separates x from $f(q)$ in D, and hence Q_{i+1} separates p from q in X and also in U. Therefore $Q_{i+1} \cap U \neq \emptyset$ for $i > I$, so $p \in Q$. Thus $Q = \operatorname{Fr} C$ and is connected.

Proof of 4.3. For any $x \in D$, we wish to show that $f^{-1}(x)$ is connected. Since x is either an end point or a cut point of D, we have two cases.

Case I. x is an end point. There exist $\{U_i\}_{i=1}^{\infty}$, a sequence of neighborhoods of x where $U_{i+1} \subset U_i$, $\bigcap_{i=1}^{\infty} U_i = \{x\}$, diameter $U_i < 1/i$, and each U_i has one boundary point x_i. Thus $\{x_i\}_{i=1}^{\infty} \to x$. Let C_i be the component of $X - f^{-1}(x_i)$ such that $f^{-1}(x) \subset C_i$. Then $f(C_i) \subset U_i$ and $\operatorname{Cl}(C_i)$ is connected. It is easy to see that $f^{-1}(x) = \bigcap_{i=1}^{\infty} \operatorname{Cl}(C_i)$ and thus is connected.

Case II. x is a cut point. Thus $f^{-1}(x)$ separates X. By Lemmas 3.1 and 4.5 we have that $X - f^{-1}(x)$ has at most a countable number of components, $\{C_i\}_{i=1}^{\infty}$ (all but a finite number of the C_i may be empty). Let $X_1 = X - C_1$. By induction define $X_i = X_{i-1} - C_i$ for $i = 2, 3, \ldots$ By Lemmas 4.6 and 4.7, X_i is connected for each i, thus $f^{-1}(x)$ is connected since $f^{-1}(x) = \bigcap_{i=1}^{\infty} X_i$.

Our final result is an application of Theorem 4.3.

THEOREM 4.8. *There is no non-alternating mapping from a simply connected Peano continuum X onto the circle S.*

Proof. Suppose there were such a mapping f. Let $x, y \in S$, by Lemma 3.1 both $X - f^{-1}(x)$ and $X - f^{-1}(y)$ are connected. But $X - \big(f^{-1}(x) \cup \cup f^{-1}(y)\big)$ must be separated. Thus there exists a component C of $f^{-1}(x) \cup \cup f^{-1}(y)$ which separates X. Since $f(C) = x$ or y, we have a contradiction.

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Construction of group topologies on abelian groups

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Introduction. In this article we discuss a few methods for constructing grouptopologies on abelian groups and the relations between these methods.

In section 1 and 2 the method of Hinrichs (intended for the ring of integers, cf. [4]) is investigated together with its relation to the construction occurring in [6]. In section 3 it is explained how the topology of a given abelian group can be refined by making a character continuous.

Next the problem is studied of finding topologies T on an abelian group G such that G becomes a complete topological group with respect to T. Generalizations of the results of section 8 of [6] and section 8 of [2] are obtained in section 4 and 5 respectively. The methods used resemble those introduced in section 1 and 2.

Finally, in section 6 it is observed that this paper basically deals with refinements. The problem is posed of reaching the aims of section 4 and 5 of this article and the result of [8], which is obtained by coarsifying, at the same time.

Notations and terminology. All groups in this article will be commutative and additively written. Let G be a group and U and V subsets of it. $U+V$ is defined by $U+V = \{a+b\colon\ a \in U,\, b \in V\}$; $1U \underset{D}{=} U$ and $nU \underset{D}{=} (n-1)U + U$, for $n > 1$. Instead of $n\{x\}$ we will write nx and instead of $n\{-x, 0, x\}$ we will write $n \cdot x$.

We will denote a topological group frequently by (G, T) in which G is a group and T a topology defined on it such that the operation $(x, y) \to x - y$ is continuous in both variables together. G_d stands for (G, D), in which D is the discrete topology on G. G alone stands for the group G without a topology. We may discuss topologies defined on it. Sometimes we will also use the notation G for a topological group, if there is no danger for confusion about the topology that is meant.

$\boldsymbol{Z}$ will denote the group of integers, $\boldsymbol{R}$ the group of reals and by $\boldsymbol{N}$ we will mean the positive integers including 0.

By a groupnorm on a group G we will mean a function from G to $\boldsymbol{R}$, $g \to \|g\|$, that satisfies the following properties:

(i) for all g, $\|g\| \geqslant 0$, $\|0\| = 0$;

(ii) for all g, $\|-g\| = \|g\|$;

(iii) for all g and h, $\|g+h\| \leqslant \|g\| + \|h\|$.

Note that we do not require the groupnorm to be positive definite. Sometimes we also speak about norm when we mean groupnorm. Let $\|.\|$ be a norm on a group G, then this norm defines a grouptopology T on G, by letting $U_\varepsilon = \{g \in G\colon \|g\| < \varepsilon\}$ and choosing $\{U_\varepsilon\colon \varepsilon > 0\}$ as neighborhood-basis at 0 in G. This topology is called $T(\|\,.\,\|)$, or, if no confusion can arise, the normtopology.

We observe that the collection of groupnorms on a given group G forms a partially ordered set with respect to the relation defined by $\|\cdot\|_1 \geqslant \|\cdot\|_2$ if and only if $\|g\|_1 \geqslant \|g\|_2$ for all $g \in G$.

If f is a function defined on a subset S of G we can speak about the set of all norms $\|\,.\,\|$ on G that satisfy

$$\|s\| \leqslant f(s) \quad \text{for all } s \in S .$$

If S generates G, then this set of norms has a maximal element which we call the maximal relationally defined norm relative to the set of relations

$$s \in S \quad \text{implies} \quad \|s\| \leqslant f(s) .$$

We usually abbreviate this expression and call it the MRD-norm. The norm topology with respect to an MRD-norm is called an MRD-topology. Usually, for arbitrary S and f, the MRD-norm will not satisfy $\|s\| = f(s)$ for all s. However, if it does, we speak about the maximal equationally defined norm relative to the set of equations

$$s \in S \quad \text{implies} \quad \|s\| = f(s) .$$

The name maximal equationally defined is abbreviated as MED. These definitions occur also in a slightly more restricted setting, in [6], p. 302.

§ 1. The Hinrichs method for groups.

In this section we will denote a sequence $\{k_i\colon i \in N\}$ by $\boldsymbol{k}$.

1.1. Definition. A countable sequence of neighborhoods of 0, $\boldsymbol{U} = \{U_i\colon i \in N\}$, of a topological group is said to *converge at rate* $\boldsymbol{k}$ if and only if $k_i U_{i+1} \subset U_i$ for all $i \in N$. $\boldsymbol{k}$ is called the *rate of convergence of* $\boldsymbol{U}$.

In any first countable topological group and for every $\boldsymbol{k}$ there is a neighborhood basis at 0, consisting of symmetric neighborhoods converging at rate $\boldsymbol{k}$.

1.2. Definition. Let G be an abelian group and let $\boldsymbol{n} = \{n_i\colon i \in N\}$ be such that $n_0 = 1$, $n_i | n_{i+1}$, $n_i < n_{i+1}$ for all i and let $\boldsymbol{F} = \{F_i\colon i \in N\}$ be a sequence of symmetric subsets of G, containing 0. Put $\varphi = \{\boldsymbol{n}, \boldsymbol{F}\}$. Let $\boldsymbol{k} = \{n_{i+1}/n_i\colon i \in N\}$. Then $T(\varphi)$ is the finest topology on G having a basis $\boldsymbol{U} = \{U_i\colon i \in N\}$ of symmetric neighborhoods of 0, such that $\boldsymbol{U}$ converges at rate $\boldsymbol{k}$ and $F_i \subset U_i$ for all i, in which $\boldsymbol{k} = \{n_{i+1}/n_i\colon i \in N\}$. Occasionally we will call $T(\varphi)$ also the φ-topology. This definition makes sense as the following lemma shows.

1.3. Lemma. *Let G, $\boldsymbol{n}$, $\boldsymbol{F}$, and φ be as in 1.2. Then a neighborhood basis $\boldsymbol{U}$ for $T(\varphi)$ is given by choosing*

$$U_m = \bigcup_{l \geqslant m} F_{l,m} \quad \textit{for each } m \in \boldsymbol{N},$$

in which

$$F_{m,m} = F_m \quad \textit{and} \quad F_{l,m} = F_m \cup k_m F_{l,m+1} \quad \textit{for all } m \in \boldsymbol{N},\ l > m .$$

Remark. A similar construction was used for ringtopologies on $\boldsymbol{Z}$ by L. A. Hinrichs [4].

Proof. We have to show that $k_m U_{m+1} \subset U_m$ for all m, as it is clear that $U_m \supset F_m$, for all $m \in N$.

First we show $F_{l+1,m} \supset F_{l,m}$, by induction on m. For $m = l$ it is clear. Suppose l is given and suppose the statement is proved for $s < m \leqslant l$. Then $F_{l+1,s} = F_s \cup k_s F_{l+1,s+1} \supset F_s \cup k_s F_{l,s+1} = F_{l,s}$, so the statement is proved for $m = s$.

Consequently, if $x_i \in U_{m+1}$ for $1 \leqslant i \leqslant k_m$, then there exists an l, such that $x_i \in F_{l,m+1}$ for $1 \leqslant i \leqslant k_m$, hence $\sum_{i=1}^{k_m} x_i \in k_m F_{l,m+1} \subset F_{l,m}$, which is contained in U_m, so $k_m U_{m+1} \subset U_m$ for all $m \in N$.

Remark. Some computations in the sequel may be visualized by arranging the elements of U_0 in a staircase diagram, putting the F_i in the squares $E_{i,i}$.

The $E_{l,m}$ are constructed row by row, each time going from right to left, as follows. The union of all squares in the staircase down and to the right of $E_{n,n}$ must be U_n. Write down in the square $E_{l,m}$, all elements of $U_m \setminus U_{m+1} \subset G$, whose presence in this set follows from

$$k_m U_{m+1} \subset U_m ,$$

$$E_{k,n} \subset U_{m+1} \text{ for } l \geqslant k \geqslant n \geqslant m+1$$

(everything to the right and above $E_{l,m+1}$),

$$E_{l,m} \cap E_{k,m} = \emptyset \text{ if } k \neq 1 ,$$

$F_{l,m}$ is then just the union of what is written down in the squares to the right and above $E_{l,m}$.

$E_{0,0}$			
$E_{1,0}$	$E_{1,1}$		
$E_{2,0}$	$E_{2,1}$	$E_{2,2}$	
$E_{3,0}$	$E_{3,1}$	$E_{3,2}$	$E_{3,3}$

1.4. LEMMA. *For φ as in 1.2, the φ-topology is an MRD-topology on G, relative to the set of relations*

(*) $x \in F_i$ *implies* $\|x\| \leqslant 1/n_i$.

Proof. Outline. Using the expression for the MRD-norm given in [6], § 3, proof of Lemma 6, we prove that for $x \in U_i$ holds $\|x\| \leqslant 1/n_i$ as well. After that we prove that $\|x\| < 1/n_i$ implies $x \in U_i$. This fact is proved by a rather complicated but not difficult induction hypothesis.

The MRD-norm that satisfies the relations (*) is defined by

(1) $\|x\| = \inf\{\sum_{i=0}^{r} a_i/n_i :\ x = \sum_{i=0}^{r}\sum_{j=1}^{a_i} x_{ij}$ with $x_{ij} \in F_i\}$.

Clearly $x \in F_{m,m} = F_m$ implies $\|x\| \leqslant 1/n_m$.

Suppose we have proved $x \in F_{j,l}$ implies $\|x\| \leqslant 1/n_l$ for $0 \leqslant j \leqslant t-1$ and also for $j = t$ and $m+1 \leqslant l \leqslant t$. Now $x \in F_{t,m}$ implies $x \in F_m$ or $x \in k_m F_{t,m+1}$; in the second case $x = \sum_{i=1}^{k_m} x_i$ with $x_i \in F_{t,m+1}$, hence $\|x_i\| \leqslant 1/n_{m+1}$. So $\|x\| \leqslant k_m/n_{m+1} = 1/n_m$. In the first case this is immediately clear.

Consequently the statement $x \in F_{l,m}$ implies $\|x\| \leqslant 1/n_m$, holds for all $l, m, l \geqslant m$ and hence $x \in U_m$ implies $\|x\| \leqslant 1/n_m$.

For the second half of the proof, we observe that, because $U_i \supset F_i$ for all i, the righthand member of (1) is larger than or equal to the same expression with U_i instead of F_i; on the other hand, because $x \in U_i$ implies $\|x\| \leqslant 1/n_i$, as proved already, the righthand member of (1) with U_i instead of F_i is smaller than or equal to $\|x\|$, it does not matter whether we put F_i or U_i in the definition of $\|x\|$.

Suppose $\|x\| < 1/n_m$, then $x = \sum_{i=0}^{r}\sum_{j=1}^{a_i} x_{ij}$ with $x_{ij} \in F_i$ and $\sum_{i=0}^{r} a_i/n_i < 1/n_m$; it follows that $a_i = 0$ for $0 \leqslant i \leqslant m$, so we have $\sum_{i=m+1}^{r} a_i/n_i < 1/n_m$. Now we make the following induction hypothesis: For $t \geqslant m+1$ and $x_{ij} \in U_i$ for $i \geqslant m+1$, such that $x_{ij} = 0$ for $m+1 \leqslant i \leqslant t$ or $j > a_i$, $\sum_{i=m+1}^{t} a_i/n_i \leqslant 1/n_m - 1/n_t$ implies

$$\sum_{i=m+1}^{t}\sum_{j=0}^{a_i} x_{ij} + U_t \subset U_m .$$

Clearly, if the induction hypothesis is true for all t and m, then $\|x\| < 1/n_m$ implies $x \in U_m$. The induction hypothesis is true for $t = m+1$, because $a_m/n_{m+1} \leqslant 1/n_m - 1/n_{m+1}$ implies $a_{m+1} \leqslant n_{m+1}/n_m - 1$, hence $x_{m+1,j} \in U_{m+1}$ for $1 \leqslant j \leqslant a_{m+1}$ implies that

$$\sum_{j=0}^{a_{m+1}} x_{m+1,j} + U_m \subset (n_{m+1}/n_m - 1)U_{m+1} + U_{m+1} = (n_{m+1}/n_m)U_{m+1} \subset U_m .$$

Suppose the induction hypothesis is proved for $m \leqslant t \leqslant s-1$. Then, let $\sum_{i=m+1}^{s} a_i/n_i \leqslant 1/n_m - 1/n_s$. For $k_s = n_s/n_{s-1}$ we put $a_s = fk_s + b_s$, with $b_s < k_s$, and $f \geqslant 0$. Suppose $x_{ij} \in U_i$ for $1 \leqslant j \leqslant a_i$, then

$$x = \sum_{i=m+1}^{s}\sum_{j=1}^{a_i} x_{ij} = \sum_{i=m+1}^{s}\sum_{j=1}^{a_i} x_{ij} + \sum_{n=0}^{f-1}\Big(\sum_{j=nk_s+1}^{(n+1)k_s} x_{sj}\Big) + \sum_{j=1+fk_s}^{a_s} x_{sj} .$$

We write the bracketed expression as y_{pq} with $p = s-1$ and $q = n + a_{s-1} + 1$, which is in U_{s-1} and we write $x_{sq} = y_{sj}$ if $q = fk_s + j$. If we put $x_{ij} = y_{ij}$ for $i < s$, we find

$$x = \Big(\sum_{i=m+1}^{s-2}\sum_{j=1}^{a_i} y_{ij}\Big) + \sum_{j=1}^{a_{s-1}+f} y_{s-1,j} + \sum_{j=1}^{b_s} x_{sj}$$

and

$$1/n_m - 1/n_s \geqslant \sum_{i=m+1}^{s} a_i/n_i = \sum_{i=m+1}^{s-1} a_i/n_i + fk_s/n_s + b_s/n_s ,$$

so

$$1/n_m - (b_s+1)/n_s \geqslant \sum_{i=m+1}^{s-2} a_i/n_i + (a_{s-1}+f)/n_{s-1} .$$

So $(x - \sum_{j=1}^{b_s} y_{sj}) + U_{s-1} \subset U_m$, by the induction hypothesis and because

$$1/n_m - 1/n_{s-1} \geqslant 1/n_m - (b_s+1)/n_s .$$

So for any $y \in U_s$, we may put $y = y_{sq}$ with $q = b_s + 1$ and we find

$$x + y = \Big(x - \sum_{j=1}^{b_s} y_{sj}\Big) + \sum_{j=1}^{b_s+1} y_{sj} \in \Big(x - \sum_{j=1}^{b_s} y_{sj}\Big) + U_{s-1} ,$$

which is contained in U_m. We have proved the induction hypothesis for $t = s$, thereby the lemma.

§ 2. Refining any first countable group topology. Now given a first countable group topology T on G, we may choose a basis of symmetric neighborhoods V for T, with a given rate of convergence $\boldsymbol{k}$. Any sequence $\boldsymbol{F}$ of symmetric subsets of V gives rise to a $\varphi = \{\boldsymbol{n}, \boldsymbol{F}\}$ with $n_i = n_{i-1}k_{i-1}$ for all $i > 0$. For this φ, $T(\varphi)$ is finer than or equal to T. The next few lemma's and definitions will show how the F_i may be chosen to obtain a $T(\varphi)$ that is strictly finer than T.

A consequence of this is, that finest nondiscrete group topologies on G (which exist by Zorn's Lemma and the existence of nondiscrete topologies on any infinite abelian G (cf. [5], also 4.1)), cannot be metrizable.

2.1. Definitions. For t a nonzero positive integer, let G_t denote $\{x\colon tx = 0\}$ and let $G_\infty = G$. For a topological group G, let $t(G)$ be the smallest t (integer or ∞) such that G_t is open.

2.2. Lemma. *Let G be a first countable Hausdorff topological group. Let $\{H_i\colon i \in L\}$ be an at most countable collection of nowhere dense closed subgroups of G. Let A be a subset of G, consisting of isolated points and not containing 0. Then for any n, $0 < n < t(G)$ and any $i \in L$, there exist arbitrarily small $b \in G \backslash H_i$, with $n \cdot b \cap A$ empty.*

Proof. For any $x \in G$, $x \neq 0$, define

$$k(x) = \min\{k\colon k \cdot x \cap A \neq \emptyset\}.$$

Then k is upper semicontinuous in points x where $k(x)$ is finite. This is clear in case $k(x) = 1$ and otherwise $\big(k(x)-1\big) \cdot x \cap A = \emptyset$, implies that for U small enough and $y \in x+U$, $\big(k(x)-1\big) \cdot y \cap A = \emptyset$, so $k(y) \geqslant k(x)$. Furthermore, it follows that $k(x)x \in A$ or $-k(x)x \in A$. When $k(x) \leqslant n$, $n < t(G)$, then there exists for every neighborhood U of 0, an $y \in U \backslash G_n$, with $k(x)(x+y) \notin A$, because A consists of isolated points and G_n, for $n < t(G)$, is nowhere dense and closed. It follows that $k(x) < k(x+y)$. Then it follows that $\{x\colon k(x) > n\}$ is everywhere dense, so has nonempty intersection with $V \backslash H_i$ for every open neighborhood V of 0 and every $i \in L$.

We now can show that on any first countable topological group a finer topology can be defined, even under preservation of the property that all $H_i, i \in L$ are nowhere dense.

2.3. Lemma. *Let (G, T) be a nondiscrete Hausdorff topological group, with topology T, satisfying the first axiom of countability. Let $A = \{g_i\colon i \in N\}$ be a sequence of elements of G converging to zero, $g_i \neq 0$ for all $i \in N$ and suppose $\{H_i\colon i \in L)$ is an at most countable collection of nowhere dense closed subgroups. Let $\boldsymbol{n} = \{n_i\colon i \in N\}$ be such that $n_0 = 1$, $n_i < n_{i+1}$, $n_i | n_{i+1}$. Then there exists in G a sequence $\{b_i\colon i \in N, i > 0\}$ and a symmetric set containing 0, F_0, such that for $F_i = \{-b_i, 0, b_i\}$ for $i > 0$ and $\varphi = \{\boldsymbol{n}, \boldsymbol{F}\}$, the sequence A does not converge in the φ-topology $T(\varphi)$ and such that H_i is nowhere dense and closed with respect to $T(\varphi)$, for all $i \in L$.*

Proof. We may suppose T induced by a norm f. Let

$$B(a) = \{x\colon f(x) < a\} \quad \text{and} \quad \tau = t(G).$$

First we choose F_0 such that it generates G. Let $\varepsilon > 0$ be such that $G \backslash B(\varepsilon)$ contains a point, x_0, and $B(\varepsilon) \subset G_\tau$. Then put

$$F_0 = \big(G \backslash B(\tfrac{1}{2}\varepsilon)\big) \cup \{0\}.$$

For $x \notin F_0$, $f(x+x_0) \geqslant f(x)-f(x_0) \geqslant \frac{1}{2}\varepsilon$, so $x_0 \in F_0$, $x+x_0 \in F_0$ and hence $G = F_0 - F_0$.

We will now choose b_i recursively, in such a way that

$$A \cap F_{1,0} = A \cap F_0,$$

which is a finite set. Define $b_1 \in G_\tau$ such that $n_1 \cdot b_1 \cap A = \emptyset$. As $F_{1,0} = F_0 \cup n_1 \cdot b_1$ we have the beginning step of the recursion already. For the next steps we need a sequence of elements of L, $\{m_i\colon i \in N\}$, such that for all k, $\{m_i\colon i > k\} = L$; it is clear such a sequence exists since L is at most countable.

Suppose $b_1, \ldots, b_l$ are already defined such that $n_1 F_{l,1} \cap A = \emptyset$. Observe that $F_{l,0} = F_0 \cup n_1 F_{l,1}$, hence $A \cap F_{l,0} = A \cap F_0$. Observe also that $n_1 F_{l,1}$ is finite and hence there exists an ε, such that $B(\varepsilon) \subset G_\tau$ and $(n_1 F_{l,1} \backslash \{0\}) + B(\varepsilon)$ contains no points of A.

Let n be such that $n+1 = \min(n_{l+1}, \tau)$ and let $b_{l+1} \in B(\varepsilon/n) \backslash H_{m_l}$, such that $n \cdot b_{l+1} \cap A = \emptyset$. Now we need an auxiliary formula, which is intuitively clear, namely:

$$F_{l,m} \subset (n_l/n_m) F_l + F_{l-1,m}.$$

This formula is evident for $l = m+1$ and one may prove it by induction on the difference between l and m:

$$\begin{aligned} F_{l+1,m} = F_m \cup k_m F_{l+1} &\subset F_m \cup \big(k_m\big((n_{l+1}/n_{m+1}) F_{l+1} + F_{l,m+1}\big)\big) \\ &\subset F_m \cup \big((n_{l+1}/n_m) F_{l+1} + k_m F_{l,m+1}\big) \\ &\subset (n_{l+1}/n_m) F_{l+1} + (F_m \cup k_m F_{l,m+1}) = (n_{l+1}/n_m) F_{l+1} + F_{l,m}. \end{aligned}$$

Consequently

$$n_1 F_{l+1,1} \subset n_1 (n_{l+1}/n_1) F_{l+1} + n_1 F_{l,1} = n_{l+1} F_{l+1} + n_1 F_{l,1}.$$

Now $n_{l+1} F_{l+1} = n_{l+1} \cdot b_{l+1} \subset B(\varepsilon)$. Hence

$$n_1 F_{l+1,1} \backslash \{0\} \subset \big(B(\varepsilon) + (n_1 F_{l,1} \backslash \{0\})\big) \cup n_{l+1} \cdot b_{l+1},$$

which does not contain points of A.

U_0 in the φ-topology is just $\bigcup_{l\in N} F_{l,0}$ and from the construction follows that $A \cap U_0 = A \cap F_0$, which is a finite set. On the other hand, every U_m contains $\{b_j: j > m\}$ hence elements from the complement of each H_i, $i \in L$. So none of the H_i is open in the φ-topology. As they are all closed in the φ-topology, they are nowhere dense.

2.4. THEOREM. *Let (G, T) be an abelian nondiscrete Hausdorff topological group. Let $\{H_i: i \in L\}$ be a finite or countable set of nowhere dense closed subgroups of G. If (G, T) is metrizable, then there exists a nondiscrete Hausdorff topology T' on G, finer than T, which does not satisfy the first axiom of countability, such that H_i is nowhere dense and closed for all $i \in L$.*

Proof. By Zorn's Lemma. Let $\{T_a: a \in A\}$ be a totally ordered set of topologies, finer than T and such that H_i is nowhere dense and closed with respect to T_a, for all $a \in A$. Then the H_i enjoy the same property for the union of the T_a. As the previous lemma shows that a maximal topology, stronger than T, such that all H_i are nowhere dense and closed, cannot be metrizable (otherwise it was not maximal), we are done.

§ 3. Refining a group topology for which there exist noncontinuous characters.

In this section, the completion of a topological group (G, T) will be denoted by $(G, T)^c$. We recall, that G_d denotes G with the discrete topology.

Let a fixed topological group (G, T) be given. Let H be the group of continuous characters of (G, T). H has the discrete topology throughout this section. The group of continuous characters of any compact or discrete group X will be denoted by $\hat{X}$, the dual of X with its usual topology.

As is wellknown, to every inclusion map of a submodule A of $(G_d)^\wedge$, there corresponds an epimorphism $a: G_d \to \hat{A}$ in the category of locally compact abelian groups. The map a is defined by $a(g)\chi = \chi(g)$. The topology of $\hat{A}$ is defined by pointwise convergence on A, so we see that a is continuous with respect to T if and only if each $\chi \in A$ is continuous with respect to T. So we have

3.1. LEMMA. a: $(G, T) \to \hat{A}$ *is continuous if and only if* $A \subset H$.

The discontinuity of a can be measured in an other way, namely the extent to which small T-neighborhoods of G are sprayed around in $\hat{A}$.

3.2. DEFINITION. For $a: (G, T) \to \hat{A}$ define $\check{A}$, the *discontinuity core of a with respect to* T, as the intersection of the closures of $a(U)$, in which U runs through all T-neighborhoods of 0 in G.

Now clearly $\check{A}$ is a closed subgroup of $\hat{A}$.

3.3. LEMMA. *Let $a: (G, T) \to \hat{A}$ be given and also $q: \hat{A} \to \hat{B}$, a morphism of compact abelian groups, then the composition $qa: (G, T) \to \hat{B}$ is continuous if and only if* $\ker q \supset \check{A}$.

Proof. Here and in the following we denote by $\mathrm{cl}(S)$ the closure of S. Because $\hat{A}$ is compact,

$$q\check{A} = \Big(\bigcup_{0\in U} \mathrm{cl}\big(a(U)\big)\Big) = \bigcup_{0\in U} q\big(\mathrm{cl}\big(a(U)\big)\big) = \bigcup_{0\in U} \mathrm{cl}\big(qa(U)\big) = \check{B}\,.$$

qa is continuous, if and only if $\check{B} = 0$.

3.4. LEMMA. $A = (H \cap A)^\perp = \big(A/(H \cap A)\big)^\wedge$.

Proof. From 3.1 follows that for surjective maps $\hat{A} \to \hat{B}$, the composition $(G, T) \to \hat{A} \to \hat{B}$ is continuous if and only if $B \subset H \cap A$, whence by duality and 3.3 the result.

3.5. DEFINITION. For a topological group (G, T) and $A \subset (G_d)^\wedge$, let T_A be the topology on G with neighborhood basis at 0 consisting of sets $U \cap a^{-1}(V)$ in which U is a T-neighborhood of $0 \in G$ and V is a neighborhood in $\hat{A}$ and a: $G_d \to \hat{A}$ the canonical map.

3.6. LEMMA. *Let $A \subset B \subset (G_d)^\wedge$; let $q: \hat{B} \to \hat{A}$, $a: G_d \to \hat{A}$ and $b: G_d \to \hat{B}$ be the canonical dual maps. Then b is continuous with respect to T_A if and only if $A + H = B + H$.*

Proof. $\mathrm{cl}\big(b(U \cap a^{-1}(V))\big) = \mathrm{cl}\big(b(U) \cap ba^{-1}(V)\big) = \mathrm{cl}\big(b(U) \cap q^{-1}(V)\big)$. Let $N = \ker q$. It is easy to compute

$$\check{B} \cap N = \big(B/(A + (H \cap B))\big)^\wedge.$$

On the other hand, for each U and V, there is a neighborhood W of 0 in $\hat{B}$ such that $(N \cap \check{B}) + W \supset \mathrm{cl}\big(b(U) \cap q^{-1}(V)\big) \supset N \cap \check{B}$. It follows that the discontinuity core of b with respect to T_A is just $N \cap \check{B}$. This is zero if and only if $A + H = B + H$.

3.7. COROLLARY. *The group of characters of G that are continuous with respect to T_A is just $H + A$.*

Proof. Obviously all characters from H and A are continuous with respect to T_A; conversely, let $\chi \notin H + A$, then $F = \chi \cdot Z + H + A \neq H + A$, hence $(G, T) \to \hat{F}$ is not continuous, so χ cannot be in the group of characters of (G, T_A) that are continuous with respect to T_A.

Now if a T_A-Cauchy-net $\{x_\lambda\}$ in G is T-convergent to zero then $a(x_\lambda)$ converges to an element of $\check{A}$. So if we complete, we have an exact sequence

$$(*) \qquad 0 \to \check{A} \to (G, T_A)^c \to (G, T)^c \to 0\,.$$

Because $(G, T_A)^c$ is topologically embedded in $(G, T)^c \oplus \hat{A}$ (by continuous extension of $1 \oplus a$: $G \to G \oplus \hat{A}$), we see that the injection $\check{A} \to (G, T_A)^c$ in $(*)$ is also a homeomorphism onto the image and moreover, if $H \subset A$ is a direct summand of A, then $\check{A}$ is a direct summand of $(G, T_A)^c$. Now let G_A be the full inverse image of G under the morphism $(G, T_A)^c \to (G, T)^c$. Then we can formulate:

3.8. THEOREM. *Let (G, T) be a topological group and H the group of continuous characters of (G, T). Then to any exact sequence* E1 *with an A that is a group of characters of $(G_d)\hat{}$ and that contains H,*

E1 $$0 \to H \to A \to A/H \to 0$$

there corresponds an exact sequence of topological groups

E2 $$0 \leftarrow (G, T) \leftarrow (G_A, T_A) \leftarrow \check{A} \to 0 ,$$

in which $\check{A} = (A/H)\hat{}$ and A is the group of continuous characters of (G_A, T_A); if G is complete with respect to T, then G_A is so with respect to T_A; if E1 *splits, then* E2 *splits topologically;* E2 *splits algebraically anyway.*

3.9. COROLLARY. *Let G be an infinite abelian group and T a finest topology on G. Then $(G_d)\hat{}$ is just the group of continuous characters of (G, T).*

§ 4. Construction of complete topologies on any infinite abelian group. In this section we will discuss topologies T on an arbitrary abelian group G, such that G is complete with respect to T; if such topologies are non-discrete, we call them *selfcomplete*.

A selfcomplete topology on Z was found in § 8 of [6] and § 8 of [2] and we will construct another example in § 5.

We will prove

4.1. THEOREM. *On any infinite abelian group G exist selfcomplete topologies.*

Outline of proof. We prove a Lemma that breaks up the proof in three parts, one of which is proved by the preceding remark and the two remaining ones are dealt with separately, roughly speaking by the method used in § 8 of [6].

First we observe the following:

Let G be an abelian group and H an infinite subgroup of G. Let T be a topology on H and let T' be the topology on G defined by taking a neighborhood basis at 0 for T as neighborhood basis at 0 for T'. Then T' is discrete if and only if T is discrete; Hausdorff, selfcomplete, with locally compact completion or satisfying the first axiom of countability respectively if and only if T is such.

We will now construct topologies on any infinite abelian group by constructing them on subgroups.

4.2. LEMMA. *For G an infinite abelian group, G has an infinite subgroup H of one of the following kinds:*

(i) $H \cong \boldsymbol{Z}$;

(ii) $H \cong \sum\{\boldsymbol{Z}/(p_i)\colon\, p_i$ *prime*, $i \in \boldsymbol{N}\}$;

(iii) $H \cong \boldsymbol{Z}/(p^\infty)$.

Remark. This is also proved in [5], see also Math. Rev., 1955, p. 11. For comprehensiveness we include here a proof.

Proof. We may suppose G is a torsiongroup, and as such the sum of its primary components. If an infinite number of these are nontrivial we may choose infinitely many primes p_i and nonzero elements $x_i \in G$, such that $p_i x_i = 0$. Hence the x_i generate a subgroup of kind (ii). Now G is infinite, so if there are a finite number of primary components, at least one of them is infinite, say C, the p-primary component. Denote by C_n the subgroup $\{x \in C\colon p^n x = 0\}$. If for some n, C_n is infinite, C_1 is infinite, hence because it is a module over $Z/(p)$, a vectorspace, so of kind (ii). Now suppose C_1 and hence C_n for all $n \geqslant 1$ are finite. Then C_1 must contain elements of infinite height, otherwise, by Corollary 33.3 of [1], C is isomorphic to a direct sum of an infinite number of cyclic p-groups, hence C_1 infinite. Put $x_0 = 0$, x_1 an element of infinite height in C_1 and suppose we have constructed $x_i \in C_i$ for $0 \leqslant i \leqslant n$, such that $px_i = x_{i-1}$ and x_i of infinite height, for $1 \leqslant i \leqslant n$. Then $x_n = p^k y_k$ has a solution for all k. So for all k, $p^{k-1}y_k \in C_{n+1}$. Because C_{n+1} is finite, it contains an element z such that $p^{k-1}y_k = z$ for infinitely many k. Hence z has also infinite height, and moreover $pz = x_n$. Put $x_{n+1} = z$. Now the subgroup generated by the x_i is of kind (iii).

4.3. LEMMA. *Let $G = \sum\{G_i\colon i \in N\}$, in which each of the G_i is a finite cyclic group. Then there exists a selfcomplete topology on G.*

Proof. Let I be an ideal of the power set of $\boldsymbol{N}$ with the property that for every infinite B, there exists an infinite $A \in I$, with $A \subset B$. Such ideals exist, see § 8, [6]. If we denote an arbitrary element of G by x, then its ith coordinate will be denoted by x^i. The product of all G_i is denoted by P, and the ith coordinate of an element of P likewise by a superscript i.

For $A \in I$ let T_A be the topology on G, defined by

$$U_{A,n} = \{x \in G\colon\, i \leqslant n \text{ or } i \in A \text{ implies } x^i = 0\} .$$

Put $G_A = (G, T_A)^c$ and let $c_A\colon G \to G_A$ be the natural embedding. As T_A is finer than the topology of coordinatewise convergence, the injection $g\colon G \to P$ factors through G_A, $g = jc_A$.

Let $x = \{x_k\colon k \in \Lambda\}$ be a Cauchy net relative T_A. Then, for each n, there is a $K \in \Lambda$ such that $x_k^i = x_l^i$ when $k \geqslant K$ and $l \geqslant K$ and $i \leqslant n$ or $i \in A$. It follows that for L such that $x_k^i = 0$ if $i \geqslant L$, holds that $x_k^i = 0$ for $k \geqslant K$, $i \geqslant L$, $i \in A$. So we see, if we put $A_L = \{i \in A\colon i \geqslant L\}$, x converges coordinatewise on the complement of A_L, whereas $x_k^i = 0$ if $k \geqslant K$ and $i \in A_L$. So in particular a Cauchy net in G which converges coordinatewise to 0 in P, converges to zero in G, hence j is injective. The image of j consists of all $x \in P$, such that there exists an L with the property that $i \in A_L$ implies $x^i = 0$. Now let T be the union of all T_A, $A \in I$. A Cauchy net for T is a Cauchy net for T_A, for all $A \in I$. Let $\{x_k: k \in \Lambda\}$ be a T-Cauchy-net. Suppose for an infinite set $B \subset N$, it holds that for $i \in B$, $\{x_k^i: k \in \Lambda\}$ does not converge to 0. Then there exists an infinite $A \in I$, $A \subset B$ and the net is not a convergent T_A-Cauchy-net. Hence the T-Cauchy-nets in G converge to an element of G, so T is selfcomplete.

4.4. Lemma. *Let q be a fixed positive integer, $q > 1$ and let $G = \boldsymbol{Z}/(q^\infty)$ (the group of rationals that have q^n for denominator, modulo 1). Then there exists a selfcomplete topology T on G.*

Proof. For A an infinite subset of N, $0 \in A$, define $\|.\|_a$ by $\|x\|_A = \max\limits_{i \in A} |q^i x|$, in which $|.|$ is the usual absolute value norm on $\boldsymbol{R}/\boldsymbol{Z}$. Let G_A be the set of all $x \in \boldsymbol{R}/\boldsymbol{Z}$, such that $\lim\limits_{i \in A} |q^i x| = 0$. Clearly $G \subset G_A$, for all A. G_A is complete for $\|.\|_A$, because $\|x\|_A \leqslant |x|$ and, moreover, if $\{x_n: n \in \Lambda\}$ is a Cauchy net, $q^i x_n$ is a bounded function of i from A to $\boldsymbol{R}/\boldsymbol{Z}$, vanishing at infinity, for each n. Furthermore, these functions form a Cauchy net with respect to the uniform norm, hence the limit is a bounded function vanishing at infinity, defined on A. The limit is given by $\{q^i x: i \in A\}$, in which $x = \lim\limits_{n \in \Lambda} x_n$. Also we see, that $A' \subset A$ implies $\|x\|_{A'} \leqslant \|x\|_A$.

For any $A \subset N$, let $g(A)$, the *maximal gap width*, be defined as the maximum of the difference between two consecutive elements of A, if this exists, and $g(A) = \infty$ otherwise. We show that $\|.\|_A$ induces the discrete topology on G, if and only if $g(A) < \infty$.

So, let $g(A) < \infty$ and let $x = \sum\limits_{j=1}^{k} a_j q^{-j} \bmod 1$ and let $a_k \neq 0$. Let a be the last element of A smaller than k, then $k - a \leqslant m$, $m = g(A)$. Hence $q^a x = \sum\limits_{j=1}^{k-a} a_{j+a} q^{-j} \bmod 1 \in q^{a-k}\boldsymbol{Z}/\boldsymbol{Z}$, hence $|q^a x| \geqslant q^{a-k} \geqslant q^{-m}$. On the other hand, let $g(A) = \infty$ and let ε be given. Choose n such that $q^{-n} < \varepsilon$ and let $k+1 \in A$ be such that for the last element a of A, preceding $k+1$ holds $k - a \geqslant n$. Then, for $x = q^{-k} \bmod 1$, $i \in A$ implies that either $q^i x = 0 \bmod 1$ or $|q^i x| = q^{i-k} \leqslant q^{a-k} \leqslant q^{-n} < \varepsilon$. Now, let $x \in \boldsymbol{R}/\boldsymbol{Z} \backslash G$ be arbitrary. So $x = \sum\limits_{j=1}^{\infty} a_j q^{-j} \bmod 1$, with not almost all a_j equal to 0 and not almost all a_j equal to $q-1$. Then the set $B = \{i: q^i x \geqslant (q-1)/q^2\}$ is infinite and hence for any infinite A such that $A \subset B$, $x \notin G_A$. This is clear, because if $a_j \neq 0$ and $a_j \neq q-1$, then $q^{j-1}x = a_j/q + \sum\limits_{i=2}^{\infty} a_{i+j-1} q^{-i} \bmod 1$, which exceeds $1/q$ in absolute value. If for almost all i holds $a_i = 0$ or $a_i = q-1$, then there are infinitely many i, such that $a_i = q-1$ and $a_{i+1} = 0$. So let $a_j = q-1$ and $a_{j+1} = 0$. Then $q^{j-1}x + (q-1)/q + \sum\limits_{i=3}^{\infty} a_{i+j-1} q^{-i} \bmod 1$, which exceeds $1/q - 1/q^2$ in absolute value.

Now suppose I is a collection of subsets of N, such that

(i) $A \in I$, $A \in I$ implies $A \cap A' \in I$,

(ii) $g(A) = \infty$ and A is infinite for all $A \in I$.

(iii) For all infinite B, there exists an $A \in I$ such that $A \subset B$.

Then the coarsest topology T, finer than all topologies induced by $\|.\|_A$, $A \in I$, has the property that it is selfcomplete. Indeed, none of the $\|.\|_A$-topologies are discrete and if $\{x_\lambda\}$ is a Cauchy net for T, then $\{x_\lambda\}$ must be a Cauchy net for all $\|.\|_A$-topologies, so $\{x_\lambda\}$ cannot converge to any element not in G.

§ 5. The finest topology on Z in which a fast increasing sequence converges.

In this section we apply the theory of MED-groups as developed in [6], to investigate topologies on $\boldsymbol{Z}$ for which a given fast increasing sequence converges.

The crucial point is again, to show that the coefficients a_i in the representation $z = \sum a_i n_i$, such that $\sum |a_i| p_i$ is minimal, can be used as a kind of coordinates to be used in statements like: convergence implies coordinatewise convergence. In [6] we were dealing with the situation that n_{i+1}/n_i was integer, so that the subgroup generated by $\{n_i: i > j\}$ did not contain n_j. Here we will investigate the situation that $\lim n_{i+1}/n_i = \infty$, so that the subgroup generated by $\{n_i: i > j\}$ may contain n_j, but only as linear combination of n_i with "large" coefficients.

5.1. Notations. $\{n_i: i \in \boldsymbol{N}\}$ will denote a given increasing sequence of positive integers. $\boldsymbol{Z}$ as a group is generated by $\{n_i: i \in \boldsymbol{N}\}$. We define k_i by $k_i = n_{i+1}/n_i$ and $p = \{p_i: i \in \boldsymbol{N}\}$ will denote a sequence of positive reals. The MRD-norm with respect to n_i, p_i is the largest group norm satisfying $\|n_i\| \leqslant p_i$. We recall that, when in all relations the equality sign holds, the MRD-norm is called an MED-norm. Let $\langle x \rangle$ denote the largest integer smaller than $x-1$, so $x - \langle x \rangle > 1$.

5.1. Lemma. *If $|a_i| < \langle k_i \rangle$ for all i, $0 \leqslant i \leqslant m$, then*

(i) $n_{l+1} > \sum\limits_{i=0}^{l} a_i n_i$ *for all $l \leqslant m$.*

(ii) *For each $l \leqslant m$ such that $a_l \neq 0$ and each s, $0 \leqslant s < l$,*

$$\Big|\sum_{i=s}^{l} a_i n_i\Big| > n_s .$$

Proof. (i) is clear. (ii) is proved by induction on t in the following statement:

$$\Big|\sum_{i=l-t}^{l} a_i n_i\Big| > n_{l-t}.$$

For $t=1$,

$$|a_l n_l + a_{l-1} n_{l-1}| \geqslant n_l - \langle k_{l-1}\rangle n_{l-1} > n_{l-1}.$$

The induction step is:

$$\Big|\sum_{i=l-t-1}^{l} a_i n_i\Big| > n_{l-t} - \langle k_{l-t-1}\rangle n_{l-t-1} > n_{l-t-1}.$$

5.2. Lemma. *If $|a_i|$, $|b_i|$ and $|c_i|$ are less than $\langle k_i\rangle/3$ for all i, $0\leqslant i\leqslant m$, then $\sum_{i=0}^{m} a_i n_i + \sum_{i=0}^{m} b_i n_i = \sum_{i=0}^{m} c_i n_i$ implies $a_i + b_i = c_i$ for all i, $0\leqslant i\leqslant m$.*

Proof. Consider $0=\sum_{i=0}^{m}(a_i+b_i-c_i)n_i$. If not all the coefficients of the right hand member of this equation are equal to 0, then at least two of them are not equal 0. We may obtain then an equation

$$\sum_{i=0}^{l} d_i n_i = \sum_{i=l+1}^{s} d_i n_i$$

in which $d_s\neq 0$ and $l+1<s$. According to 5.1 (i) the left hand member is less than n_{l+1}, according to 5.1 (ii) the right hand member is more than n_{l+1}, which is a contradiction.

5.3. Lemma. *Suppose for all i, $1\geqslant p_i\geqslant 1/\langle k_i\rangle$ and $p_i\geqslant p_{i+1}$. Then there exists a norm $\|.\|$ such that $\|n_i\|=p_i$.*

Proof. Suppose $n_j=\sum_{i\neq j} a_i n_i$. We have to prove (cf. proof of Lemma 6 in [6]) that $p_j\leqslant\sum_{i\neq j}|a_i|p_i$. In view of the monotonicity of the sequence p, we may assume

$$n_j=\sum_{i>j} a_i n_i.$$

Then, because of 5.1 (ii), for some $s>j$, $|a_s|>\langle k_s\rangle$, hence

$$\sum_{i>j}|a_i|p_i > p_s\langle k_s\rangle \geqslant 1 \geqslant p_j.$$

From now on we will assume that $k_i>2$ for all i, $\lim k_i=\infty$ and

(*) $$p_i\geqslant p_{i+1};\quad \lim p_i=0;\quad 1\geqslant p_i\geqslant 1/\langle k_i\rangle.$$

We denote the MED-norm defined by

$$\|n_i\|=p_i \quad\text{for all } i$$

by $\|.\|_p$. We denote $\{x\colon \|x\|_p<r\}$ by $B_p(r)$, the ball of radius r. When $\|z\|_p=\sum|a_i|p_i$ and $z=\sum a_i n_i$, then we call a_i the ith coordinate of z with respect to p. The coordinates of z may not be uniquely determined, but within a closed ball of radius 1/3, they are. Consequently, convergence in $B_p(1/3)$ implies coordinatewise convergence and when $q_i\leqslant p_i$, the coordinates with respect to p and q coincide in $B_p(1/3)$. Now observe that for every norm f on Z, such that $\lim f(n_i)=0$, there is an MED-norm g, defined with respect to n_i and p_i as in (*).

5.4. Theorem. *Let n_i be a given sequence of positive integers such that $\lim(n_{i+1}/n_i)=\infty$. Let T be the finest topology such that n_i converges to 0 relative to T. Then $(\boldsymbol{Z}, T)$ is a complete topological group.*

Remark 1. This theorem was first conjectured by H. Freudenthal.

Remark 2. M. I. Graev in [2], § 8, constructs for any topology T on the set $A=\{0,1,2!,3!,4!,\dots\}$ such that A is compact with respect to T, the finest group topology T' on $\boldsymbol{Z}$, such that T' restricted to A is T. Such a topology exists and is always selfcomplete.

Proof. Suppose not. Then there exists an MED-metric $\|.\|_p$ defined relative to n_i and p_i as in (*) such that $B_p(1/3)$ in $(\boldsymbol{Z},\|.\|)^c$ contains an image $x\notin\boldsymbol{Z}$ of $y\in(\boldsymbol{Z},T)^c$ under the natural injection. Let $\{x_\lambda\}$ be a T-Cauchy-net in $\boldsymbol{Z}$, converging to y. Then $\{x_\lambda\}$ converges coordinatewise to x in $(\boldsymbol{Z},\|.\|_p)^c$. As $x\notin\boldsymbol{Z}$, the set $A\subset\boldsymbol{N}$ of i such that the ith coordinate of x_λ does not converge to 0, is infinite. Now choose $1\geqslant q_i\geqslant p_i$ such that for all i,

$$q_i\geqslant q_{i+1};\quad \lim q_i=0;\quad \sum_{i\in A} q_i=\infty.$$

$\{x_\lambda\}$ converges to $x'\in(Z,\|.\|_q)^c\setminus\boldsymbol{Z}$. Choose in a ball of radius 1/3 around x' a $z\in\boldsymbol{Z}\cap B_p(1/3)$ and consider the net $w_\lambda=z-x_\lambda$. We denote by w the limit of w_λ relative to $\|.\|_p$ and by w' the limit relative to $\|.\|_q$. $w'\notin\boldsymbol{Z}$, $w\notin\boldsymbol{Z}$, $w'=z-x'$ and $w=z-x$. Furthermore, $\|w'\|_q$, $\|w\|_p$, $\|z\|_p$, $\|x\|_p$ are all less than 1/3. This means that the ith coordinate of w, a_i, is just the difference between the ith coordinate of z and b_i, the ith coordinate of x. So for i large enough, the coordinates of w_λ relative p and relative to q are equal to a_i, for $\lambda>\lambda_0$. So $\|w_\lambda\|_q$ tends to infinity by the choice of q. We have arrived at a contradiction.

§ 6. A problem. In [8], an example of a monothetic group on which no continuous characters exist, was constructed. This was done by factoring out a discrete infinite cyclic subgroup from the monothetic group constructed by S. Rolewicz in [9].

This process may be described also as "changing a given topology into a coarser one".

In this paper, in the preceding section, a class of selfcomplete topologies T on $\boldsymbol{Z}$ was constructed. It is not difficult to prove, that for each of these the set of continuous characters contains a Cantor set $C \subset \hat{\boldsymbol{Z}} = \boldsymbol{R}/\boldsymbol{Z}$.

The method used for the construction of selfcomplete topologies can be very roughly described as "constructing so fine a topology on $\boldsymbol{Z}$, that all possible Cauchy nets are convergent to an element of $\boldsymbol{Z}$".

So it seems difficult to reconcile the two aims in the following

6.1. Problem. Does there exist a minimally almost periodic and selfcomplete topology on $\boldsymbol{Z}$?

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Fixed point sets of homeomorphisms on dendrites (1)

by

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1. Introduction. It has been known for some time that not every non-empty closed subset of a dendrite D can be the fixed point set of a homeomorphism. G. E. Schweigert [5] proved that such a fixed point set cannot consist of one end point only, and several further restrictions can be found in [4]. These restrictions are mainly concerned with the behaviour of the fixed point set on the end points and branch points of D.

Here we show that the fixed point set of a homeomorphism of D is in fact to a large extent determined by the end points and branch points which it contains. More precisely: if the fixed point set F of a homeomorphism f of D contains points of the closure V of the set of all end points and branch points of D, then we can construct an isotopy relative to V which transforms f into a homeomorphism which is fixed point free on $D \setminus V$ (Theorem 1). If, on the other hand, F contains no end points and branch points, then F consists of a single point of order two (Theorem 2).

Many, but not all, of the known restrictions on the fixed point set of a homeomorphism of D also hold for monotone surjective self-maps [4], [6]. It is shown in § 4 that Theorem 1 cannot be extended to monotone maps. I do not know whether Theorem 2 (suitably modified) is still true in the monotone case.

2. Dendrites. The purpose of this paragraph is to collect the properties of dendrites needed in this paper. They can be found in [2], [4], [6], [7] and [8].

A dendrite D is a metric continuum (i.e. compact connected Hausdorff space) in which every pair of distinct points is separated by a third point. It has a partial order structure which was developed by L. E. Ward,

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