X}_{\omega 1}) = \widetilde{X}_{\omega 2}$ and $r_{\omega 2}f|\widetilde{X}_{\omega 1} = r_{\omega 1}|\widetilde{X}_{\omega 1}$. Thus the overlays $e_\omega\colon \widetilde{X}_\omega\to X$ are all equivalent to one another. This shows that the representation $\boldsymbol{\omega}$ determines an overlay $e = O(\boldsymbol{\omega})$ of X. From (2.6) it follows easily that $O(\boldsymbol{\omega}) = O(\boldsymbol{\omega}')$ whenever $\boldsymbol{\omega}\simeq\boldsymbol{\omega}'$.

If $e\colon \widetilde{X}\to X$ is any d-fold overlay of X then, according to the extension theorem (5.2), e can be extended to an overlay $r\colon \widetilde{U}\to U$ of some connected neighborhood U of X in P, and this determines a representation ω of $\pi(U)$, and thereby a representation $\boldsymbol{\omega}$ of $\overline{\pi}(X)$ in Σ_d. The constructed representation $\boldsymbol{\omega}$ is obviously such that $O(\boldsymbol{\omega}) = e$.

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(9) This theorem has an alternative form that is stated in $(\mathfrak{J},=)$. Let x be a basepoint of X and $\tilde{x}$ a basepoint of $\tilde{X}$. The statement reads: *The d-fold overlays $(\tilde{X},\tilde{x})$ of a pointed connected separable metrizable space (X,x) are in biunique correspondence with the $(\mathfrak{J},=)$-equivalence classes of representations in Σ_d of the $(\mathfrak{J},=)$ inverse system* $\overline{\pi}(U^c(X,P,x))$.

In order to complete the proof that $\boldsymbol{\omega}\to O(\boldsymbol{\omega})$ establishes a biunique correspondence between the homotopy classes of representations of $\overline{\pi}(X)$ in Σ_d and the d-fold overlays of X it remains only to show that if $O(\boldsymbol{\omega}) = O(\boldsymbol{\omega}')$ then $\boldsymbol{\omega}\simeq\boldsymbol{\omega}'$. Let us consider, for purpose, a constituent $\omega\colon \pi(U)\to\Sigma_d$ of $\boldsymbol{\omega}$ and a concurrent constituent $\omega'\colon \pi(U)\to\Sigma_d$ of $\boldsymbol{\omega}'$. Let $r\colon \widetilde{U}\to U$ and $r'\colon \widetilde{U}'\to U$ be the overlays determined respectively by ω and ω'. Since $O(\boldsymbol{\omega}) = O(\boldsymbol{\omega}')$ we may identify both $r^{-1}(X)$ and $r'^{-1}(X)$ with $\widetilde{X}$. With this identification, what we have to show is that there is a connected neighborhood U_0 of X in U such that $\omega u_* = \omega' u_*$, where u denotes the inclusion $U_0\subset U$.

Let $\boldsymbol{B}$ be an open cover of U that is evenly overlaid by an open cover $\widetilde{\boldsymbol{B}}$ of $\widetilde{U}$, and let $\boldsymbol{B}'$ be an open cover of U that is evenly overlaid by an open cover $\widetilde{\boldsymbol{B}}'$ of $\widetilde{U}'$. Let $\boldsymbol{M}$ be a countable locally finite open cover of X that is evenly overlaid by an open cover $\widetilde{\boldsymbol{M}}$ of $\widetilde{X}$. Replacing $\boldsymbol{M}$ by a suitable refinement if necessary, we may assume that $\boldsymbol{M}$ is a common refinement of $\boldsymbol{B}$ and $\boldsymbol{B}'$. By (5.5) $\boldsymbol{M}$ may be extended to a cover $\boldsymbol{A}$ of U by open sets of U which is such that its nerve is naturally isomorphic to the nerve of $\boldsymbol{M}$. After trimming the various sets A_i as the occasion demands, we may assume that $\boldsymbol{A}$ is also a common refinement of $\boldsymbol{B}$ and $\boldsymbol{B}'$, and that each component of each A_i intersects X. Thus $\bigcup A_i$ is a connected neighborhood of X, and this will be our choice for U_0.

A loop g in U_0 is represented by ωu_* as a permutation of the form, say, $\left(\begin{smallmatrix}\ldots & \alpha & \ldots\\ \ldots & \beta & \ldots\end{smallmatrix}\right)$ if and only if it is lifted by r to a path in $\widetilde{U}$ from $\tilde{x}^\alpha$ to $\tilde{x}^\beta$, and it is represented by $\omega' u_*$ as a permutation of the form $\left(\begin{smallmatrix}\ldots & \alpha & \ldots\\ \ldots & \beta & \ldots\end{smallmatrix}\right)$ if and only if it is lifted by r' to a path in $\widetilde{U}'$ from $\tilde{x}^\alpha$ to $\tilde{x}^{\beta'}$. Now there can be found a finite sequence $A_{i_1},\ldots,A_{i_n}$ of sets belonging to the open cover $\boldsymbol{A}$ that form a chain about g, i.e. g is the product of n paths which are to be found respectively in $A_{i_1},\ldots,A_{i_n}$. Since the nerve of $\boldsymbol{A}$ is naturally isomorphic to the nerve of $\boldsymbol{M}$, the sets $M_{i_1},\ldots,M_{i_n}$ also form a chain, i.e. $M_{i_j}\cap M_{i_{j+1}}\neq 0$ for $j = 1,\ldots,n-1$, and consequently lift uniquely to form a chain of sets of $\widetilde{\boldsymbol{M}}$ from $\tilde{x}^\alpha$ to, say, $\tilde{x}^\gamma$. Since this determines the chains from $\tilde{x}^\alpha$ to which the chain $A_{i_1},\ldots,A_{i_n}$ is lifted by r and r', it follows that β and β' must both be equal to γ. Thus the loop g must be represented by ωu_* and $\omega' u_*$ as the same permutation. This shows that $\omega u_* = \omega' u_*$. ∎

EXAMPLE 1. Let X be the figure made up of the part of the graph of the function $s = \cos(\pi/t)$ that lies between $t = -1$ and $t = +1$ together with the graph of the function $s = -1-\sqrt{1-t^2}$ (cf. diagram 3). In the plane P there is a neighborhood sequence $U_1, U_2, \ldots$ for X such that each U_n is a region homeomorphic to an annulus. Thus $\pi(U_n)$ is an infinite cyclic group $Z^{(n)} = |z_n\colon|$, and the injection $\xi\colon Z^{(n+1)}\to Z^{(n)}$ is defined by $\xi(z_{n+1}) = z_n$. For each $d\leqslant\infty$ there is a unique homotopy class of transitive representations in Σ_d, and to each of these there corresponds the d-fold connected overlay of X that is the restriction to X of the d-fold cyclic covering of an arbitrary neighborhood U_n. The disconnected overlays correspond to the intransitive representation classes. Thus the theory of the overlays of this space is just the same as the theory of the covering spaces of a simple closed curve.

EXAMPLE 2. Let $U_1, U_2, \ldots$ be a sequence of (open) solid tori, each one contained in the preceding one. Let us suppose that the winding number of U_{n+1} in U_n is a prime number p_n. The continuum $X = \bigcap U_n$ is a solenoid (the dyadic solenoid if all $p_n = 2$). A neighborhood system of X in the absolute neighborhood retract $P = U_1$ is the sequence $U_1, U_2, \ldots$, and every overlay of X is the restriction of some overlay of some U_n. A fundamental trope $\bar{\pi}(X)$ is the well-known sequence

$$Z^{(1)} \xleftarrow{p_1} Z^{(2)} \xleftarrow{p_2} Z^{(3)} \xleftarrow{p_3} \ldots, \tag{6.2}$$

where p_n denotes the homomorphy $Z_n \xleftarrow{p_n} Z_{n+1}$. For each $d < \infty$ that is not divisable by any of the primes that occur infinitely often in the sequence $p_1, p_2, \ldots$ there is just one homotopy class of transitive representations of this system in Σ_d, and correspondingly there is just one connected d-fold overlay of X, and these are the only connected overlays. (This fact can also be seen geometrically.) As in example 1, the disconnected overlays correspond to the intransitive representation classes. There follows immediately the known fact that the number of topological types of solenoids is uncountable. (Cf. [8].) It should be observed that the inverse limit of the sequence (6.2) is trivial (either in the sense of $(\mathfrak{J}, \simeq)$ or in the classical sense of $(\mathfrak{J}, =)$).

According to the fundamental theorem of covering space theory connected covering spaces correspond to transitive representations, and conversely. However contrary to expectation, although connected overlays do determine transitive representations, transitive representations do not always determine connected overlays.

EXAMPLE 3. Let X be the 1-dimensional planar continuum that consists of two concentric circles and a doubly infinite spiral between them, (cf. diagram 5) and let $\widetilde{X}$ consist of the lines $s = -1$ and $s = +1$ together with the graphs of the functions $s = \frac{2}{\pi}\,\mathrm{Arctan}(t-n)$, $n = \ldots, -1, 0, 1, \ldots$ Then X is contained in the annulus P that is bounded by the two concentric circles, and the strip $-1 \leqslant s \leqslant 1$ is the universal covering space $\widetilde{P}$ of P. The part of $\widetilde{P}$ that lies over X is $\widetilde{X}$, which is thereby seen to be an overlay of X. Although X is connected, $\widetilde{X}$ is not connected, on the other hand it may be verified that the corresponding

representation in Σ_∞ of a fundamental trope $\bar{\pi}(X)$ is transitive.[10] (X has a neighborhood sequence $U_1, U_2, \ldots$ each member of which is a triply connected region, the corresponding trope is the sequence

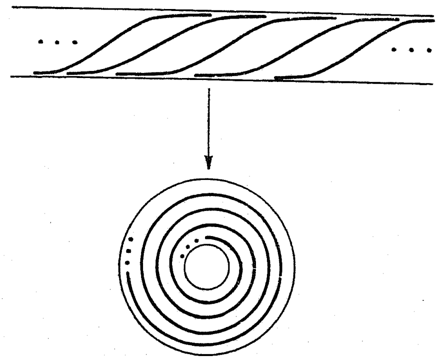

Diagram 5

$|x, y{:}\ | \leftarrow |x, y{:}\ | \leftarrow \ldots$ in which each of the homomorphies is an isomorphism. The overlay $\widetilde{X} \to X$ corresponds to the representation $x, y \to (\ldots -1, 0, 1, \ldots)$.)

A moment's thought will show that *an overlay $e{:}\ \widetilde{X} \to X$ corresponds to a transitive representation if and only if it is not possible to separate $\widetilde{X}$ into disjoint open subsets $\widetilde{X}_1$ and $\widetilde{X}_2$ such that $e_1 = e|\widetilde{X}_1$ and $e_2 = e|\widetilde{X}_2$ are each overlays of X.*

EXAMPLE 4. Let X be the set of those points (s, t) of the plane for which either s or t or both is rational. Since the planar complement of any point both of whose coordinates are irrational is a connected open neighborhood of X in the plane P, the number of connected overlays of X is uncountable. The fact that in this example X is not a closed subset of the absolute neighborhood retract P does not invalidate this conclusion.[6]

In the complete theory of overlays (which will be published in another place) the theorems of covering space theory appear in their ultimate form. For example the important *lifting theorem* which gives the condition under which a mapping $f{:}\ (X, x) \to (Y, y)$ can be lifted to a mapping $\tilde{f}{:}\ (X, x) \to (\widetilde{Y}, \tilde{y})$ requires no hypothesis on the separable metrizable spaces X, Y other than that they be connected.[11]

[10] This example has other strange properties. For instance, what are its covering translations?

[11] Example 6.6.14, pp. 258–259 of [11] would seem to contradicts this. However what this example really shows is the inadequacy for spaces that are not locally connected of the fundamental group. For, although the condition $f_*(\pi(X, x_0)) = g_*(\pi(Y, y_0))$ is satisfied, the condition $f_*(\bar{\pi}(X, x_0)) = g_*(\bar{\pi}(Y, y_0))$ is not.

The results of §§ 5 and 6 have been generalized to the theory of fibre bundles by M. G. Scharlemann [17], unfortunately using an earlier version of § 6 in which the definitions of § 2 were not quite the same. The extension theorem (5.2) generalizes nicely, provided the group of the bundle is an absolute neighborhood retract, and a counterexample shows that this condition is necessary. There are no further surprises, and Scharlemann shows how to use the generalized extension theorem to classify the bundles over a given (possibly pathological) base space, using an appropriate universal bundle that in many cases is much simpler than Milnor's.

7. Local theory. Let J and J' be subsets of respective absolute neighborhood retracts P and P', and let X and X' be homeomorphic subsets of J and J' that are closed in P and P' respectively. Following Lomonaco ([14], p. 323) let us say that J at X is of the same *local type* as J' at X' if there are neighborhoods U of X and U' of X' such that $(U, U\cap J)$ is homeomorphic to $(U', U'\cap J')$. When $X = J$ and $X' = J'$ (which cannot happen unless J and J' are homeomorphic) this is Noguchi's isoneighboring property [16]; the case where X and X' are points was the one considered by Lomonaco (see below).

Now consider the neighborhood system $\boldsymbol{U}(X, P)$, and delete J from it, obtaining the inverse system $\boldsymbol{U}(X, P-J)$ made up of the sets $U-J$, $U\in\boldsymbol{U}(X, P)$. Clearly J at X cannot be of the same local type as J' at X' unless the systems $\boldsymbol{U}(X, P-J)$ and $\boldsymbol{U}(X', P'-J')$ are of the same homotopy type. Among the invariants of the local type of J at X the invariants of the homotopy type of $\boldsymbol{U}(X, P-J)$ will be the first to come into consideration.

The fundamental group concept can be used effectively when X has arbitrarily small connected neighborhood U that are not disconnected by J. Thus X should be connected, and J should not separate P locally. We can then consider $\boldsymbol{U}^c(X, P-J)$, the inverse system made up of the connected sets $U-J$, $U\in\boldsymbol{U}(X, P-J)$ together with the indigenous inclusions. In this system it is, of course, impossible to choose a common base point, and this is a compelling reason for the use of the concept of homotopy of homomorphies. Let us choose in each $U-J$ a base point o, and corresponding to each inclusion $U_1-J\subset U_2-J$ a path l_{12} in U_2-J from o_2 to o_1. Then the fundamental groups $\pi(U-J, o)$, together with the homomorphies induced by the paths l form a trope $\bar\pi(\boldsymbol{U}^c(X, P-J, \boldsymbol{o}, \boldsymbol{l}))$. The homotopy type of this trope does not depend on the choice of $\boldsymbol{o}$ and $\boldsymbol{l}$, although, of course, it does depend on P, J and X. It is the *fundamental group shape* $\Pi(X, P-j)$ *at X of the imbedding* $J\subset P$. It is obviously an invariant of the local type of J at X. The various tropes $\bar\pi(\boldsymbol{U}^c(X, P-J, \boldsymbol{o}, \boldsymbol{l}))$

may be called the *fundamental tropes at X of the imbedding $J\subset P$, and denoted by* $\bar\pi(X, P-J)$.

It should be observed that each of the tropes $\bar\pi(\boldsymbol{U}^c(X, P-j, \boldsymbol{o}, \boldsymbol{l}))$ is not only an inverse system in $(\mathfrak{I}, \simeq)$ but also in $(\mathfrak{I}, =)$. In $(\mathfrak{I}, \simeq)$ the inverse limit, when it exists, is a group $\lim\Pi(X, P-J)$ that does not depend on the choice of $\boldsymbol{o}$ and $\boldsymbol{l}$, and in $(\mathfrak{I}, =)$ the inverse limit (which always exists) is a group $\lim\bar\pi(\boldsymbol{U}^c(X, P-J, \boldsymbol{o}, \boldsymbol{l}))$ that does depend on this choice. For example if P is R^3, X is a point, and J is a simple arc that is locally tame at every point of $J-X$, and if J is contained in a double cone C whose vertex is X (such as is shown for example in [14], p. 340, figure 4) then $\lim\bar\pi\boldsymbol{U}^c(X, P-J, \boldsymbol{o}, \boldsymbol{l})$ is the infinite cyclic group Z if, crudely speaking, all of the paths l that are close to X can be pulled free of P and into the exterior of C without moving their endpoints (which are supposed to lie outside C), and it is the trivial group 1 otherwise; on the other hand $\lim\Pi(X, P-J)$ is just Z.

Example 5. Let J be the "remarkable arc" (cf. [5]) of diagram 6.

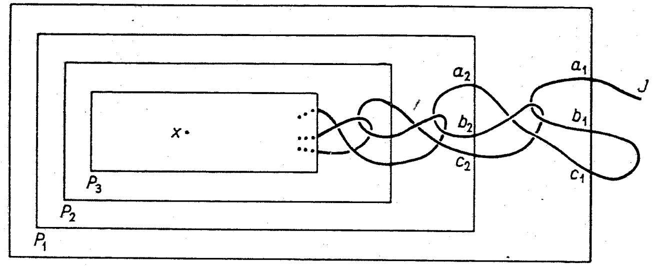

Diagram 6

The group

$$G_N = \pi(U_N-J) = \{a_n, b_n, c_n\colon\ c_n b_n^{-1} a_n = 1,\ b_n = b_{n+1}^{-1} a_n b_{n+1},$$
$$c_{n+1} = c_n b_n^{-1} b_{n+1} b_n c_n^{-1},\ a_{n+1} = b_{n+1} c_n b_{n+1},\ n \geqslant N\}$$
$$= \{b_n\colon\ b_{n+1} b_n b_{n+1}^{-2} b_{n+2} b_{n+1} = b_n b_{n+1}^{-1} b_{n+2},\ n \geqslant N\}$$

has the representation

$$b_n \to \begin{cases} (12345) & \text{for} \quad n \equiv 0 \pmod 3, \\ (14352) & \text{for} \quad n \equiv 1 \pmod 3, \\ (13254) & \text{for} \quad n \equiv 2 \pmod 3, \end{cases}$$

from which it follows that $\bar\pi(X, R^3-J)$ is not homotopic to the rudimentary trope 1. This shows that J at x is not of the same local type as a line segment at one of its endpoints. Thus J is a wild arc.

EXAMPLE 6. The proof on p. 984 of [7] shows that the arc X of example 1.1 of [7] at either of its endpoints is not of the same local type as a line segment at one of its endpoints. Using the representation of p. 983 of [7] the same statement can be made about the arc X^* of example 1.1*. Whether or not X at p is of the same local type as X^* at p is a question that does not appear to be easy to answer.

EXAMPLE 7. The proof on p. 988 of [7] shows that the arc H^4 of example 1.4 is not of the same local type as a line segment at one of its interior points.

EXAMPLE 8. The proof in [3] shows that the n-frame D of [3] at its vertex is not of the local type of a standard n-frame at its vertex.

In [14] Lomonaco calls two representations $\varrho_1\colon \Gamma_1 \to \Sigma$ and $\varrho_2\colon \Gamma_2 \to \Sigma$ *locally equivalent* if there are finitely implicated normal subgroups Δ_1 of Γ_1 and Δ_2 of Γ_2 such that $\varrho_1(\Delta_1) = 1$ and $\varrho_2(\Delta_2) = 1$, and Γ_1/Δ_1 is isomorphic to Γ_2/Δ_2. He then shows that if J is any arc imbedded in the interior of a 3-dimensional manifold P that is locally tame at every point of J except possibly at a point x, then for any acceptable system $\boldsymbol{U}$ of neighborhoods of x (where "acceptable" is an invariantly defined concept) the representations $\pi(U) \to Z$ are all of the same local equivalence class, and moreover if J and J' are two such arcs then J at x cannot be of the same local type as J' at x' unless the representations $\pi(U) \to Z$ are locally equivalent to the representations $\pi(U') \to Z$. Lomonaco then constructs algebraic invariants of local equivalence that are based on the (infinite) Alexander matrices of the groups $\pi(U)$.

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