On shape and fundamental deformation retracts I [1]

by

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Introduction. The well known theorem of Fox gives a necessary and sufficient condition for a map to be a homotopy equivalence (see [5]):

THEOREM OF FOX. *The map $f\colon X \to Y$ is a homotopy equivalence iff X is homeomorphic to a deformation retract of the mapping cylinder C_f.*

The purpose of this paper is to give a necessary and sufficient condition for a map to generate a fundamental equivalence. As the main result we obtain the following

THEOREM (8.1). *Let X, Y be two compact metric spaces. The map $f\colon X \to Y$ generates a fundamental equivalence iff X is homeomorphic to a fundamental deformation retract of the mapping cylinder C_f.*

The proof is based on the idea of S. Mardešić and J. Segal (see [6], [7]).

We introduce the notions of retract and deformation retract for inverse systems (§ 1, 2), and define the mapping cylinder for the usual map of inverse systems (§ 3). Then, we establish for inclusion-ANR-systems an analogue of the Fox Theorem, (4.5).

As was shown by K. Borsuk in [4], the basic notions of Shape Theory introduced in [1], [2] for compact subsets of the Hilbert cube can be equivalently defined for compact subsets of arbitrary AR-spaces. This approach is studied in §§ 5, 6 where the class of so called convenient absolute retracts (CAR) is defined and the results of [7] are generalized to compact subsets of CARs. Next, we establish the connection between the notion of retract for inverse systems and the notion of fundamental retract (§ 7). This enables us to prove the main Theorem 8.1.

Some remarks concerning a limit map and a mapping cylinder are given in the Appendix.

1. Similarity of maps of inverse systems. Usual maps. We are concerned with *inverse systems* of arbitrary Hausdorff spaces, i.e. systems of the

(1) When the paper was in press, some stronger results were obtained. These results will be published as the part II.

form $X = (X_\alpha, p_\alpha^{\alpha'}, \mathcal{A})$, $\mathcal{A}$ being a closure finite directed set with respect to the relation $\geqslant$ (see [6]), and $p_\alpha^{\alpha'}\colon X_{\alpha'} \to X_\alpha$ $(\alpha' \geqslant \alpha)$ being continuous functions satisfying the following two conditions: if $\alpha'' \geqslant \alpha' \geqslant \alpha$ then $p_\alpha^{\alpha'} p_{\alpha'}^{\alpha''} = p_\alpha^{\alpha''}$, and $p_\alpha^\alpha = 1_{X_\alpha}$ for every $\alpha \in \mathcal{A}$.

According to the definitions introduced by S. Mardešić and J. Segal in [6] (2), given two inverse systems $X = (X_\alpha, p_\alpha^{\alpha'}, \mathcal{A})$, $Y = (Y_\beta, q_\beta^{\beta'}, \mathcal{B})$, the system $f = (\varphi, f_\beta)$ is said to be a *map of* X *into* Y whenever $\varphi\colon \mathcal{B} \to \mathcal{A}$ is an increasing function and $f_\beta\colon X_{\varphi(\beta)} \to Y_\beta$ satisfy the condition $f_\beta p_{\varphi(\beta)}^{\varphi(\beta')} \simeq q_\beta^{\beta'} f_{\beta'}$ for every $\beta' \geqslant \beta$, i.e. the diagram

$$\begin{tikzcd} X_{\varphi(\beta)} \arrow[d, "f_\beta"'] & X_{\varphi(\beta')} \arrow[l, "p_{\varphi(\beta)}^{\varphi(\beta')}"'] \arrow[d, "f_{\beta'}"] \\ Y_\beta & Y_{\beta'} \arrow[l, "q_\beta^{\beta'}"] \end{tikzcd}$$

commutes up to homotopy for $\beta' \geqslant \beta$.

The *identity map* $1_X\colon X \to X$ is defined as a map $(1_{\mathcal{A}}, 1_{X_\alpha})$.

The *composition of two maps* $f = (\varphi, f_\beta)\colon X \to Y$, $g = (\psi, g_\gamma)\colon Y \to Z$ is defined as a map $gf = (\varphi\psi, g_\gamma f_{\psi(\gamma)})$ (see [6]).

The maps $f = (\varphi, f_\beta)$, $f' = (\varphi', f'_\beta)$ are said to be *homotopic* (in symbols $f \simeq f'$) whenever

$$\bigwedge_{\beta \in \mathcal{B}} \bigvee_{\alpha \geqslant \varphi(\beta), \varphi'(\beta)} f_\beta p_{\varphi(\beta)}^\alpha \simeq f'_\beta p_{\varphi'(\beta)}^\alpha .$$

The map $f\colon X \to Y$ is said to be a *homotopy equivalence* iff there is a map $g\colon Y \to X$ which is a homotopy inverse of f, i.e. $fg \simeq 1_Y$ and $gf \simeq 1_X$.

Now, let us define the following relation $\cong$ in the class of all maps of inverse systems.

Given two inverse systems $X = (X_\alpha, p_\alpha^{\alpha'}, \mathcal{A})$, $Y = (Y_\beta, q_\beta^{\beta'}, \mathcal{B})$ and two maps $f = (\varphi, f_\beta)$, $f' = (\varphi', f'_\beta)$, we say that f, f' are *similar* (in symbols $f \cong f'$) iff

$$\bigwedge_{\beta \in \mathcal{B}} \bigvee_{\alpha \geqslant \varphi(\beta), \varphi'(\beta)} f_\beta p_{\varphi(\beta)}^\alpha = f'_\beta p_{\varphi'(\beta)}^\alpha .$$

Obviously $f \cong f' \Rightarrow f \simeq f'$.

One can easily prove $\cong$ to be an equivalence relation.

In the class of all maps of systems let us distinguish the so called usual maps.

(2) In [6] the authors are concerned with ANR-systems. Obviously their definitions can be extended to arbitrary inverse systems of topological spaces.

The map $f = (\varphi, f_\beta)\colon X \to Y$ is said to be a *usual* one whenever the maps f_β satisfy the condition $f_\beta p_{\varphi(\beta)}^{\varphi(\beta')} = q_\beta^{\beta'} f_{\beta'}$, i.e. the diagram

$$\begin{tikzcd} X_{\varphi(\beta)} \arrow[d, "f_\beta"'] & X_{\varphi(\beta')} \arrow[l, "p_{\varphi(\beta)}^{\varphi(\beta')}"'] \arrow[d, "f_{\beta'}"] \\ Y_\beta & Y_{\beta'} \arrow[l, "q_\beta^{\beta'}"] \end{tikzcd}$$

is commutative for every $\beta' \geqslant \beta$.

The implication

$$f \simeq f' \wedge g \simeq g' \Rightarrow gf \simeq g'f',$$

which holds for arbitrary maps of systems ([6]) whenever these compositions are defined, enables us to consider the category $\mathfrak{T}^*_{\simeq}$ of inverse systems with homotopy classes of maps as morphisms. Analogically, the implication

$$f \cong f' \wedge g \cong g' \Rightarrow gf \cong g'f',$$

which holds for usual maps, enables us to consider the category $\hat{\mathfrak{T}}^*_{\cong}$ of inverse systems with classes of similar usual maps as morphisms.

2. Extension of maps of inverse systems. Retraction. Let $f = (\varphi, f_\beta)\colon X \to Y$ be any map of the inverse system $X = (X_\alpha, p_\alpha^{\alpha'}, \mathcal{A})$ into $Y = (Y_\beta, q_\beta^{\beta'}, \mathcal{B})$. According to the definition given in [6], f is said to be *regular* whenever φ is strictly increasing. We say that f is *cofinal* whenever the set $\varphi(\mathcal{B})$ is cofinal to $\mathcal{A}$. Notice that

2.1. *If* $\mathcal{A} = \mathcal{B} = \mathcal{N}$ *— the set of natural numbers, then*

$$f \text{ is regular} \Rightarrow f \text{ is cofinal}.$$

Proof. If $\varphi\colon \mathcal{N} \to \mathcal{N}$ is strictly increasing, then $\varphi(n) \geqslant n$ for every $n \in \mathcal{N}$. In fact, $\varphi(1) \geqslant 1$ and $\varphi(n-1) \geqslant n-1 \Rightarrow \varphi(n) > \varphi(n-1) \geqslant n-1 \Rightarrow \varphi(n) \geqslant n$. Thus $\bigwedge_n \bigvee_{n'} \varphi(n') \geqslant n$, i.e. $\varphi(\mathcal{N})$ is cofinal to $\mathcal{N}$. ∎

Let us define an inclusion of inverse systems as follows.

Given two inverse systems $X = (X_\alpha, p_\alpha^{\alpha'}, \mathcal{A})$, $\hat{X} = (\hat{X}_\alpha, \hat{p}_\alpha^{\alpha'}, \hat{\mathcal{A}})$, the map $i = (\tau, i_\alpha)\colon X \to \hat{X}$ is said to be an *inclusion* of X into $\hat{X}$ iff i is a usual cofinal one and $i_\alpha\colon X_{\tau(\alpha)} \to \hat{X}_\alpha$ is an inclusion for every $\alpha \in \hat{\mathcal{A}}$.

The map $\hat{f}\colon \hat{X} \to Y$ is said to be an *extension* of $f\colon X \to Y$ (in symbols $f \subset \hat{f}$) whenever there is an inclusion $i\colon X \to \hat{X}$ such that $\hat{f} i \cong f$.

Recall that the inverse system $(X_\alpha, p_\alpha^{\alpha'}, \mathcal{A})$ is called an ANR*-system* iff $X_\alpha \in$ ANR for every $\alpha \in \mathcal{A}$.

Let us prove the following theorem on extension of a homotopy for ANR-systems [3].

2.2. THEOREM. *Let $f, g\colon X \to Y$ be two maps of inverse systems, Y being an ANR-system. If $f \simeq g$ and $f \subset \hat{f}\colon \hat{X} \to Y$, then there exists a $\hat{g}\colon \hat{X} \to Y$ such that $\hat{f} \simeq \hat{g}$ and $g \subset \hat{g}$.*

Proof. Let $X = (X_\alpha, p_\alpha^{\alpha'}, \mathfrak{A})$, $Y = (Y_\beta, q_\beta^{\beta'}, \mathfrak{B})$, Y_β being ANR-spaces for $\beta \in \mathfrak{B}$; let $f = (\varphi, f_\beta)$, $g = (\psi, g_\beta)$, $\varphi, \psi\colon \mathfrak{B} \to \mathfrak{A}$ being two increasing functions, $f_\beta\colon X_{\varphi(\beta)} \to Y_\beta$, $g_\beta\colon X_{\psi(\beta)} \to Y_\beta$. Assume $f \simeq g$, i.e.

$$(1) \qquad \bigwedge_{\beta\in\mathfrak{B}} \bigvee_{\alpha \geqslant \varphi(\beta), \psi(\beta)} f_\beta p^\alpha_{\varphi(\beta)} \simeq g_\beta p^\alpha_{\psi(\beta)} .$$

Let $f \subset \hat{f} = (\hat{\varphi}, \hat{f}_\beta)\colon \hat{X} \to Y$, where $\hat{X} = (\hat{X}_\alpha, \hat{p}_\alpha^{\alpha'}, \hat{\mathfrak{A}})$, $\hat{\varphi}\colon \mathfrak{B} \to \hat{\mathfrak{A}}$, $\hat{f}_\beta\colon \hat{X}_{\hat{\varphi}(\beta)} \to Y_\beta$, and let $i = (\tau, i_\alpha)\colon X \to \hat{X}$ be a corresponding inclusion, i.e. i is usual and cofinal, and i_α is an inclusion for every $\alpha \in \mathfrak{A}$. Moreover, $f \simeq \hat{f} i$, i.e.

$$(2) \qquad \bigwedge_{\beta\in\mathfrak{B}} \bigvee_{\alpha \geqslant \varphi(\beta), \tau\hat{\varphi}(\beta)} f_\beta p^\alpha_{\varphi(\beta)} = \hat{f}_\beta i_{\hat{\varphi}(\beta)} p^\alpha_{\tau\hat{\varphi}(\beta)} .$$

Since i is cofinal, we have

$$(3) \qquad \bigwedge_{\alpha\in\mathfrak{A}} \bigvee_{\alpha' \in \hat{\mathfrak{A}}} \tau(\alpha') \geqslant \alpha .$$

Notice that in both of the conditions (1), (2) the index α can be replaced by any greater one; so by (1), (2), (3) we obtain

$$(4) \qquad \bigwedge_{\beta\in\mathfrak{B}} \bigvee_{\alpha' \in \hat{\mathfrak{A}}} \big(\tau(\alpha') \geqslant \varphi(\beta), \psi(\beta), \tau\hat{\varphi}(\beta)\big) \wedge \big(f_\beta p^{\tau(\alpha')}_{\varphi(\beta)} \simeq g_\beta p^{\tau(\alpha')}_{\psi(\beta)}\big) \wedge$$
$$\wedge \big(f_\beta p^{\tau(\alpha)}_{\varphi(\beta)} = \hat{f}_\beta i_{\hat{\varphi}(\beta)} p^{\tau(\alpha')}_{\tau\hat{\varphi}(\beta)}\big) .$$

Thus

$$(4') \qquad \bigwedge_{\beta\in\mathfrak{B}} \bigvee_{\alpha' \in \hat{\mathfrak{A}}} \big(\alpha' \geqslant \hat{\varphi}(\beta)\big) \wedge \big(\tau(\alpha') \geqslant \varphi(\beta), \psi(\beta)\big) \wedge \big(f_\beta p^{\tau(\alpha')}_{\varphi(\beta)} \simeq g_\beta p^{\tau(\alpha')}_{\psi(\beta)}\big) \wedge$$
$$\wedge \big(f_\beta p^{\tau(\alpha')}_{\varphi(\beta)} = \hat{f}_\beta i_{\hat{\varphi}(\beta)} p^{\tau(\alpha')}_{\tau\hat{\varphi}(\beta)}\big) .$$

Since $i_{\hat{\varphi}(\beta)} p^{\tau(\alpha')}_{\tau\hat{\varphi}(\beta)} = \hat{p}^{\alpha'}_{\hat{\varphi}(\beta)} i_{\alpha'}$, i.e. the diagram

$$\begin{tikzcd}
X_{\tau\hat{\varphi}(\beta)} \arrow[d, "i_{\hat{\varphi}(\beta)}"'] & X_{\tau(\alpha')} \arrow[l, "p^{\tau(\alpha')}_{\tau\hat{\varphi}(\beta)}"'] \arrow[d, "i_{\alpha'}"] \\
\hat{X}_{\hat{\varphi}(\beta)} & \hat{X}_{\alpha'} \arrow[l, "\hat{p}^{\alpha'}_{\hat{\varphi}(\beta)}"]
\end{tikzcd}$$

[3] This statement is related to Patkowska's Theorem on the extension of a homotopy for fundamental sequences (see [9]).

is commutative, we obtain

$$(5) \qquad \bigwedge_{\beta\in\mathfrak{B}} \bigvee_{\alpha' \in \hat{\mathfrak{A}}} \big(\alpha' \geqslant \hat{\varphi}(\beta)\big) \wedge \big(\tau(\alpha') \geqslant \varphi(\beta), \psi(\beta)\big) \wedge \big(f_\beta p^{\tau(\alpha')}_{\varphi(\beta)} \simeq g_\beta p^{\tau(\alpha')}_{\psi(\beta)}\big) \wedge$$
$$\wedge \big(f_\beta p^{\tau(\alpha')}_{\varphi(\beta)} \subset \hat{f}_\beta \hat{p}^{\alpha'}_{\hat{\varphi}(\beta)}\big) .$$

Let us fix such α' for every $\beta \in \mathfrak{B}$ and define $\hat{\psi}'\colon \mathfrak{B} \to \hat{\mathfrak{A}}$, $\hat{\psi}'(\beta) \underset{\mathrm{Df}}{=} \alpha'$. By Lemma 5 of [6] there is an increasing function $\hat{\psi}\colon \mathfrak{B} \to \hat{\mathfrak{A}}$ such that $\hat{\psi}(\beta) \geqslant \hat{\psi}'(\beta)$ for every $\beta \in \mathfrak{B}$. We have

$$(6) \qquad \bigwedge_{\beta\in\mathfrak{B}} \big(\tau\hat{\psi}(\beta) \geqslant \psi(\beta)\big) \wedge \big(\hat{\psi}(\beta) \geqslant \hat{\varphi}(\beta)\big) .$$

Since $Y_\beta \in \mathrm{ANR}$ for every $\beta \in \mathfrak{B}$, we can apply the Borsuk Theorem on extension of homotopy (see [3], p. 94): $f_\beta p^{\tau\hat{\psi}(\beta)}_{\varphi(\beta)}, g_\beta p^{\tau\hat{\psi}(\beta)}_{\psi(\beta)}\colon X_{\tau\hat{\psi}(\beta)} \to Y_\beta$ are homotopic and $f_\beta p^{\tau\hat{\psi}(\beta)}_{\varphi(\beta)} \subset \hat{f}_\beta \hat{p}^{\hat{\psi}(\beta)}_{\hat{\varphi}(\beta)}\colon \hat{X}_{\hat{\varphi}(\beta)} \to Y_\beta$, so there exists a $\hat{g}_\beta\colon \hat{X}_{\hat{\psi}(\beta)} \to Y_\beta$ such that

$$(7) \qquad g_\beta p^{\tau\hat{\psi}(\beta)}_{\psi(\beta)} \subset \hat{g}_\beta$$

and

$$(8) \qquad \hat{f}_\beta \hat{p}^{\hat{\psi}(\beta)}_{\hat{\varphi}(\beta)} \simeq \hat{g}_\beta .$$

It remains to prove $\hat{g} = (\hat{\psi}, \hat{g}_\beta)$ to be a map of systems and to satisfy the conditions:

$$g \subset \hat{g} \quad \text{and} \quad \hat{f} \simeq \hat{g} .$$

Let us consider two diagrams:

$$\begin{tikzcd}
\hat{X}_{\hat{\varphi}(\beta)} \arrow[d, "\hat{f}_\beta"'] & \hat{X}_{\hat{\varphi}(\beta')} \arrow[l, "\hat{p}^{\hat{\varphi}(\beta')}_{\hat{\varphi}(\beta)}"'] \arrow[d, "\hat{f}_{\beta'}"] \\
Y_\beta & Y_{\beta'} \arrow[l, "q^{\beta'}_\beta"]
\end{tikzcd} , \qquad
\begin{tikzcd}
\hat{X}_{\hat{\psi}(\beta)} \arrow[d, "\hat{g}_\beta"'] & \hat{X}_{\hat{\psi}(\beta')} \arrow[l, "p^{\hat{\psi}(\beta')}_{\hat{\psi}(\beta)}"'] \arrow[d, "\hat{g}_{\beta'}"] \\
Y_\beta & Y_{\beta'} \arrow[l, "q^{\beta'}_\beta"]
\end{tikzcd} ,$$

the first one being commutative up to homotopy. By (8),

$$\hat{g}_\beta \hat{p}^{\hat{\psi}(\beta')}_{\hat{\psi}(\beta)} \simeq \hat{f}_\beta \hat{p}^{\hat{\psi}(\beta)}_{\hat{\varphi}(\beta)} \hat{p}^{\hat{\psi}(\beta')}_{\hat{\psi}(\beta)} = \hat{f}_\beta \hat{p}^{\hat{\psi}(\beta')}_{\hat{\varphi}(\beta)}$$

and

$$q^{\beta'}_\beta \hat{g}_{\beta'} \simeq q^{\beta'}_\beta \hat{f}_{\beta'} \hat{p}^{\hat{\psi}(\beta')}_{\hat{\varphi}(\beta')} \simeq \hat{f}_\beta \hat{p}^{\hat{\varphi}(\beta')}_{\hat{\varphi}(\beta)} \hat{p}^{\hat{\psi}(\beta')}_{\hat{\varphi}(\beta')} = \hat{f}_\beta \hat{p}^{\hat{\psi}(\beta')}_{\hat{\varphi}(\beta)} ;$$

thus the second diagram commutes up to homotopy as well, and then $\hat{g}$ is a map of systems.

We have to prove $g \subset \hat{g}$, i.e.

$$\bigwedge_{\beta\in\mathfrak{B}} \bigvee_{\alpha \geqslant \psi(\beta), \tau\hat{\psi}(\beta)} g_\beta p^\alpha_{\psi(\beta)} = \hat{g}_\beta i_{\hat{\psi}(\beta)} p^\alpha_{\tau\hat{\psi}(\beta)} .$$

Take $\beta \in \mathcal{B}$ and let $\alpha \underset{\mathrm{Df}}{=} \tau\hat{\psi}(\beta)$. By (6), $\alpha \geqslant \psi(\beta)$. By (7), $\hat{g}_\beta i_{\hat{\psi}(\beta)} = g_\beta p_{\psi(\beta)}^{\tau\hat{\psi}(\beta)}$, so

$$\hat{g}_\beta i_{\hat{\psi}(\beta)} p^{\alpha}_{\tau\hat{\psi}(\beta)} = g_\beta p_{\psi(\beta)}^{\tau\hat{\psi}(\beta)} p^{\alpha}_{\tau\hat{\psi}(\beta)} = g_\beta p^{\alpha}_{\psi(\beta)};$$

thus $\boldsymbol{g} \subset \hat{\boldsymbol{g}}$.

Finally, we prove that $\hat{\boldsymbol{f}} \simeq \hat{\boldsymbol{g}}$, i.e.

$$\bigwedge_{\beta \in \mathcal{B}} \bigvee_{\alpha'' \geqslant \hat{\varphi}(\beta), \hat{\psi}(\beta)} \hat{f}_\beta \hat{p}^{\alpha''}_{\hat{\varphi}(\beta)} \simeq \hat{g}_\beta \hat{p}^{\alpha''}_{\hat{\psi}(\beta)} .$$

In fact, putting $\alpha'' \underset{\mathrm{Df}}{=} \hat{\psi}(\beta)$ for $\beta \in \mathcal{B}$, we obtain by (6) $\alpha'' \geqslant \hat{\varphi}(\beta)$ and, by (8),

$$\hat{f}_\beta p^{\alpha''}_{\hat{\varphi}(\beta)} = \hat{f}_\beta \hat{p}^{\hat{\psi}(\beta)}_{\hat{\varphi}(\beta)} \simeq \hat{g}_\beta = \hat{g}_\beta \hat{p}^{\hat{\psi}(\beta)}_{\hat{\psi}(\beta)} = \hat{g}_\beta \hat{p}^{\alpha''}_{\hat{\psi}(\beta)} .$$

Thus the proof is complete. ∎

The map $\boldsymbol{r}$: $\boldsymbol{X} \to \boldsymbol{X}$ is said to be a *retraction* iff $\mathbf{1}_X \subset \boldsymbol{r}$, i.e. there is an inclusion $\boldsymbol{i}$ such that $\boldsymbol{ri} \cong \mathbf{1}_X$. If, moreover, $\boldsymbol{ir} \simeq \mathbf{1}_{\hat{X}}$, then $\boldsymbol{r}$ is said to be a *deformational retraction*. The inverse system $\boldsymbol{X}$ is called a *retract* (*deformation retract*) of $\hat{\boldsymbol{X}}$ whenever a retraction (deformational retraction) $\boldsymbol{r}$: $\hat{\boldsymbol{X}} \to \boldsymbol{X}$ does exist.

If the ANR-system $\boldsymbol{X} = (X_\alpha, p_\alpha^{\alpha'}, \mathcal{A})$ is an *inclusion*-ANR-*system*, i.e. all the maps $p_\alpha^{\alpha'}$ are inclusions, then obviously any two inclusions $\boldsymbol{i}, \boldsymbol{i}'$: $\boldsymbol{X} \to \hat{\boldsymbol{X}}$ are similar. This enables us to prove the following

2.3. Proposition. *If* $\boldsymbol{X}$ *is an inclusion*-ANR-*system, and* $\boldsymbol{i}$: $\boldsymbol{X} \to \hat{\boldsymbol{X}}$ *is an inclusion, then*

(i) $\boldsymbol{X}$ *is a retract of* $\hat{\boldsymbol{X}} \Leftrightarrow \boldsymbol{i}$ *has a left homotopy inverse,*

(ii) $\boldsymbol{X}$ *is a deformation retract of* $\hat{\boldsymbol{X}} \Leftrightarrow \boldsymbol{i}$ *is a homotopy equivalence.*

Proof. The implications $\Rightarrow$ obviously hold for arbitrary inverse systems. Let us prove $\Leftarrow$.

Assume $\boldsymbol{h}$: $\hat{\boldsymbol{X}} \to \boldsymbol{X}$ to be a left homotopy inverse of $\boldsymbol{i}$, i.e.

$$\boldsymbol{hi} \simeq \mathbf{1}_X . \tag{1}$$

Obviously we have

$$\boldsymbol{hi} \subset \boldsymbol{h} . \tag{2}$$

By Theorem 2.2 (on extension of homotopy for inverse systems), since $\boldsymbol{X}$ is an ANR-system, it follows by (1), (2) that there exists an $\boldsymbol{r}$: $\hat{\boldsymbol{X}} \to \boldsymbol{X}$, such that

$$\mathbf{1}_X \subset \boldsymbol{r} \tag{3}$$

and

$$\boldsymbol{r} \simeq \boldsymbol{h} . \tag{4}$$

By (3), $\boldsymbol{r}$ is a retraction, which proves (i).

Assume now that $\boldsymbol{h}$ is a homotopy inverse of $\boldsymbol{i}$, i.e. that it satisfies (1) and

$$\boldsymbol{ih} \simeq \mathbf{1}_{\hat{X}} . \tag{5}$$

By (4), (5) the retraction $\boldsymbol{r}$ satisfies

$$\boldsymbol{ir} \simeq \mathbf{1}_{\hat{X}} , \tag{6}$$

which proves (ii). ∎

Finally, let us notice that

2.4. *If* $\boldsymbol{r} = (\varrho, r_\alpha)$: $\hat{\boldsymbol{X}} \to \boldsymbol{X}$ *is a retraction,* $\boldsymbol{i} = (\tau, i_\beta)$: $\boldsymbol{X} \to \hat{\boldsymbol{X}}$ *is a corresponding inclusion,* $X = \varprojlim \boldsymbol{X}$ *and* p_α: $X \to X_\alpha$, $\alpha \in \mathcal{A}$, *are projections, then*

$$\bigwedge_{\alpha} r_\alpha i_{\varrho(\alpha)} p_{\tau\varrho(\alpha)} = p_\alpha .$$

Proof. By the assumption, $\boldsymbol{ri} \cong \mathbf{1}_X$, i.e.

$$\bigwedge_{\alpha} \bigvee_{\alpha' \geqslant \alpha, \tau\varrho(\alpha)} r_\alpha i_{\varrho(\alpha)} p^{\alpha'}_{\tau\varrho(\alpha)} = p^{\alpha'}_{\alpha} ,$$

so

$$\bigwedge_{\alpha} \bigvee_{\alpha' \geqslant \alpha, \tau\varrho(\alpha)} r_\alpha i_{\varrho(\alpha)} p^{\alpha'}_{\tau\varrho(\alpha)} p_{\alpha'} = p^{\alpha'}_{\alpha} p_{\alpha'} .$$

Since $p^{\alpha'}_{\alpha} p_{\alpha'} = p_\alpha$ and $p^{\alpha'}_{\tau\varrho(\alpha)} p_{\alpha'} = p_{\tau\varrho(\alpha)}$, we obtain

$$r_\alpha i_{\varrho(\alpha)} p_{\tau\varrho(\alpha)} = p_\alpha \quad \text{for every } \alpha \in \mathcal{A} . \ \blacksquare$$

3. Mapping cylinder for a usual map of inverse systems. Consider two arbitrary inverse systems of topological spaces, $\boldsymbol{X} = (X_\alpha, p_\alpha^{\alpha'}, \mathcal{A})$, $\boldsymbol{Y} = (Y_\beta, q_\beta^{\beta'}, \mathcal{B})$ and a usual map $\boldsymbol{f} = (\varphi, f_\beta)$: $\boldsymbol{X} \to \boldsymbol{Y}$. By a *mapping cylinder of* $\boldsymbol{f}$ we understand the system $\boldsymbol{Z} = (Z_\beta, r_\beta^{\beta'}, \mathcal{B})$ defined as follows:

Z_β is a mapping cylinder of f_β: $X_{\varphi(\beta)} \to Y_\beta$, i.e. $Z_\beta = \big((X_{\varphi(\beta)} \times I) \cup Y_\beta\big)|_\sim$, where $I = \langle 0, 1\rangle$ and the equivalence relation $\sim$ is defined by the conditions:

$$\begin{aligned}
&\text{for } (x, t), (x', t') \in X_{\varphi(\beta)} \times I, && (x, t) \sim (x', t') \Leftrightarrow \big[(x, t) = (x', t') \vee \\
& && \vee \big(t = t' = 1 \wedge f_\beta(x) = f_\beta(x')\big)\big],\\
&\text{for } y, y' \in Y_\beta && y \sim y' \Leftrightarrow y = y',\\
&\text{for } (x, t) \in X_{\varphi(\beta)} \times I,\ y \in Y_\beta, && (x, t) \sim y \Leftrightarrow t = 1 \wedge f_\beta(x) = y;
\end{aligned}$$

$$r_\beta^{\beta'}(z) \underset{\mathrm{Df}}{=} \begin{cases} [p^{\varphi(\beta')}_{\varphi(\beta)}(x), t] & \text{if } z = [x, t] \\ [q_\beta^{\beta'}(y)] & \text{if } z = [y] \end{cases} \quad \text{for every } \beta' \geqslant \beta .$$

Since the mapping cylinder of a map of spaces f: $X \to Y$ is usually denoted by C_f, we shall use the symbol $\boldsymbol{C}_{\boldsymbol{f}}$ to denote the mapping cylinder of the map of inverse systems $\boldsymbol{f}$: $\boldsymbol{X} \to \boldsymbol{Y}$.

Let us prove that

3.1. *C_f is an inverse system.*

Proof. First let us verify the continuity of $r_\beta^{\beta'}$. Since $p_\alpha^{\alpha'}$, $q_\beta^{\beta'}$ are continuous, it suffices to show that

$$r_\beta^{\beta'}[x,1] = r_\beta^{\beta'}[f_{\beta'}(x)] \quad \text{for} \quad x \in X_{\varphi(\beta')},\ \beta' \geqslant \beta .$$

Since $\big(p_{\varphi(\beta)}^{\varphi(\beta')}(x), 1\big) \sim f_\beta p_{\varphi(\beta)}^{\varphi(\beta')}(x) = q_\beta^{\beta'} f_{\beta'}(x)$, we have

$$r_\beta^{\beta'}[x,1] = [p_{\varphi(\beta)}^{\varphi(\beta')}(x), 1] = [q_\beta^{\beta'} f_{\beta'}(x)] = r_\beta^{\beta'}[f_{\beta'}(x)] .$$

Take $\beta'' \geqslant \beta' \geqslant \beta$ and show that $r_\beta^{\beta'} r_{\beta'}^{\beta''} = r_\beta^{\beta''}$. If $z = [x, t]$, then

$$r_\beta^{\beta'} r_{\beta'}^{\beta''}(z) = [p_{\varphi(\beta)}^{\varphi(\beta')} p_{\varphi(\beta')}^{\varphi(\beta'')}(x), t] = [p_{\varphi(\beta)}^{\varphi(\beta'')}(x), t] = r_\beta^{\beta''}(z);$$

if $z = [y]$, then

$$r_\beta^{\beta'} r_{\beta'}^{\beta''}(z) = [q_\beta^{\beta'} q_{\beta'}^{\beta''}(y)] = [q_\beta^{\beta''}(y)] = r_\beta^{\beta''}(z) .$$

Obviously $r_\beta^{\beta}(z) = z$. ■

3.2. *If $\boldsymbol{X}$, $\boldsymbol{Y}$ are both* ANR*-systems, then $\boldsymbol{C}_f$ is an* ANR*-system as well.*

Proof. By the Borsuk Theorem on the matching of ANRs (see [3], p. 116), $C_{f_\beta} \in \mathrm{ANR}$ for every $\beta \in \mathfrak{B}$. Thus $\boldsymbol{C}_f = (C_{f_\beta}, r_\beta^{\beta'}, \mathfrak{B})$ is an ANR-system.

Obviously

3.3. *If all the maps $p_\alpha^{\alpha'}$, $q_\beta^{\beta'}$ are inclusions, then $r_\beta^{\beta'}$ are inclusions as well.*

4. Fox Theorem for inverse systems. Consider a usual cofinal map of inverse systems, $\boldsymbol{f} = (\varphi, f_\beta)\colon \boldsymbol{X} \to \boldsymbol{Y}$, and let $\boldsymbol{Z}$ be the mapping cylinder of $\boldsymbol{f}$. We define the following two maps $\boldsymbol{i}\colon \boldsymbol{X} \to \boldsymbol{Z}$, $\boldsymbol{j}\colon \boldsymbol{Y} \to \boldsymbol{Z}$:

$$\boldsymbol{i} = (\varphi, i_\beta), \quad i_\beta\colon X_{\varphi(\beta)} \to Z_\beta, \quad i_\beta(x) = [x, 0] \quad \text{for} \quad x \in X_{\varphi(\beta)},\ \beta \in \mathfrak{B},$$

$$\boldsymbol{j} = (1_{\mathfrak{B}}, j_\beta), \quad j_\beta\colon Y_\beta \to Z_\beta, \quad j_\beta(y) = [y] \quad \text{for} \quad y \in Y_\beta,\ \beta \in \mathfrak{B} .$$

Let us notice that

4.1. *Both $\boldsymbol{i}, \boldsymbol{j}$ are usual maps.*

Proof. Consider the diagrams

$$\begin{array}{ccc} X_{\varphi(\beta)} & \xleftarrow{p_{\varphi(\beta)}^{\varphi(\beta')}} & X_{\varphi(\beta')} \\ {\scriptstyle i_\beta}\downarrow & & \downarrow{\scriptstyle i_{\beta'}} \\ Z_\beta & \xleftarrow[r_\beta^{\beta'}]{} & Z_{\beta'} \end{array}, \qquad \begin{array}{ccc} Y_\beta & \xleftarrow{q_\beta^{\beta'}} & Y_{\beta'} \\ {\scriptstyle j_\beta}\downarrow & & \downarrow{\scriptstyle j_{\beta'}} \\ Z_\beta & \xleftarrow[r_\beta^{\beta'}]{} & Z_{\beta'} \end{array} \quad \text{for} \quad \beta' \geqslant \beta .$$

We have

$$i_\beta p_{\varphi(\beta)}^{\varphi(\beta')}(x) = [p_{\varphi(\beta)}^{\varphi(\beta')}, 0] = r_\beta^{\beta'}[x, 0] = r_\beta^{\beta'} i_{\beta'}(x) \quad \text{for every} \quad x \in X_{\varphi(\beta')},$$

and

$$j_\beta q_\beta^{\beta'}(y) = [q_\beta^{\beta'}(y)] = r_\beta^{\beta'}[y] = r_\beta^{\beta'} j_{\beta'}(y) \quad \text{for every} \quad y \in Y_{\beta'} .$$

So $\boldsymbol{i}, \boldsymbol{j}$ are usual maps.

We are going to generalize the Fox Theorem to the inverse systems (4.5) [4]. First, let us establish two lemmas.

4.2. *$\boldsymbol{i} \simeq \boldsymbol{jf}$.*

Proof. We have to show that

$$\bigwedge_\beta \bigvee_{\alpha \geqslant \varphi(\beta)} i_\beta p_{\varphi(\beta)}^{\alpha} \simeq j_\beta f_\beta p_{\varphi(\beta)}^{\alpha}, \qquad \begin{tikzcd} X_{\varphi(\beta)} \arrow[d, "f_\beta"'] \arrow[dd, bend left, "i_\beta"] & X_\alpha \arrow[l, "p_{\varphi(\beta)}^{\alpha}"'] \\ Y_\beta \arrow[d, "j_\beta"'] & \\ Z_\beta & \end{tikzcd}$$

Let us take $\alpha = \varphi(\beta)$ for any $\beta \in \mathfrak{B}$.

For $x \in X_{\varphi(\beta)}$, we have

$$i_\beta(x) = [x, 0], \qquad j_\beta f_\beta(x) = [f_\beta(x)] = [x, 1] .$$

Obviously, the map $\xi_\beta\colon X_{\varphi(\beta)} \times I \to Z_\beta$, defined by the formula $\xi_\beta(x, t) = [x, t]$, is the required homotopy. ■

4.3. *There exists a usual map $\boldsymbol{h}\colon \boldsymbol{Z} \to \boldsymbol{Y}$ which is a homotopy inverse of $\boldsymbol{j}\colon \boldsymbol{Y} \to \boldsymbol{Z}$.*

Proof. Let

$$\boldsymbol{h} = (1_{\mathfrak{B}}, h_\beta), \qquad h_\beta(z) \underset{\mathrm{Df}}{=} \begin{cases} f_\beta(x) & \text{for} \quad z = [x, t],\ x \in X_{\varphi(\beta)}, \\ y & \text{for} \quad z = [y], \quad x \in Y_\beta . \end{cases}$$

Consider the diagram

$$\begin{array}{ccc} Z_\beta & \xleftarrow{r_\beta^{\beta'}} & Z_{\beta'} \\ {\scriptstyle h_\beta}\downarrow & & \downarrow{\scriptstyle h_{\beta'}} \\ Y_\beta & \xleftarrow[q_\beta^{\beta'}]{} & Y_{\beta'} \end{array} .$$

We have

$$h_\beta r_\beta^{\beta'}(z) = \left\{ \begin{array}{lll} f_\beta p_{\varphi(\beta)}^{\varphi(\beta')}(x) & \text{for} & z = [x, t] \\ q_\beta^{\beta'}(y) & \text{for} & z = [y] \end{array} \right\} = \left\{ \begin{array}{lll} q_\beta^{\beta'} f_{\beta'}(x) & \text{for} & z = [x, t] \\ q_\beta^{\beta'}(y) & \text{for} & z = [y] \end{array} \right\}$$

$$= q_\beta^{\beta'} h_{\beta'}(z) \quad \text{for} \quad \beta' \geqslant \beta .$$

Thus $\boldsymbol{h}$ is a usual map.

[4] The proof of 4.5 is a modification of the proof given by Fox in [5].

The map $\boldsymbol{h}$ is a left homotopy inverse of $\boldsymbol{j}$ (moreover, it is a left inverse):

$$\boldsymbol{hj} = (1_{\mathfrak{B}}, h_\beta j_\beta), \quad h_\beta j_\beta(y) = y; \quad \text{so } \boldsymbol{hj} = \boldsymbol{1}_Y \quad \text{and thus } \boldsymbol{hj} \simeq \boldsymbol{1}_Y.$$

Finally, prove $\boldsymbol{h}$ to be a right homotopy inverse of $\boldsymbol{j}$. We have

$$\boldsymbol{jh} = (1_{\mathfrak{B}}, j_\beta h_\beta),$$

$$j_\beta h_\beta(z) = \begin{Bmatrix} [f_\beta(x)] & \text{for} & z = [x, t] \\ [y] & \text{for} & z = [y] \end{Bmatrix} = \begin{cases} [x, 1] & \text{for} \quad z = [x, t], \\ [y] & \text{for} \quad z = [y]. \end{cases}$$

Since the map $\zeta_\beta\colon Z_\beta \times I \to Z_\beta$ defined by the formula

$$\zeta_\beta(z, s) \underset{\mathrm{Df}}{=} \begin{cases} [x, s+t(1-s)] & \text{for} \quad z = [x, t], \\ [y] & \text{for} \quad z = [y], \end{cases} \qquad \beta \in B,$$

satisfies the conditions: $\zeta_\beta(z, 0) = z$, $\zeta_\beta(z, 1) = j_\beta h_\beta(z)$, we have $j_\beta h_\beta \simeq 1_{Z_\beta}$ for every $\beta \in \mathfrak{B}$. Thus $\boldsymbol{jh} \simeq \boldsymbol{1}_{\boldsymbol{Z}}$. ■

By 4.2 and 4.3 it follows that

4.4. *$\boldsymbol{f}$ is a homotopy equivalence $\Leftrightarrow$ $\boldsymbol{i}$ is a homotopy equivalence.*

Proof. By 4.3, there is a map $\boldsymbol{h}\colon \boldsymbol{Z} \to \boldsymbol{Y}$ which is a homotopy inverse of $\boldsymbol{j}$; thus $\boldsymbol{j}, \boldsymbol{h}$ are both homotopy equivalences.

By 4.2, we have

$$\boldsymbol{i} \simeq \boldsymbol{jf}; \tag{1}$$

and thus $\boldsymbol{hi} \simeq \boldsymbol{hjf}$. Since $\boldsymbol{hj} \simeq \boldsymbol{1}_Y$, we get

$$\boldsymbol{f} \simeq \boldsymbol{hi}. \tag{2}$$

Obviously the composition of two homotopy equivalences is a homotopy equivalence again; hence, by (1), we obtain the implication $\Rightarrow$, and, by (2), the implication $\Leftarrow$. ■

Remark. The maps i_β, j_β are both topological imbeddings for every $\beta \in \mathfrak{B}$. Thus, identifying $[x, 0]$ with x and $[y]$ with y for $x \in X_{\varphi(\beta)}$, $y \in Y_\beta$, one can assume $X_{\varphi(\beta)}, Y_\beta$ to be subsets of Z_β. Then the maps $\boldsymbol{i}, \boldsymbol{j}$ are both inclusions.

By the statements 2.3, 4.4 and the above remark, we obtain the following

4.5. Corollary ([5]). *Let $\boldsymbol{X}$ be an inclusion-ANR-system, $\boldsymbol{Y}$ — an arbitrary inverse system, and $\boldsymbol{f}\colon \boldsymbol{X} \to \boldsymbol{Y}$ a usual cofinal map. Then $\boldsymbol{f}$ is a homotopy equivalence iff $\boldsymbol{X}$ is a deformation retract of the mapping cylinder $C_{\boldsymbol{f}}$.*

([5]) If $X_{\varphi(\beta)}, Y_\beta$ are not assumed to be subsets of Z_β, then Corollary 4.5 should be formulated as follows: $\boldsymbol{f}$ is a homotopy equivalence iff $\boldsymbol{X}$ is isomorphic (in the sense of the category $\hat{\mathfrak{T}}^*_{=}$) to a deformation retract of the mapping cylinder $C_{\boldsymbol{f}}$.

5. Convenient absolute retracts. Let us recall the basic notions of Borsuk's shape theory (see [1], [2], [4]). Consider two pairs of compacta (M, X), (N, Y), the spaces M, N being arbitrary ARs. The sequence of maps $f^n\colon M \to N$ is said to be a *fundamental sequence from X to Y relative to (M, N)* (in symbols $\underline{f} = (f^n, X, Y)_{M,N}$) whenever for any neighbourhood V of Y in N there exist a neighbourhood U of X in M and an index n_0 such that

$$f^n|U \simeq f^{n+1}|U \quad \text{in } V \text{ for every } n > n_0.$$

Two fundamental sequences $\underline{f} = (f^n, X, Y)_{M,N}$, $\underline{f}' = (f'^n, X, Y)_{M,N}$ are said to be *homotopic* ($\underline{f} \simeq \underline{f}'$) iff for any neighbourhood V of Y in N there are a neighbourhood U of X in M and an index n_0 such that

$$f^n|U \simeq f'^n|U \quad \text{in } V \text{ for } n > n_0.$$

Two compacta X, Y are said to be of the *same shape*, ($X \underset{F}{\simeq} Y$), iff there exist $M, N \in \mathrm{AR}$ and two fundamental sequences $\underline{f} = (f^n, X, Y)_{M,N}$, $\underline{g} = (g^n, Y, X)_{N,M}$ such that $\underline{fg} \simeq \underline{1}_X$, $\underline{gf} \simeq \underline{1}_Y$ (i.e. $\underline{f}$ is a *fundamental equivalence rel. M, N*).

X is a *fundamental retract* of $\hat{X}$ iff there exist $M, \hat{M} \in \mathrm{AR}$ and a fundamental sequence $\underline{r} = (r^n, \hat{X}, X)_{\hat{M},M}$ such that $r^n(x) = x$ for every $x \in X$ (i.e. $\underline{r}$ is a *fundamental retraction rel. $\hat{M}, M$*). If, moreover, $\underline{r}$ satisfies the condition $\underline{ir} \simeq \underline{1}_{\hat{X}}$, $\underline{i} = (i, X, \hat{X})_{M,\hat{M}}$ being the inclusion (i.e. $i(x) = x$ for every $x \in M$) then $\underline{r}$ is a *fundamental deformational retraction* and X is a *fundamental deformation retract* of $\hat{X}$.

S. Mardešić and J. Segal in [7] are concerned with the case of $M = N = Q$, Q being the Hilbert cube. They study the so called inclusion-ANR-sequences associated with compact subsets of Q. One can easily verify that the only properties of the Hilbert cube needed in [7] are the following two:

1° $Q \in \mathrm{AR}$,

2° for every compact subset X of Q there is a decreasing sequence of ANRs, $\{X_n\}_{n=1,2,\ldots}$, such that

$$X = \bigcap_{n=1}^{\infty} X_n, \qquad \text{every } X_n \text{ being a neighbourhood of } X \text{ in } Q.$$

Hence, all the results of [7] remain valid if any pair (Q, X) is replaced by the pair (M, X), where M satisfies both of the conditions 1°, 2°. Such a space M will be referred to as a *convenient absolute retract* (in symbols $M \in \mathrm{CAR}$) (⁶).

Obviously

5.1. CAR *is a topological invariant.*

We are interested in preserving the class CAR under the operation of matching (see [3], p. 116). We start with proving three Lemmas 5.2–5.4.

Consider two arbitrary disjoint spaces M, N, the closed subset M_0 of M and the map $g\colon M_0 \to N$.

Let $\pi\colon M \cup N \to M \underset{g}{\cup} N$ be the natural projection.

5.2. *If $X = \overline{X} \subset M$, $Y = \overline{Y} \subset N$, the sets U, V are neighbourhoods of X, Y in M, N respectively, and $X \cap M_0 = g^{-1}(Y)$, $U \cap M_0 = g^{-1}(V)$, then for any open neighbourhood V' of Y in N such that $V' \subset V$ there is an open neighbourhood U' of X in M such that $U' \subset U$ and $\pi^{-1}\pi(U') \subset U' \cup V'$.*

Proof. Take an open neighbourhood V' of Y in N such that $V' \subset V$, and put $U_0 \underset{\mathrm{Df}}{=} g^{-1}(V')$. Since g is continuous, the set U_0 is open in M_0. Since $X \cap M_0 = g^{-1}(Y)$ and $U \cap M_0 = g^{-1}(V)$, we have

$$X \cap M_0 \subset U_0 \subset U. \tag{1}$$

Notice that

$$\pi^{-1}\pi(U_0) \subset U_0 \cap V' \tag{2}$$

Let U'' be an open neighbourhood of X in M, $U'' \subset U$. Put

$$U' = U_0 \cup [(M - M_0) \cap U'']. \tag{3}$$

The set U is open in M;

$$X = (M_0 \cap X) \cup [(M-M_0) \cap X] \subset U_0 \cup [(M-M_0) \cap U''] = U'$$

and $U' \subset U$;

moreover,

$$\pi^{-1}\pi(U') = \pi^{-1}\pi(U_0) \cup \pi^{-1}\pi\big((M-M_0) \cap U''\big),$$

where

$\pi^{-1}\pi(U_0) \subset U_0 \cup V'$ by (2), and $\pi^{-1}\pi\big((M-M_0) \cap U''\big) = (M - M_0) \cap U''$

since $\pi | M - M_0$ is a topological imbedding. Thus

$$\pi^{-1}\pi(U') \subset U_0 \cup V' \cup [(M-M_0) \cap U''] = U' \cup V'. \ \blacksquare$$

5.3. *W is a neighbourhood of a compactum Z in $M \underset{g}{\cup} N \Leftrightarrow \pi^{-1}(W)$ is a neighbourhood of $\pi^{-1}(Z)$ in $M \cup N$.*

(⁶) As was shown by K. Borsuk in [3], p. 156, there exist absolute retracts which are not convenient ones.

Proof. The implication $\Rightarrow$ is an immediate consequence of the continuity of π.

Let us prove $\Leftarrow$. Assume $\pi^{-1}(W)$ to be a neighbourhood of $\pi^{-1}(Z)$ in $M \cup N$. Put

$$X = \pi^{-1}(Z) \cap M, \quad Y = \pi^{-1}(Z) \cap N$$

and

$$U = \pi^{-1}(W) \cap M, \quad V = \pi^{-1}(W) \cap N.$$

Since $M \cap N = O$, U is a neighbourhood of X in M, and V is a neighbourhood of Y in N. Moreover,

$$X \cap M_0 = g^{-1}(Y) \quad \text{and} \quad U \cap M_0 = g^{-1}(V).$$

Let V' be an open neighbourhood of Y in N such that $V' \subset V$. By 5.2, there is an open neighbourhood U' of X in M such that $U' \subset U$ and $\pi^{-1}\pi(U') \subset U' \cup V'$.

Let $W' \underset{\mathrm{Df}}{=} U' \cup V'$. The set W' is an open subset of $M \cup N$ and $\pi^{-1}(Z) \subset W' \subset \pi^{-1}(W)$; so $Z \subset \pi(W') \subset W$. Let us show that $\pi(W')$ is open in $M \underset{g}{\cup} N$. In fact, by the definition of quotient topology, it suffices to prove that $\pi^{-1}\pi(W') = W'$. Since $\pi^{-1}\pi(W') = \pi^{-1}\pi(U') \cup \pi^{-1}\pi(V')$, $\pi|N$ being a topological imbedding, we have $\pi^{-1}\pi(W') \subset U' \cup V' = W'$; thus $\pi^{-1}\pi(W') = W'$.

Hence, $\pi(W')$ is an open neighbourhood of Z contained in W, and therefore W is a neighbourhood of Z in $M \underset{g}{\cup} N$. $\blacksquare$

5.4. *Let Z be a compact subset of $M \underset{g}{\cup} N$, $X = \pi^{-1}(Z) \cap M$, $Y = \pi^{-1}(Z) \cap N$. If $X_n \subset M$, $Y_n \subset N$, $g(X_n \cap M_0) \subset Y_n$ for $n = 1, 2, \ldots$, $X = \bigcap_{n=1}^{\infty} X_n$, $Y = \bigcap_{n=1}^{\infty} Y_n$ and $Z_n = \pi(X_n \cup Y_n)$, then $Z = \bigcap_{n=1}^{\infty} Z_n$.*

Proof. We have

$$\begin{aligned} Z &= \pi(X \cup Y) \\ &= \pi\Big(\bigcap_{n=1}^{\infty} X_n \cup \bigcap_{n=1}^{\infty} Y_n\Big) \subset \pi\Big(\bigcap_{n=1}^{\infty} (X_n \cup Y_n)\Big) \subset \bigcap_{n=1}^{\infty} \pi(X_n \cup Y_n) = \bigcap_{n=1}^{\infty} Z_n. \end{aligned}$$

On the other hand, since $g(X_n \cap M_0) \subset Y_n$, we have

$$\begin{aligned} \pi^{-1}\Big(\bigcap_{n=1}^{\infty} Z_n\Big) &= \bigcap_{n=1}^{\infty} \big(\pi^{-1}\pi(X_n \cup Y_n)\big) = \bigcap_{n=1}^{\infty} \big(\pi^{-1}\pi(X_n) \cup \pi^{-1}\pi(Y_n)\big) \\ &= \bigcap_{n=1}^{\infty} \big(Y_n \cup g^{-1}(Y_n)\big) = \bigcap_{n=1}^{\infty} Y_n \cup \bigcap_{n=1}^{\infty} g^{-1}(Y_n) \\ &= Y \cup g^{-1}(Y) \subset X \cup Y = \pi^{-1}(Z); \end{aligned}$$

thus $\bigcap_{n=1}^{\infty} Z_n \subset Z$. $\blacksquare$

It is useful to generalize the notion of CAR to pairs of spaces. For $M_0 \neq \emptyset$:

$(M, M_0) \in \mathrm{CAR} \underset{\mathrm{Df}}{\Leftrightarrow} M \in \mathrm{AR}$, $M_0 \in \mathrm{AR}$ and for every compact pair $(X, X_0) \subset (M, M_0)$ such that $X_0 = X \cap M_0$, there exists a decreasing sequence of pairs $(X_n, X_{n0}) \subset (M, M_0)$ such that $X_{n0} = X_n \cap M_0$, X_n is a neighbourhood of X in M, moreover $X_n, X_{n0} \in \mathrm{ANR}$ and $X = \bigcap_{n=1}^{\infty} X_n$ (7);

$(M, \emptyset) \in \mathrm{CAR} \underset{\mathrm{Df}}{\Leftrightarrow} M \in \mathrm{CAR}$.

Obviously the following implication holds

5.5. *If $(M, M_0) \in \mathrm{CAR}$, then for every pair of neighbourhoods (U, U_0) of a compact pair (X, X_0) in (M, M_0) such that $U_0 = M_0 \cap U$ and $X_0 = M_0 \cap X$, there is a pair of neighbourhoods $(U', U_0') \subset (U, U_0)$ such that $U', U_0' \in \mathrm{ANR}$ and $U_0' = U_0 \cap U'$* (8).

Now, let us establish the theorem on the matching of CARs:

5.6. THEOREM. *Let $M_0 \subset M$ and $M \cap N = \emptyset$. For any map $g: M_0 \to N$, if $(M, M_0) \in \mathrm{CAR}$ and $N \in \mathrm{CAR}$ then $M \underset{g}{\cup} N \in \mathrm{CAR}$.*

Proof. By the Borsuk Theorem on the matching of ARs ([3], p. 121, (9.17)), $M \underset{g}{\cup} N \in \mathrm{AR}$.

Given a compactum $Z \subset M \underset{g}{\cup} N$, prove that there is a decreasing sequence of neighbourhoods $\{Z_n\}_{n=1,2,...}$ of Z in $M \underset{g}{\cup} N$ such that $Z_n \in \mathrm{ANR}$ for $n = 1, 2, \ldots$, and $Z = \bigcap_{n=1}^{\infty} Z_n$.

Let $X = \pi^{-1}(Z) \cap M$, $X_0 = \pi^{-1}(Z) \cap M_0$ and $Y = \pi^{-1}(Z) \cap N$, π: $M \cup N \to M \underset{g}{\cup} N$ being the natural projection. Since $N \in \mathrm{CAR}$, there is a decreasing sequence $\{Y_n\}_{n=1,2,...}$ of neighbourhoods of Y in N such that $Y_n \in \mathrm{ANR}$ for $n = 1, 2, \ldots$ and $Y = \bigcap_{n=1}^{\infty} Y_n$. By the continuity of g, for every n there is a neighbourhood $\tilde{X}_{n0}$ of X_0 in M_0 such that $g(\tilde{X}_{n0}) \subset Y_n$.

(7) Using the terminology introduced for pairs in [8], by Theorems A.1, B.1 of [8] we can express this definition as follows:

$(M, M_0) \in \mathrm{CAR} \underset{\mathrm{Df}}{\Longleftrightarrow} (M, M_0) \in_1 \mathrm{AR}$ and for every compact pair $(X, X_0) \subset (M, M_0)$ there is a decreasing sequence of neighbourhoods (X_n, X_{n0}) of (X, X_0) in (M, M_0) such that $(X_n, X_{n0}) \in_1 \mathrm{ANR}$ and $(\bigcap_{n=1}^{\infty} X_n, \bigcap_{n=1}^{\infty} X_{n0}) = (X, X_0)$.

(8) Using the terminology for pairs: if $(M, M_0) \in \mathrm{CAR}$, then for every neighbourhood (U, U_0) of a compact pair (X, X_0) in (M, M_0) there is a neighbourhood $(U', U_0') \subset (U, U_0)$ such that $(U', U_0') \in_1 \mathrm{ANR}$.

Let $\tilde{X}_n$ be a neighbourhood of X in M such that $\tilde{X}_n \cap M_0 = \tilde{X}_{n0}$. Since $(M, M_0) \in \mathrm{CAR}$, by (5.5) there is a pair (X_n, X_{n0}) of ANRs such that $X_n \subset \tilde{X}_n$ and $X_{n0} = X_n \cap M_0$, the set X_n being a neighbourhood of X in M. Obviously $\{X_n\}_{n=1,2,...}$ can be made decreasing. Let g_n: $X_{n0} \to Y_n$ be defined by the formula

$$g_n(x) = g(x) \quad \text{for } x \in X_{n0}.$$

Since

$$X_{n0} \subset X_n, \quad X_u \cap Y_n = \emptyset \quad \text{and} \quad X_n, X_{n0}, Y_n \in \mathrm{ANR},$$

thus by the theorem on the matching of ANRs ([3], p. 116),

$$X_n \underset{g_n}{\cup} Y_n \in \mathrm{ANR}, \quad \text{i.e.} \quad \pi(X_n \cup Y_n) \in \mathrm{ANR}.$$

Setting

$$Z_n \underset{\mathrm{Df}}{=} \pi(X_n \cup Y_n) \quad \text{for} \quad n = 1, 2, \ldots$$

we obtain the required sequence of ANRs, $\{Z_n\}_{n=1,2,...}$. In fact, we have $\pi^{-1}(Z_n) \supset X_n \cup Y_n$, so $\pi^{-1}(Z_n)$ is a neighbourhood of $X \cup Y$ in $M \cup N$ and thus, by 5.3. Z_n is a neighbourhood of Z in $M \underset{g}{\cup} N$; obviously $\{Z_n\}_{n=1,2,...}$ is decreasing and, by 5.4, $\bigcap_{n=1}^{\infty} Z_n = Z$. ■

Let us notice that the following condition is sufficient for (M, M_0) to be a CAR-pair:

(*) For any $\varepsilon > 0$ there exists a finite collection of pairs $(A_\nu, A_{\nu 0})$ such that:

$$M = \bigcup_\nu A_\nu, \quad \bigwedge_\nu M_0 \cap A_\nu = A_{\nu 0},$$

$$\bigwedge_{\nu,\nu'} [A_\nu \cap A_{\nu'} \in \mathrm{AR} \vee A_\nu \cap A_{\nu'} = \emptyset] \wedge [A_{\nu 0} \cap A_{\nu' 0} \in \mathrm{AR} \vee A_{\nu 0} \cap A_{\nu' 0} = \emptyset],$$

and

$$\bigwedge_\nu \delta(A_\nu) < \varepsilon.$$

Of course, the pair $(Q^n \times I, Q^n \times (1))$, Q^n being the n-dimensional cube, satisfies condition (*). On the other hand, for any $\varepsilon > 0$ there is a natural number n such that $Q = Q^n \times Q'$, where $Q' \underset{\mathrm{top}}{=} Q$ and $\delta(Q') < \varepsilon$.

Thus, we have the following example of a CAR-pair:

5.7. $(Q \times I, Q \times (1)) \in \mathrm{CAR}$.

By 5.6 and 5.7 we obtain

5.8. COROLLARY. *For any map f: $Q \to Q$*

$$C_f \in \mathrm{CAR}.$$

Proof. Obviously, $C_f = Q\times I \cup_g Q$, where $g\colon Q\times(1)\to Q$, $g(x,1)=f(x)$ for every $x\in Q$. Setting, in 5.6, $M=Q\times I$, $M_0=Q\times(1)$, $N=Q$, we infer that $C_f\in \mathrm{CAR}$. ■

6. ANR-sequence associated with a mapping cylinder. Let us generalize the notions of [7] as follows:

Given the pair (M, X), the space M being a convenient absolute retract and X — a compact subset of M, the inclusion ANR-sequence $\boldsymbol{X}=(X_n, p_n^{n'}, \mathcal{N})$ is said to be *associated with* X *in* M whenever $\{X_n\}_{n=1,2,\dots}$ is a decreasing sequence of neighbourhoods of X in M and $X=\bigcap_{n=1}^{\infty} X_n$. We write simply $\boldsymbol{X}=(X_n, p_n^{n'})$.

According to the definitions given in [7], $\boldsymbol{X}$ is associated with X iff $\boldsymbol{X}$ is associated with X in Q.

Let $\boldsymbol{X}, \boldsymbol{Y}$ be two inclusion-ANR-sequences associated with X, Y in M, N respectively, $\boldsymbol{X}=(X_n, p_n^{n'})$, $\boldsymbol{Y}=(Y_n, q_n^{n'})$. The regular map $\boldsymbol{f}=(\varphi, f_n)\colon \boldsymbol{X}\to\boldsymbol{Y}$ and the fundamental sequence $\underline{f}=(f^n, X, Y)_{M,N}$ are said to be *related* one to another provided the following two conditions are satisfied:

(i) $k, k'\geqslant \varphi(n) \Rightarrow f^k|X_{\varphi(n)}\simeq f^{k'}|X_{\varphi(n)}$ in Y_n,

(ii) $f_n(x)=f^{\varphi(n)}(x)$ for every $x\in X_{\varphi(n)}$.

6.1. Proposition. *Let X, Y be two compact subsets of the Hilbert cube Q. Let $\boldsymbol{X}$, $\boldsymbol{Y}$ be two inclusion-ANR-sequences associated with X, Y respectively and let $\underline{f}$ be a fundamental sequence (rel. (Q,Q)) generated by the map $f\colon X\to Y$. Then there exist a usual map $\boldsymbol{f}\colon \boldsymbol{X}\to\boldsymbol{Y}$ related to $\underline{f}$ and a space $R\in \mathrm{CAR}$ such that $\boldsymbol{C}_f$ is an inclusion-ANR-sequence associated with C_f in R.*

Proof. Since $\underline{f}$ is generated by $f\colon X\to Y$, it is of the form $\underline{f}=(\tilde{f}, X, Y)$, the map $\tilde{f}\colon Q\to Q$ being an extension of f. By Lemma 2 of [7], there is a map of sequences $\boldsymbol{f}=(\varphi, f_n)\colon \boldsymbol{X}\to\boldsymbol{Y}$ related to $\underline{f}$, i.e. $f_n(x)=\tilde{f}(x)$ for $x\in X_{\varphi(n)}$.

Since $p_n^{n'}, q_n^{n'}$ are inclusions, we have

$$f_n p_{\varphi(n)}^{\varphi(n')}(x)=\tilde{f}(x)=q_n^{n'} f_{n'}(x) \quad \text{for} \quad x\in X_{\varphi(n')};$$

thus $\boldsymbol{f}$ is usual.

Let us put

$$R \underset{\mathrm{Df}}{=} C_{\tilde{f}}\,.$$

It follows by 5.8, that $R\in\mathrm{CAR}$.

Prove $\boldsymbol{C}_f$ to be an inclusion-ANR-sequence associated with C_f in R.

In fact, by 3.1–3.3, $\boldsymbol{C}_f$ is an inclusion-ANR-sequence. It remains to show that $\boldsymbol{C}_f$ is associated with C_f in R, i.e. $C_f=\bigcap_{n=1}^{\infty} C_{f_n}$, $\{C_{f_n}\}_{n=1,2,\dots}$ being a decreasing sequence of neighbourhoods of C_f in R. It follows by 5.4 that

$$C_f=\bigcap_{n=1}^{\infty} C_{f_n}\,.$$

Since $\{X_n\}, \{Y_n\}$ are both decreasing, $\{C_{f_n}\}$ is decreasing as well.

Finally, by 5.3, since $X_n\times I$ is a neighbourhood of $X\times I$ in $Q\times I$ and Y is a neighbourhood of Y in Q, the set C_{f_n} is a neighbourhood of C_f in R, which completes the proof.

7. Retracts of ANR-systems and fundamental retracts of compacta. We are going to establish the connection between the notion of retract of ANR-system (see § 2) and that of fundamental retract (see [1], [2], or § 5 of this paper). We start with proving proposition 7.1, which states the connection between the notion of inclusion of ANR-system, as defined in § 2, and the notion of inclusion in the sense of Borsuk (see [2]). Given two convenient absolute retracts $M, \hat{M}$, let us consider the inclusion ANR-sequences $\boldsymbol{X}=(X_n, p_n^{n'})$, $\hat{\boldsymbol{X}}=(\hat{X}_n, \hat{p}_n^{n'})$ associated with the compacta $X, \hat{X}$ in $M, \hat{M}$ respectively. Recall that the fundamental sequence $\underline{i}=(i^n, X, \hat{X})_{M,\hat{M}}$ is said to be an *inclusion* (in the sense of Borsuk) whenever $i^n(x)=x$ for every $x\in M$ (then obviously $M\subset\hat{M}$).

7.1. *If a regular map $\boldsymbol{i}=(\tau, i_n)\colon \boldsymbol{X}\to\hat{\boldsymbol{X}}$ is related to a fundamental sequence $\underline{i}=(i^n, X, \hat{X})_{M,\hat{M}}$ then*

(1) *$\underline{i}$ is the inclusion in the sense of Borsuk $\Rightarrow$ $\boldsymbol{i}$ is an inclusion,*

(2) *$\boldsymbol{i}$ is an inclusion $\Rightarrow$ $\underline{i}$ is homotopic to the inclusion in the sense of Borsuk.*

Proof. Since $\boldsymbol{i}$ and $\underline{i}$ are related one to another, we have $i_n(x)=i^{\tau(n)}(x)$ for every $x\in X_{\tau(n)}$.

(1): Let $\underline{i}$ be the inclusion in the sense of Borsuk, i.e. $i^n(x)=x$ for every $x\in M$. Then $i_n(x)=x$ for $x\in X_{\tau(n)}$, and thus i_n are inclusions. By 2.1, since $\boldsymbol{i}$ is regular, it is cofinal. Moreover, since $p_n^{n'}$ are inclusions, $\boldsymbol{i}$ is usual. Thus $\boldsymbol{i}$ is an inclusion.

(2): Let $\boldsymbol{i}$ be an inclusion, i.e. $i_n(x)=x$ for $x\in X_{\tau(n)}$. Let $i_M\colon M\to\hat{M}$ be the inclusion. Obviously, the inclusion $\underline{i}'=(i_M, X, \hat{X})_{M,\hat{M}}$ is related to $\boldsymbol{i}$. So $\underline{i}, \underline{i}'$ are both related to $\boldsymbol{i}$ and thus, by Lemma 5 of [7], $\underline{i}\simeq\underline{i}'$. ■

Let us prove that

7.2. *If a regular map $\boldsymbol{r}=(\varrho, r_n)\colon \hat{\boldsymbol{X}}\to\boldsymbol{X}$ is related to a fundamental sequence $\underline{r}=(r^n, \hat{X}, X)_{\hat{M},M}$, $\boldsymbol{i}=(\tau, i_n)\colon \boldsymbol{X}\to\hat{\boldsymbol{X}}$ is an inclusion and $p_n\colon X\to X_n$ are inclusions for $n=1,2,\dots$, then*

$$[\bigwedge_n r_n i_{\varrho(n)} p_{\tau\varrho(n)} = p_n]$$

$$\Rightarrow [\underline{r} \textit{ has a subsequence } \underline{r}' \textit{ which is a fundamental retraction}]\,.$$

Proof. By the assumption, $r_n i_{\varrho(n)} p_{r\varrho(n)}(x) = p_n(x)$ for every $x \in X$, the map $p_n\colon X \to X_n$ being an inclusion for every n; thus $r_n(x) = x$ for every $x \in X$. Since r and $\underline{r}$ are related one to another, $r_n(x) = r^{\varrho(n)}(x)$ for every $x \in X_{\varrho(n)}$ and thus $r_n(x) = r^{\varrho(n)}(x)$ for $x \in X$.

Let $\underline{r}' \underset{\mathrm{Df}}{=} (r^{\varrho(n)}, \hat{X}, X)_{\hat{M},M}$. Since r is regular, $\underline{r}'$ is a subsequence of $\underline{r}$. So as $r^{\varrho(n)}(x) = r_n(x) = x$ for every $x \in X$, $\underline{r}'$ is a fundamental retraction. ■

7.3. *If a regular map $r = (\varrho, r_n)\colon \hat{X} \to X$ is related to a fundamental sequence $\underline{r} = (r^n, \hat{X}, X)_{\hat{M},M}$, then $\underline{r}$ has a subsequence $\underline{r}'$ such that*

(i) *r is a retraction $\Rightarrow$ $\underline{r}'$ is a fundamental retraction,*

(ii) *r is a deformational retraction $\Rightarrow$ $\underline{r}'$ is a deformational fundamental retraction.*

Proof. By 2.4 and 7.2 we obtain (i).

Prove (ii). Let $i\colon X \to \hat{X}$ be an inclusion and let $\underline{i}$ be a related fundamental sequence. By 7.1, $\underline{i}$ is homotopic to the inclusion $\underline{i}'$ in the sense of Borsuk.

If r is a deformational retraction, then r is a retraction and $ir \simeq 1_{\hat{X}}$. Then, by (i), $\underline{r}$ has a subsequence $\underline{r}'$ which is a fundamental retraction; moreover, by Lemma 6 of [7], $\underline{ir} \simeq \underline{1}_{\hat{X}}$. Thus, by the statement (3.4) of [1], $\underline{i}'\underline{r}' \simeq \underline{1}_{\hat{X}}$. Hence $\underline{r}'$ is a deformational fundamental retraction. ■

Finally, let us establish

7.4. Theorem. *Let $M, \hat{M} \in$ CAR. Let $X, \hat{X}$ be two inclusion-ANR-sequences associated with the compacta $X, \hat{X}$ in $M, \hat{M}$ respectively. Then X is a (deformation) retract of $\hat{X} \Leftrightarrow X$ is a (deformation) fundamental retract of $\hat{X}$.*

Proof. By the statement 7.3 we obtain the implication $\Rightarrow$.

Let us prove $\Leftarrow$.

Assume X to be a fundamental retract of $\hat{X}$. Since the notion of retract does not depend on the choice of $M, \hat{M}$, there are an inclusion $\underline{i} = (i^n, X, \hat{X})_{M,\hat{M}}$ and a fundamental retraction $\underline{r} = (r^n, \hat{X}, X)_{\hat{M},M}$. As proved in [2], $\underline{ri} \simeq \underline{1}_X$. Let $i\colon X \to \hat{X}$, $r\colon \hat{X} \to X$ be related to $\underline{i}, \underline{r}$. Then, by 7.1, i is an inclusion; on the other hand, by Lemma 6 of [7], $ri \simeq 1_X$. Thus, by 2.3, X is a retract of $\hat{X}$.

If, moreover, $\underline{ir} \simeq \underline{1}_{\hat{X}}$, then $ir \simeq 1_{\hat{X}}$, and thus X is a deformation retract of $\hat{X}$. ■

8. Fox Theorem in the Theory of Shape. Let us establish the following

8.1. Theorem. *Let X, Y be two compacta. The map $f\colon X \to Y$ generates a fundamental equivalence iff X is homeomorphic to a fundamental deformation retract of the mapping cylinder C_f.*

Proof. We can assume X, Y to be subsets of the Hilbert cube Q. Let $Z = C_f$; we can also assume X to be a subset of Z.

Let $\underline{f}$ be a fundamental sequence generated by f. Take inclusion-ANR-sequences X, Y associated with X, Y respectively. By 6.1, there is a usual map $f\colon X \to Y$ related to $\underline{f}$ and a space $R \in$ CAR such that C_f is an inclusion-ANR-sequence associated with Z in R. This maps f can easily be made cofinal. Let $Z = C_f$. Since $\underline{f}$ and f are related one to another, it follows by the arguments used in the proof of Theorem in [7] that

(1) $\underline{f}$ is a fundamental equivalence $\Leftrightarrow$ f is a homotopy equivalence.

By 4.5,

(2) f is a homotopy equivalence $\Leftrightarrow$ X is a deformation retract of Z.

Finally, setting in 7.4 $M = Q$, $\hat{M} = R$, $\hat{X} = Z$, $\hat{X} = Z$, we obtain

(3) X is a deformation retract of $Z \Leftrightarrow X$ is a deformation fundamental retract of Z.

Thus the proof is complete. ■

Appendix. It is known that for any usual map $f\colon X \to Y$ there exists a limit map $f = \underleftarrow{\lim} f\colon \underleftarrow{\lim} X \to \underleftarrow{\lim} Y$. Take

$$X = (X_\alpha, p_\alpha^{\alpha'}, \mathcal{A}), \quad Y = (Y_\beta, q_\beta^{\beta'}, \mathcal{B}), \quad f = (\varphi, f_\beta)$$

and let $X = \underleftarrow{\lim} X$, $Y = \underleftarrow{\lim} Y$. Then

$$X = \Big\{\{x_\alpha\} \in \mathsf{P}_{\alpha \in \mathcal{A}} X_\alpha\colon \bigwedge_{\alpha' \geqslant \alpha} p_\alpha^{\alpha'}(x_{\alpha'}) = x_\alpha\Big\}, \qquad Y = \Big\{\{y_\beta\} \in \mathsf{P}_{\beta \in \mathcal{B}} Y_\beta\colon \bigwedge_{\beta' \geqslant \beta} q_\beta^{\beta'}(y_{\beta'}) = y_\beta\Big\}$$

and the map $f\colon X \to Y$ is defined by the formula $f(\{x_\alpha\}) = \{f_\beta(x_{\varphi(\beta)})\}$. Moreover, if $p_\alpha\colon X \to X_\alpha$, $q_\beta\colon Y \to Y_\beta$ are projections, then f is a unique map satisfying the condition: $q_\beta f = f_\beta p_{\varphi(\beta)}$ for every $\beta \in \mathcal{B}$.

We are now concerned in the properties of the limit map.

Proposition 1. *For any two usual maps $f, f'\colon X \to Y$*

$$f \cong f' \Rightarrow \underleftarrow{\lim} f = \underleftarrow{\lim} f'.$$

Proof. Take two usual maps $f, f'\colon X \to Y$, $f = (\varphi, f_\beta)$, $f' = (\varphi', f'_\beta)$ and let $X = \underleftarrow{\lim} X$, $Y = \underleftarrow{\lim} Y$, $f = \underleftarrow{\lim} f$.

Take $p_\alpha\colon X \to X_\alpha$, $q_\beta\colon Y \to Y_\beta$. We have

$$\bigwedge_{\alpha' \geqslant \alpha} p_\alpha = p_\alpha^{\alpha'} p_{\alpha'} \quad \text{and} \quad \bigwedge_{\beta' \geqslant \beta} q_\beta = q_\beta^{\beta'} q_{\beta'}.$$

The map f is a unique one satisfying the condition $q_\beta f = f_\beta p_{\varphi(\beta)}$ for every $\beta \in \mathcal{B}$;

the map f' is a unique one satisfying the condition $q_\beta f' = f'_\beta p_{\varphi'(\beta)}$ for every $\beta \in \mathcal{B}$.

Take an arbitrary $\beta \in B$. Since $\boldsymbol{f} \simeq \boldsymbol{f}'$, there is an $\alpha \geqslant \varphi(\beta), \varphi'(\beta)$ such that

$$f_\beta p^\alpha_{\varphi(\beta)} = f'_\beta p^\alpha_{\varphi'(\beta)}\,.$$

We thus have

$$q_\beta f = f_\beta p_{\varphi(\beta)} = (f_\beta p^\alpha_{\varphi(\beta)})p_\alpha = (f'_\beta p^\alpha_{\varphi'(\beta)})p_\alpha = f'_\beta p_{\varphi'(\beta)} = q_\beta f'\,.$$

Hence, by the uniqueness of f, we obtain $f = f'$. ■

Proposition 2. *If i: $X \to \hat{X}$ is an inclusion, then there exist $\boldsymbol{X}, \hat{\boldsymbol{X}}$ such that $X = \varprojlim \boldsymbol{X}$, $\hat{X} = \varprojlim \hat{\boldsymbol{X}}$ and $\varprojlim \boldsymbol{i}$: $X \to \hat{X}$ is an inclusion.*

Proof. Let $\boldsymbol{X} = (X_\alpha, p^{\alpha'}_\alpha, \mathcal{A})$, $\hat{\boldsymbol{X}} = (\hat{X}_\alpha, \hat{p}^{\alpha'}_\alpha, \hat{\mathcal{A}})$. Take the inclusion $\boldsymbol{i}$: $\boldsymbol{X} \to \hat{\boldsymbol{X}}$, $\boldsymbol{i} = (\tau, i_\beta)$. Put

$$X = \{\{x_\alpha\} \in \mathop{\mathsf{P}}_{\alpha \in \tau(\hat{\mathcal{A}})} X_\alpha \colon \bigwedge_{\alpha' \geqslant \alpha} p^{\alpha'}_\alpha(x_{\alpha'}) = x_\alpha\}, \qquad \hat{X} = \{\{x_\alpha\} \in \mathop{\mathsf{P}}_{\alpha \in \hat{\mathcal{A}}} \hat{X}_\alpha \colon \bigwedge_{\alpha' \geqslant \alpha} \hat{p}^{\alpha'}_\alpha(x_{\alpha'}) = x_\alpha\}.$$

Since $\boldsymbol{i}$ is cofinal, $\tau(\hat{\mathcal{A}})$ is cofinal with $\mathcal{A}$ and thus $X = \varprojlim \boldsymbol{X}$. Obviously, $\hat{X} = \varprojlim \hat{\boldsymbol{X}}$.

Let $i = \varprojlim \boldsymbol{i}$: $X \to \hat{X}$; we have $i\{x_{\tau(\beta)}\} = \{i_\beta(x_{\tau(\beta)})\} = \{x_{\tau(\beta)}\}$ for every $\{x_{\tau(\beta)}\} \in X$, and so i is an inclusion. ■

Proposition 3. *If $\boldsymbol{r}$: $\hat{X} \to X$ is a usual retraction, then there exist $\boldsymbol{X}, \hat{\boldsymbol{X}}$ such that $X = \varprojlim \boldsymbol{X}$, $\hat{X} = \varprojlim \hat{\boldsymbol{X}}$ and $\varprojlim \boldsymbol{r}$: $\hat{X} \to X$ is a retraction.*

Proof. Take the usual retraction $\boldsymbol{r}$: $\hat{\boldsymbol{X}} \to \boldsymbol{X}$; by the definition, there is an inclusion $\boldsymbol{i}$: $\boldsymbol{X} \to \hat{\boldsymbol{X}}$ such that $\boldsymbol{ri} \simeq \boldsymbol{1_X}$. By Proposition 2, there are $\boldsymbol{X}, \hat{\boldsymbol{X}}$ which are inverse limits of $\boldsymbol{X}, \hat{\boldsymbol{X}}$ and are such that $i = \varprojlim \boldsymbol{i}$: $X \to \hat{X}$ is an inclusion. By Proposition 1, since $\varprojlim$ is a covariant functor, we obtain $ri = 1_X$, and so r is a retraction. ■

Now, let us establish the connection between the mapping cylinder of $\boldsymbol{f}$ and the mapping cylinder of $\varprojlim \boldsymbol{f}$.

Proposition 4. *If $\boldsymbol{f}$: $\boldsymbol{X} \to \boldsymbol{Y}$ is a usual cofinal map and $f = \varprojlim \boldsymbol{f}$, then $C_f = \varprojlim \boldsymbol{C_f}$.*

Proof. Let $Z = C_f$, $\boldsymbol{Z} = \boldsymbol{C_f}$. We have to find a homeomorphism χ: $\varprojlim \boldsymbol{Z} \to Z$.

Recall that

1° $z \in \varprojlim \boldsymbol{Z} \Leftrightarrow z = \{z_\beta\} \in \mathop{\mathsf{P}}_{\beta \in \mathcal{B}} Z_\beta, r^{\beta'}_\beta(z_{\beta'}) = z_\beta$ for $\beta' \geqslant \beta$, where $z_\beta = [x_{\varphi(\beta)}, t]$ or $z_\beta = [y_\beta]$,

the class $[x_{\varphi(\beta)}, t]$, $t < 1$, consisting of a single point $(x_{\varphi(\beta)}, t) \in X_{\varphi(\beta)} \times (0, 1)$, $[y_\beta]$ consisting of a single point $y_\beta \in Y_\beta$,

$[x_{\varphi(\beta)}, t] = [y_\beta]$ iff $t = 1$ and $f_\beta(x_{\varphi(\beta)}) = y_\beta$;

Since $\varphi(\mathcal{B})$ is cofinal with $\mathcal{A}$, $X = \varprojlim (X_{\varphi(\beta)}, p^{\varphi(\beta')}_{\varphi(\beta)}, \mathcal{B})$. Thus

2° $z \in Z \Leftrightarrow z = [x, t]$ or $z = [y]$, the class $[x, t]$, $t < 1$, consisting of a single point $(x, t) \in X \times \langle 0, 1)$, $[y]$ consisting of a single point $y \in Y$, $[x, t] = [y]$ iff $t = 1$ and $f(x) = y$, where

$$x = \{x_\alpha\} \in \mathop{\mathsf{P}}_{\alpha \in \varphi(\mathcal{B})} X_\alpha, \quad p^{\alpha'}_\alpha(x_{\alpha'}) = x_\alpha \quad \text{for} \quad \alpha' \geqslant \alpha,$$

$$y = \{y_\beta\} \in \mathop{\mathsf{P}}_{\beta \in \mathcal{B}} Y_\beta, \quad q^{\beta'}_\beta(y_{\beta'}) = y_\beta \quad \text{for} \quad \beta' \geqslant \beta,$$

and $f(\{x_{\varphi(\beta)}\}) = \{f_\beta(x_{\varphi(\beta)})\}$.

Define χ by the formula

$$\chi(z) \underset{\mathrm{Df}}{=} \begin{cases} [\{x_{\varphi(\beta)}\}, t] & \text{for} \quad z = \{[x_{\varphi(\beta)}, t]\}, \\ [\{y_\beta\}] & \text{for} \quad z = \{[y_\beta]\}. \end{cases}$$

It is easy to see that the restrictions of χ are both continuous; moreover

$$\begin{aligned} &\{[x_{\varphi(\beta)}, t]\} = \{[y_\beta]\} \\ &\quad \Leftrightarrow \bigwedge_{\beta \in \mathcal{B}} [x_{\varphi(\beta)}, t] = [y_\beta] \Leftrightarrow t = 1 \wedge \bigwedge_{\beta \in \mathcal{B}} f_\beta(x_{\varphi(\beta)}) = y_\beta \\ &\quad \Rightarrow [\{x_{\varphi(\beta)}\}, t] = [\{x_{\varphi(\beta)}\}, 1] = [\{f_\beta(x_{\varphi(\beta)})\}] = [f(\{x_{\varphi(\beta)}\})] = [\{y_\beta\}]\,; \end{aligned}$$

thus χ is continuous.

Define now χ': $Z \to \varprojlim \boldsymbol{Z}$ as follows:

$$\chi'(z) \underset{\mathrm{Df}}{=} \begin{cases} \{[x_{\varphi(\beta)}, t]\} & \text{for} \quad z = [\{x_{\varphi(\beta)}\}, t], \\ \{[y_\beta]\} & \text{for} \quad z = [\{y_\beta\}]. \end{cases}$$

It is easy to see that the restrictions of χ' are both continuous; moreover

$$\begin{aligned} [\{x_{\varphi(\beta)}\}, t] = [\{y_\beta\}] &\Leftrightarrow t = 1 \wedge f\{x_{\varphi(\beta)}\} = \{y_\beta\} \Leftrightarrow t = 1 \wedge \bigwedge_{\beta \in \mathcal{B}} f_\beta(x_{\varphi(\beta)}) = y_\beta \\ &\Rightarrow \{[x_{\varphi(\beta)}, t]\} = \{[x_{\varphi(\beta)}, 1]\} = \{[f_\beta(x_{\varphi(\beta)})]\} = \{[y_\beta]\}\,; \end{aligned}$$

thus χ' is continuous.

Obviously, χ' is an inverse of χ, whence χ is a homeomorphism.

Let us notice that Proposition 4 fails if $\boldsymbol{f}$ is not assumed to be cofinal. In fact, let us consider the following

EXAMPLE. Take the subsets of the 3-dimensional Cartesian space:

$$X_n = \left\{(x_1, x_2, x_3)\colon x_1^2 + x_2^2 \leqslant \frac{1}{n^2} \wedge x_3 = 1\right\},$$

$$Y_n = \left\{(x_1, x_2, x_3)\colon x_1^2 + x_2^2 \leqslant \frac{1}{n^2} \wedge x_3 = 0\right\}, \quad n = 1, 2, \ldots$$

and let $p_n^{n'}\colon X_{n'} \to X_n$, $q_n^{n'}\colon Y_{n'} \to Y_n$ $(n' \geqslant n)$ be the inclusions.

Obviously $\boldsymbol{X} = (X_n, p_n^{n'})$, $\boldsymbol{Y} = (Y_n, q_n^{n'})$ are both inverse systems. Define $\boldsymbol{f} = (\varphi, f_n)\colon \boldsymbol{X} \to \boldsymbol{Y}$ as follows:

$$\varphi(n) = 1 \quad \text{for every } n,$$

$$f_n(x_1, x_2, 1) = \left(\frac{x_1}{n}, \frac{x_2}{n}, 0\right) \quad \text{for} \quad (x_1, x_2, 1) \in X_1, \ n = 1, 2, \ldots$$

Obviously $\varprojlim \boldsymbol{C}_f = \bigcap_{n=1}^{\infty} C_{f_n}$; this is a cone with the base X_1 and the vertex $(0, 0, 0)$. On the other hand, since $\varprojlim \boldsymbol{X}$ and $\varprojlim \boldsymbol{Y}$ both consist of single points, $C_{\varprojlim f}$ is a segment. Thus $C_{\varprojlim f} \neq \varprojlim \boldsymbol{C}_f$.

Remark 1. Let $\mathfrak{T}$ be the category of topological spaces with continuous functions as morphisms and let $\mathfrak{T}^*$ be the category of inverse systems in $\mathfrak{T}$ with usual cofinal maps of systems as morphisms. Obviously $\mathfrak{T}$ can be considered as a subcategory of $\mathfrak{T}^*$ consisting of constant systems.

We can define the operation $\mathsf{C}\colon \operatorname{Mor}\mathfrak{T}^* \to \operatorname{Ob}\mathfrak{T}^*$:

$$\mathsf{C}(\boldsymbol{f}) \underset{\mathrm{Df}}{=} \boldsymbol{C}_f.$$

Then Proposition 4 can be expressed in the form

$$\varprojlim \mathsf{C}(\boldsymbol{f}) = \mathsf{C}(\varprojlim \boldsymbol{f}),$$

which means that the operation C is continuous with respect to the inverse limit.

Remark 2. In the case of maps of spaces, the mapping cylinder can be defined by means of the Cartesian product and the matching of spaces ([3], p. 116): given $f\colon X \to Y$ we have $C_f = X \times I \underset{g}{\cup} Y$, where $g\colon X \times (1) \to Y$ is defined by the formula $g(x, 1) = f(x)$.

In a similar way the mapping cylinder of a usual map of inverse systems could be defined.

First, let us define the Cartesian product $\boldsymbol{X} \times \boldsymbol{T}$ of inverse systems $\boldsymbol{X} = (X_a, p_a^{a'}, \mathcal{A})$, $\boldsymbol{T} = (T_a, s_a^{a'}, \mathcal{A})$ as the system $(X_a \times T_a, p_a^{a'} \times s_a^{a'}, \mathcal{A})$, where $(p_a^{a'} \times s_a^{a'})(x, t) = \big(p_a^{a'}(x), s_a^{a'}(t)\big)$; one can easily prove $\boldsymbol{X} \times \boldsymbol{T}$ to be an inverse system. Define in turn the matching of inverse systems: given two inverse systems $(\boldsymbol{X}, \boldsymbol{X}^0)$, $\boldsymbol{Y}$, the first one being an inverse system of pairs (X_a, X_a^0), $X_a^0 = \overline{X}_a^0 \subset X_a$, let us take a usual map $\boldsymbol{g} = (\varphi, g_\beta)\colon \boldsymbol{X}^0 \to \boldsymbol{Y}$; we consider the system $(X_{\varphi(\beta)} \underset{g_\beta}{\cup} Y_\beta, r_\beta^{\beta'}, \mathcal{B})$, where

$$r_\beta^{\beta'}(z) = \begin{cases} [p_{\varphi(\beta)}^{\varphi(\beta')}(x)] & \text{for} \quad z = [x], \ x \in X_{\varphi(\beta')}, \\ [q_\beta^{\beta'}(y)] & \text{for} \quad z = [y], \ y \in Y_{\beta'}. \end{cases}$$

Denote this system by $\boldsymbol{X} \underset{g}{\cup} \boldsymbol{Y}$. One can easily prove $\boldsymbol{X} \underset{g}{\cup} \boldsymbol{Y}$ to be an inverse system.

We have

(I) $\varprojlim(\boldsymbol{X} \times \boldsymbol{T}) = \varprojlim \boldsymbol{X} \times \varprojlim \boldsymbol{T}$.

It was proved by E. Puzio that

(II) if $\boldsymbol{g}$ is cofinal and prefect (i.e. g_β is closed and $g_\beta^{-1}(y)$ is compact for every $y \in Y_\beta$, $\beta \in \mathcal{B}$), then $\varprojlim(\boldsymbol{X} \underset{g}{\cup} \boldsymbol{Y}) = \varprojlim \boldsymbol{X} \underset{\varprojlim g}{\cup} \varprojlim \boldsymbol{Y}$.

Obviously, if $\boldsymbol{f} = (\varphi, f_\beta)$, then $\boldsymbol{C}_f = \boldsymbol{X} \times I \underset{g}{\cup} \boldsymbol{Y}$, where

$$\boldsymbol{g} = (\varphi, g_\beta)\colon \boldsymbol{X} \times (1) \to \boldsymbol{Y} \quad \text{and} \quad g_\beta(x, 1) = f_\beta(x) \text{ for } x \in X_{\varphi(\beta)}, \ \beta \in \mathcal{B}.$$

Hence, in the case of perfect maps, Proposition 4 can be obtained as a corollary of both (I) and (II).

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