Non-singular set-valued compact fields in locally convex spaces

by

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Contents

Introduction. Corresponding to vanishing degree, the notion of non-singularity of set-valued compact fields in locally convex spaces is introduced. Results parallel to degree theory in [16] are derived. This notion of non-singularity in some application works as well as degree theory. The domains of maps in degree theory must be the closures of open sets but the domain of our maps can be any closed sets. The difference between ours and Cellina [3], Granas [10] is that our locally convex spaces are in general, not necessarily metrizable.

§ 1. Preliminary. Let E be a separated locally convex space and $\mathcal{K}E$ the family of all non-empty compact convex subsets of E. A set-valued map F defined on a Hausdorff space X into $\mathcal{K}E$ is said to be *upper semi-continuous at a point* $a \in X$ if for each open subset W of E with $F(a) \subset W$, there exists a neighbourhood V of $a \in X$ such that $F(V) \subset W$ where $F(V) = \bigcup \{F(x)\colon x \in V\}$ and F is said to be *upper semicontinuous on* X

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if F is upper semicontinuous at every point of X. An upper semicontinuous set-valued map $F\colon X \to \mathcal{K}E$ is said to be a *compact map* if $F(X)$ is relatively compact in E and F is said to be *finite dimensional* if $F(X)$ is contained in some finite dimensional vector subspace of E.

Let E be a separated locally convex space and $\mathcal{K}E$ the family of all non-empty compact convex subsets of E. Let Y be a subset of E. A set-valued map $f\colon Y \to \mathcal{K}E$ is called a *set-valued compact field* if the map $F\colon Y \to \mathcal{K}E$ defined by $F(y) = y - f(y)$ for $y \in Y$ is a compact map. Let I denote the closed unit interval $[0, 1]$. A set-valued map $h\colon Y \times I \to \mathcal{K}E$ is called a *homotopy* if the map $H\colon Y \times I \to \mathcal{K}E$ defined by $H(y, t) = y - h(y, t)$ for $(y, t) \in Y \times I$ is a compact map. The maps F, H mentioned above are said to be *compact maps corresponding to* f and h respectively. A set-valued compact field or homotopy are said to be *finite dimensional* if their corresponding compact maps are finite dimensional. Let $X \subset Y$ be two subsets of E and $p \in E$. We shall denote by $C(X, Y, p)$ the family of all set-valued compact fields $f\colon Y \to \mathcal{K}E$ such that $p \notin f(X)$.

Two set-valued compact fields f, g in $C(X, Y, p)$ are said to be *homotopic in* $C(X, Y, p)$ if there exists a homotopy $h\colon Y \times I \to \mathcal{K}E$ such that $h(y, 0) = f(y)$, $h(y, 1) = g(y)$ for all $y \in Y$ and $p \notin h(X \times I)$. For general properties of set-valued compact fields in locally convex spaces, see [16]. The following theorems are significant.

(1.1) THEOREM ([16]; 4.1, 4.2). *Let $X \subset Y$ be two closed subsets of a separated locally convex space E, $p \in E$ and $\mathcal{K}E$ the family of all non-empty compact convex subsets of E. Then every set-valued compact field in $C(X, Y, p)$ is homotopic to a finite dimensional set-valued compact field in $C(X, Y, p)$. Furthermore if f, g are finite dimensional set-valued compact fields homotopic in $C(X, Y, p)$, then f, g are homotopic under a finite dimensional set-valued homotopy in $C(X, Y, p)$.*

(1.2) THEOREM ([16]; 5.2, 5.3). *Let $X \subset Y$ be two closed subsets of a finite dimensional vector space E, $p \in E$ and $\mathcal{K}E$ the family of all non-empty compact convex subsets of E. Then every set-valued compact field in $C(X, Y, p)$ is homotopic to a single-valued compact field in $C(X, Y, p)$. Furthermore if f, g are two single-valued compact fields homotopic in $C(X, Y, p)$, then f, g are homotopic under a single-valued homotopy in $C(X, Y, p)$.*

(1.3) GENERALIZATION OF DUGUNDJI'S EXTENSION THEOREM ([16], 2.1). *Let A be a closed subset of a metrizable space X, E a separated locally convex space and $\mathcal{K}E$ the family of all non-empty compact convex subsets of E. If $f\colon A \to \mathcal{K}E$ is an upper semicontinuous set-valued map on A, then f has an upper semicontinuous set-valued extension $g\colon X \to \mathcal{K}E$ such that $g(X)$ is contained in the convex hull of $f(A)$.*

When E is metrizable, it is due to Cellina ([3], 2). In Dugundji [5], E is not necessarily metrizable.

(1.4) HOMOTOPY EXTENSION THEOREM (Cellina [3], 2.4). *Let $X \subset Y$ be two closed subsets of a complete metrizable locally convex space E, $p \in E$ and $\mathcal{K}E$ the family of all non-empty compact convex subsets of E. Let f, g be homotopic set-valued compact field in $C(X, X, p)$. If f^* is a set-valued compact field in $C(Y, Y, p)$ such that $f^*|_X = f$, then there exists a set-valued compact field g^* in $C(Y, Y, p)$ such that $g^*|_X = g$ and $f^* \simeq g^*$ in $C(Y, Y, p)$.*

(1.5) LEMMA. *Let $X \subset Y$ be two closed subsets of a metrizable locally convex space E, $\mathcal{K}E$ the family of all non-empty compact convex subsets of E, E_0 a complete vector subspace of E and $p \in E_0$. Let f be a set-valued compact field in $C(X, X, p)$ with $F(X) \subset E_0$ where F is the compact map corresponding to f. If $f|_{X \cap E_0}$ can be extended over $C(Y \cap E_0, Y \cap E_0, p)$, then f can be extended over $C(Y, Y, p)$.*

Proof. Let g be a set-valued compact field in $C(Y \cap E_0, Y \cap E_0, p)$ with $g|_{X \cap E_0} = f|_{X \cap E_0}$. Then the compact map $G\colon Y \cap E_0 \to \mathcal{K}E$ corresponding to g is an extension of $F|_{X \cap E_0}$ such that $G(Y \cap E_0) \subset E_0$. Let $T = X \cup (Y \cap E_0)$ and let $G'\colon T \to \mathcal{K}E$ be defined by $G'|_X = F$ and $G'|_{Y \cap E_0} = G$. Then G' is a compact map. By (1.3), there exists an upper semicontinuous set-valued map $H\colon Y \to \mathcal{K}E$ such that $H|_T = G'$ and $H(Y) \subset \operatorname{co} G'(T)$. Since $G'(T)$ is relatively compact, it is precompact and so $\operatorname{co} G'(T)$ is a precompact subset of the complete vector subspace E_0. Hence $\operatorname{co} G'(T)$ is relatively compact in E and H is a compact map. Define $h(y) = y - H(y)$ for $y \in Y$. Then h is an extension of $f|_X$. If for some $y \in Y$, $y - p \in H(y) \subset E_0$, then $y \in Y \cap E_0$ and $y - p \in G(y)$, contrary to $p \notin g(Y \cap E_0)$. Thus h is the required extension.

Generalization of Granas [10] to set-valued compact fields in Banach spaces can be dervied from (1.1), (1.2), (1.4) and (1.5). However influenced by Leray–Schauder [15], we shall introduce the notion of non-singular set-valued compact fields which is different from Granas [10]. It will allow us to handle compact fields in *non-metrizable* locally convex spaces. We shall use (1.4), (1.5) only when E is finite dimensional.

§ 2. Definition of non-singularity and its homotopy invariance.

Let $X \subset Y$ be two closed subsets of a separated locally convex space E, $\mathcal{K}E$ the family of all non-empty compact convex subsets of E and $p \in E$. A finite dimensional set-valued compact field f in $C(X, Y, p)$ with corresponding compact map F is said to be *bad* if there exists a finite dimensional vector subspace E_1 containing $F(Y)$ and the point p such that $f|_{X \cap E_1}$ can be extended over $C(Y \cap E_1, Y \cap E_1, p)$. A set-valued compact field g in $C(X, Y, p)$ is said to be *non-singular in* $C(X, Y, p)$ if there exists a bad finite dimensional set-valued compact field f homotopic to g in $C(X, Y, p)$. A set-valued compact field in $C(X, Y, p)$ is said to be *singular* in $C(X, Y, p)$ if it is not non-singular in $C(X, Y, p)$. Note that the terminology "*bad* finite dimensional set-valued compact field" will be abandoned after (2.2). We introduced it temporarily only for our convenience to state the definition of non-singularity and to prove (2.1) and (2.2). The following lemma is to clarify this.

(2.1) Lemma. *Let $X \subset Y$ be closed subsets of a separated locally convex space E, $p \in E$ and $\mathcal{K}E$ the family of all non-empty compact convex subsets of E. Let f be a finite dimensional set-valued compact field in $C(X, Y, p)$ with corresponding compact map F. Then f is non-singular in $C(X, Y, p)$ iff there exists a finite dimensional vector subspace E_1 containing $F(Y)$ and p such that $f|_{X\cap E_1}$ can be extended over $C(Y \cap E_1, Y \cap E_1, p)$.*

Proof. Suppose that f is non-singular in $C(X, Y, p)$. By definition, there exists a bad finite dimensional set-valued compact field g homotopic to f in $C(X, Y, p)$. By (1.1), f, g are homotopic under a finite dimensional homotopy h in $C(X, Y, p)$. Let G, H be the compact maps corresponding to g and h respectively. Since g is bad, there exists a finite dimensional vector subspace E_1 containing the point p and $G(Y)$ such that $g|_{X\cap E_1}$ can be extended over $C(Y \cap E_1, Y \cap E_1, p)$. Let E_2 be a finite dimensional vector subspace containing E_1 and $H(Y\times I)$. By (1.5), $g|_{X\cap E_2}$ can be extended over $C(Y \cap E_2, Y \cap E_2, p)$. Since $h|_{(X\cap E_2)\times I}$ is a homotopy for $f|_{X\cap E_2} \simeq g|_{X\cap E_2}$ in $C(X \cap E_2, X \cap E_2, p)$, by (1.4), $f|_{X\cap E_2}$ can be extended over $C(Y \cap E_2, Y \cap E_2, p)$. The converse is trivial.

(2.2) Theorem. *Let $X \subset Y$ be closed subsets of a separated locally convex space E, $\mathcal{K}E$ the family of all non-empty compact convex subsets of E and $p \in E$. Let f, g be set-valued compact fields in $C(X, Y, p)$. If f, g are homotopic in $C(X, Y, p)$ and if f is non-singular in $C(X, Y, p)$, then g is non-singular in $C(X, Y, p)$.*

Proof. Since f is non-singular in $C(X, Y, p)$, there exists a bad finite dimensional set-valued compact field h homotopic to f in $C(X, Y, p)$. Then h is also homotopic to g in $C(X, Y, p)$. Hence g is non-singular in $C(X, Y, p)$.

§ 3. Existence of roots and reduction theorems.

(3.1) Theorem. *Let $X \subset Y$ be two closed subsets of a separated locally convex space, $\mathcal{K}E$ the family of all non-empty compact convex subsets of E and $p \in E$. If f is a singular set-valued compact field in $C(X, Y, p)$, then we have $p \in f(Y)$.*

Proof. Suppose that $p \notin f(Y)$. Then f is a set-valued compact field in $C(Y, Y, p)$. By (1.1), f is homotopic to some finite dimensional set-valued compact field g in $C(Y, Y, p)$. Let E_1 be a finite dimensional vector subspace containing the point p and $G(Y)$ where G is the compact map corresponding to g. Then $g|_{Y\cap E_1}$ is an extension of $g|_{X\cap E_1}$ over $C(Y \cap E_1, Y \cap E_1, p)$. By (2.1), g is non-singular in $C(X, Y, p)$. Clearly $f \simeq g$ in $C(X, Y, p)$ and hence f is non-singular. This completes the proof.

(3.2) Theorem. *Let $X \subset Y$ be two closed subsets of a separated locally convex space, $\mathcal{K}E$ the family of all non-empty compact convex subsets of E and $p \in E$. Let f be a set-valued compact field in $C(X, Y, p)$ with corresponding compact map F and E_0 a vector subspace of E containing the closure of $F(Y)$ and the point p. If $f|_{Y\cap E_0}$ is non-singular in $C(X \cap E_0, Y \cap E_0, p)$, then f is non-singular in $C(X, Y, p)$.*

Proof. By [16], 3.2, let V be a convex symmetric neighbourhood of $o \in E$ such that $(p+V) \cap f(\partial A) = \emptyset$. By [16], 3.1, let E_2 be a finite dimensional vector subspace of E and $G\colon Y \to \mathcal{K}E$ a compact map such that the following conditions hold:

$$p \in E_2 \subset E_0, \quad G(Y) \subset E_2,$$

and

$$G(Y) \subset F(y)+V \quad \text{for all } y \in Y.$$

Define

$$g(y) = y - G(y) \quad \text{for} \quad y \in Y,$$

and

$$h(t) = (1-t)f(y)+tg(y) \quad \text{for} \quad (y, t) \in Y \times I.$$

Then h is a homotopy for $f \simeq g$ in $C(X, Y, p)$ and $h|_{(Y\cap E_0)\times I}$ is a homotopy for $f|_{Y\cap E_0} \simeq g|_{Y\cap E_0}$ in $C(X \cap E_0, Y \cap E_0, p)$ with respect to the vector subspace E_0. Since $f|_{Y\cap E_0}$ is non-singular in $C(X \cap E_0, Y \cap E_0, p)$, by (2.2), $g|_{Y\cap E_0}$ is a non-singular finite dimensional set-valued compact field in $C(X \cap E_0, Y \cap E_0, p)$. There exists a finite dimensional vector subspace E_1 of E such that $p \in E_1 \subset E_0$, $G(Y) \subset E_1$ and $g|_{X\cap E_1}$ can be extended over $C(Y \cap E_1, Y \cap E_1, p)$. Let E_3 be a finite dimensional vector subspace of E_0 containing E_1 and E_2. By (1.5), $g|_{X\cap E_3}$ can be extended over $C(Y \cap E_3, Y \cap E_3, p)$. Hence g is non-singular in $C(X, Y, p)$. By (2.1), f is non-singular in $C(X, Y, p)$. This completes the proof.

(3.3) Theorem. *Let A be an open subset of a separated locally convex space E, $p \in E$ and $\mathcal{K}E$ the family of all non-empty compact convex subsets of E. Let $\{A_j\colon j \in J\}$ be a family of disjoint open subsets of A and $f\colon \bar{A} \to \mathcal{K}E$ a set-valued compact field with $p \in E \setminus f(\bar{A} \setminus \bigcup_j A_j)$. Then for all except only a finite number of indices $j \in J$, $f|_{\bar{A}_j}$ is non-singular in $C(\partial A_j, A_j, p)$. Furthermore if for each $j \in J$, $f|_{\bar{A}_j}$ is non-singular in $C(\partial A_j, \bar{A}_j, p)$, then f is non-singular in $C(\partial A, \bar{A}, p)$.*

Proof. Let $X = \bar{A} \setminus U_j A_j$. Then f is a set-valued compact field in $C(X, \bar{A}, p)$. By (1.1), let f^* be a finite dimensional set-valued compact field homotopic to f in $C(X, \bar{A}, p)$. Then for each $j \in J$, $f^*|_{\bar{A}_j} \simeq f|_{\bar{A}_j}$ in $C(\partial A_j, \bar{A}_j, p)$. By [16], 15.2, there exists a finite subset J_0 of J such that $p \in E \setminus f^*(A_j)$ for all $j \in J \setminus J_0$. By (3.1), $f^*|_{\bar{A}_j}$ is non-singular in $C(\partial A_j, \bar{A}_j, p)$ for all $j \in J \setminus J_0$. By (2.2), $f|_{\bar{A}_j}$ is non-singular in $C(\partial A_j, \bar{A}_j, p)$ for all $j \in J \setminus J_0$. Furthermore, if for all $j \in J$, $f|_{\bar{A}_j}$ is non-singular in $C(\partial A_j, \bar{A}_j, p)$, then by (2.2), $f^*|_{\bar{A}_j}$ is non-singular in $C(\partial A_j, \bar{A}_j, p)$. Let F^* be the compact map corresponding to g^*. By (2.1), for each $j \in J$, there exists a finite dimension vector subspace E_j containing $F^*(\bar{A}_j)$ and the point p such that $f^*|_{\partial A_j \cap E_j}$ can be extended over $C(\bar{A}_j \cap E_j, \bar{A}_j \cap E_j, p)$. Let E^* be a finite dimensional vector subspace containing $\bigcup_{j \in J_0} E_j$. By (1.5), for each $j \in J_0$, $f^*|_{\partial A_j \cap E^*}$ can be extended to some g_j^* in $C(\bar{A}_j \cap E^*, \bar{A}_j \cap E^*, p)$

Define

$$g(y) = \begin{cases} f^*(y) & \text{for} \quad y \in \bar{A} \cap E^* \setminus \bigcup_{j \in J_0} A_j, \\ g_j^*(y) & \text{for} \quad y \in \bar{A}_j \cap E^* \text{ and } j \in J_0. \end{cases}$$

Then g is a set-valued compact field in $C(\bar{A} \cap E^*, \bar{A} \cap E^*, p)$ with $g|_{\partial A \cap E^*} = f^*|_{\partial A \cap E^*}$. By (2.1), f^* is non-singular in $C(\partial A, \bar{A}, p)$. By (2.2), f is non-singular in $C(\partial A, \bar{A}, p)$. This completes the proof.

§ 4. Translation invariance and component dependence.

(4.1) THEOREM. *Let $X \subset Y$ be two closed subsets of a separated locally convex space E, $\mathcal{K}E$ the family of all non-empty compact convex subsets of E, $p \in E$ and f a set-valued compact field in $C(X, Y, p)$. If $g(y) = f(y) - p$ for all $y \in Y$, then g is a set-valued compact field in $C(X, Y, p)$. Furthermore g is non-singular in $C(X, Y, o)$ iff f is non-singular in $C(X, Y, p)$.*

Proof. Suppose that f is non-singular in $C(X, Y, p)$. By (1.1), let h be a homotopy of f to a finite dimensional set-valued compact field f^* in $C(X, Y, p)$. By (2.2), f^* is non-singular in $C(X, Y, p)$. Let G^* be the compact map corresponding to f^*. By (2.1), there exists a finite dimensional vector subspace E_1 containing $F^*(Y)$ and the point p such that $f^*|_{X \cap E_1}$ can be extended over $C(Y \cap E_1, Y \cap E_1, p)$. Define

$$h^*(y, t) = h(y, t) - p \quad \text{for} \quad (y, t) \in Y \times I$$

and

$$g^*(y) = f^*(y) - p \quad \text{for} \quad y \in Y.$$

Then h^* is a homotopy of g to the finite dimensional set-valued compact field g^* in $C(X, Y, o)$ and $g^*|_{X \cap E_1}$ can be extended over $C(Y \cap E_1, Y \cap E_1, o)$. By (2.1), g is also non-singular. The converse follows by similar argument.

(4.2) THEOREM. *Let $X \subset Y$ be two closed subsets of a separated locally convex space E, $\mathcal{K}E$ the family of all non-empty compact convex subsets of E and $f\colon Y \to \mathcal{K}E$ a set-valued compact field. Let a, b belong to the same component of $E \setminus f(X)$. Then f is non-singular in $C(X, Y, a)$ iff f is non-singular in $C(X, Y, b)$.*

Proof. Since E is locally connected, there exists a continuous single-valued map $s\colon I \to E \setminus f(X)$ such that $s(0) = a$ and $s(1) = b$. Define $h(y, t) = f(y) - s(t)$ for $(y, t) \in Y \times I$. Then h is a homotopy in $C(X, Y, o)$. Now f is non-singular in $C(X, Y, a)$ iff h_0 is non-singular in $C(X, Y, o)$ iff h_1 is non-singular in $C(X, Y, o)$ iff f is non-singular in $C(X, Y, b)$.

§ 5. Fixed point theorems.

(5.1) LEMMA. *Let X be a non-empty closed subset of a separated locally convex space E, $a \in E \setminus X$ and A a component of $E \setminus X$. Let $\varphi_a(y) = y - a$ for all $y \in E$ and $Y = X \cup A$. Then $\varphi_a|_Y$ is non-singular compact field in $C(X, Y, o)$ iff $a \notin A$.*

Proof. If $a \notin A$, then $o \notin \varphi_a(Y)$ and by (4.1), $\varphi_a|_Y$ is non-singular. Conversely if $\varphi_a|_Y$ is non-singular in $C(X, Y, o)$, then by (2.1), there exists a finite dimensional vector subspace E_1 containing the point a such that $\varphi_a|_{X \cap E_1}$ can be extended to some f^* in $C(Y \cap E_1, Y \cap E_1, o)$ with respect to E_1. Let F^* be the compact map corresponding to f^*. Define

$$G(y) = \begin{cases} F^*(y) & \text{for} \quad y \in Y \cap E_1, \\ a & \text{for} \quad y \in E_1 \setminus A. \end{cases}$$

Then G is a compact map defined on E_1 into the family of all non-empty compact convex subsets of E_1. Applying Kakutani Fixed Point Theorem [13] to the restriction of G to the closed convex hull of $G(E_1)$, there exists $b \in E_1$ such that $b \in G(b)$. The fact $o \notin f^*(Y \cap E_1)$ shows that $b \in E_1 \setminus A$, i.e. $a = b \notin A$. This completes the proof.

(5.2) THEOREM. *Let A be an open neighbourhood of the origin in a separated locally convex space E and $\mathcal{K}E$ the family of all non-empty compact convex subsets of E. If f is non-singular set-valued compact field in $C(\partial A, \bar{A}, o)$, then there exists $x \in \partial A$ and $\lambda > 0$ such that $-\lambda x \in f(x)$.*

Proof. Let B be the component of A containing $o \in E$ and let $Y = B \cup (\partial A)$. Suppose that for each $x \in \partial A$ and $\lambda > 0$, we have $-\lambda x \notin f(x)$. Define $\psi(y) = y$ and $h(y, t) = y - tF(y)$ for all $y \in Y$ and $t \in I$. Then h is a homotopy for $\psi \simeq f$ in $C(\partial A, Y, o)$. Since f is non-singular in $C(\partial A, \bar{A}, o)$, $f|_Y$ is also non-singular in $C(\partial A, Y, o)$. By (2.2), ψ is non-singular in $C(\partial A, Y, o)$. By (5.1), we have $o \notin B$ which is a contradiction. This completes the proof.

Note that from (5.2) and (3.1) we can derive a fixed point theorem for set-valued compact map defined on *non-convex* neighbourhood of the origin in a separated locally convex space. See [16], 16.1.

(5.3) THEOREM. *Let A be an open convex symmetric neighbourhood of the origin in a separated locally convex space E, $\mathcal{K}E$ the family of all non-empty compact convex subsets of E and f a set-valued compact field in $C(\partial A, \bar{A}, o)$. If f is a non-singular set-valued compact field in $C(\partial A, \bar{A}, o)$, then there exist $x \in \partial A$ and $\lambda > 0$ with $f(x) \cap \lambda f(-x) \neq \emptyset$.*

Proof. Suppose that for each $x \in \partial A$ and $\lambda > 0$, we have $f(x) \cap \lambda f(-x) = \emptyset$. By [16], 9.4, let f^* be a finite dimensional set-valued antipodal compact field homotopic to f in $C(\partial A, \bar{A}, o)$. Since f is non-singular in $C(\partial A, \bar{A}, o)$, so is f^*. Let F^* be the compact map corresponding to f^*. By (2.1), there exists a finite dimensional vector subspace containing $F^*(\bar{A})$ such that $f^*|_{\partial A \cap E_1}$ can be extended to some g^* in $C(\bar{A} \cap E_1, \bar{A} \cap E_1, o)$. Since $g^*|_{\partial A \cap E_1} = f^*|_{\partial A \cap E_1}$, g^* is antipodal. By [16], 9.5, there exists a bounded open convex symmetric neighbourhood B of $o \in E_1$ and a set-valued compact field g_0 in $C(\partial B, \bar{B}, o)$ such that the following conditions hold:

(a) $$g_0(\bar{B}) \subset E_1,$$

(b) $$g_0(y) = g^*(y) \quad \text{for all } y \in \overline{A \cap E_1} \cap \bar{B},$$

(c) $$0 \notin g^*(\overline{A \cap E_1} \setminus B) \cup g_0(\bar{B} \setminus A),$$

(d) $$g_0(x) \cap \lambda g_0(-x) = \emptyset \quad \text{for all } x \in B,\ \lambda > 0.$$

Define $h(x, t) = [g_0(x) - t g_0(-x)]/(1+t)$ for $(x, t) \in \partial B \times I$. Then $h_0 \simeq h_1$ in $C(\partial B, \partial B, o)$. By (b) and (c), g_0 is an extension of h_0 over $C(\bar{B}, \bar{B}, o)$. By (1.4), h_1 can be extended to some h^* in $C(\bar{B}, \bar{B}, o)$. Since h_1 is antipodal, so is h^*. By [16], 9.6, $X = Y$, there exists a single-valued antipodal compact field g in $C(\bar{B}, \bar{B}, o)$. In other words, we find a continuous single-valued antipodal non-vanishing map defined on $\bar{B}$ into E_1. This well-known contradiction completes the proof.

Note that from (5.3) and (3.1), we can derive a fixed point theorem for antipodal compact map. See [16], 16.3.

§ 6. Extension of Borsuk's sweeping theorem.

(6.1) Lemma. *Let X be a closed subset of a separated locally convex space E and a, b two points in $E \setminus X$. Let $\psi_a(x) = x - a$ and $\psi_b(x) = x - b$ for all $x \in E$. Then a, b belong to the same component of $E \setminus X$ iff $\psi_a|_X \simeq \psi_b|_X$ in $C(X, X, o)$.*

Proof. If a, b belong to the same component of $E \setminus X$, there exists a single-valued continuous map $s\colon I \to E \setminus X$ such that $s(0) = a$, $s(1) = b$. Consequently $h(x, t) = x - s(t)$ for $(x, t) \in X \times I$ is a homotopy for $\psi_a|_X \simeq \psi_b|_X$ in $C(X, X, o)$. Conversely suppose that $\psi_a|_X \simeq \psi_b|_X$ in $C(X, X, o)$ and suppose that a, b belong to different components of $E \setminus X$. By (1.1), there exists a finite dimensional homotopy h for $\psi_a|_X \simeq \psi_b|_X$ in $C(X, X, o)$. Let H be the compact map corresponding to h and E_1 a finite dimensional vector subspace containing $H(X)$ and the points a, b. Then a, b belong to different components of $E_1 \setminus X$. Let A be a component of $E_1 \setminus X$ with $a \in A$ and $b \notin A$. Let $X_1 = X \cap E_1$ and $Y_1 = X_1 \cup A$.

Define

$$K^*(y, t) = \begin{cases} a & \text{for } y \in Y_1,\ t = 0, \\ H(y, t) & \text{for } y \in X_1,\ t \in I, \\ b & \text{for } y \in Y_1,\ t = 1. \end{cases}$$

Then $K^*\colon (X_1 \times I) \cup Y_1 \times \{0, 1\} \to \mathcal{K}E$ is a compact map. By (1.3), let $K\colon Y_1 \times I \to \mathcal{K}E$ be a compact map extending K^*. Define $k(y, t) = y - K(y, t)$ for $(y, t) \in Y_1 \times I$. Then k is a homotopy for $\psi_a|_{Y_1} \simeq \psi_b|_{Y_1}$ in $C(X_1, Y_1, o)$ with respect to the vector subspace E_1. By (5.1), $\psi_b|_{Y_1}$ is non-singular but $\psi_a|_{Y_1}$ is singular in $C(X, Y, o)$. This contradiction to (2.2) completes the proof.

With this lemma, we can prove the following extension of Borsuk's sweeping theorem without appealing to degree theory.

(6.2) Theorem ([16], 17.1). *Let Y be a closed subset of a separated locally convex space E, $\mathcal{K}E$ the family of all non-empty compact convex subsets of E and $h\colon Y \times I \to \mathcal{K}E$ a set-valued homotopy such that $h(y, o) = y$ for all $y \in Y$. If a, b belong to different components of $E \setminus Y$ but the same component of $E \setminus h(Y, 1)$, then $a \in h(Y \times I)$ or $b \in h(Y \times I)$.*

When h is single-valued map in Banach space, see Granas [10].

§ 7. Extension of Borsuk-Ulam's theorem.

(7.1) Theorem. *Let A be an open convex symmetric neighbourhood of the origin in a separated locally convex space E and $\mathcal{K}E$ the family of all non-empty compact convex subsets of E. If f is a singular set-valued compact field in $C(\partial A, \bar{A}, 0)$, then for every $b \in \partial A$, there exists $\lambda > 0$ such that $\lambda b \in f(\partial A)$.*

Proof. Let $L_b = \{\lambda b\colon \lambda > 0\}$ for all $b \in \partial A$. Suppose that there exists $b \in \partial A$, such that $L_b \cap f(\partial A) = \emptyset$. By [16], 10.3, there exists a finite dimensional set-valued compact field g homotopic to f in $C(\partial A, \bar{A}, 0)$ with $g(\partial A) \cap L_b = \emptyset$. Let E_1 be a finite dimensional vector subspace containing the point b and $G(\bar{A})$ where G is the compact map corresponding to g. By [16], 10.4, there exists a bounded open convex symmetric neighbourhood B of $o \in E_1$ such that $b \in \partial B$, $B \subset A$, $o \notin g(\bar{A} \setminus B)$ and $g|_{\bar{B}}$ is homotopic to some non-zero constant map k in $C(\partial B, \bar{B}, o)$. Then $g|_{\partial B}$ is homotopic to $k|_{\partial B}$ in $C(\partial B, \partial B, 0)$. By (1.4), $g|_{\partial B}$ can be extended to some g^* in $C(\bar{B}, \bar{B}, o)$. Define

$$h(y) = \begin{cases} g(y) & \text{for } y \in \bar{A} \setminus B, \\ g^*(y) & \text{for } y \in \bar{B}. \end{cases}$$

Then h is an extension for $g|_{\partial A \cap E_1}$ to $C(\bar{A} \cap E_1, \bar{A} \cap E_1, o)$. By (2.1) g is non-singular in $C(\partial A, \bar{A}, o)$. By (2.2), f is non-singular in $C(\partial A, \bar{A}, o)$. This completes the proof.

We can derive the following extension of Borsuk-Ulam's theorem without using degree theory.

(7.2) THEOREM ([16], 18.1). *Let A be an open convex symmetric neighbourhood of the origin in a separated locally convex space E, $\mathcal{K}E$ the family of all non-empty compact convex subsets of E and $f\colon \bar{A}\to\mathcal{K}E$ a set-valued compact field. If there exists a vector subspace M of E such that $f(\partial A)\subset M$ and $\partial A\not\subset M$, then there exists $x\in\partial A$ such that $f(x)\cap f(-x)\neq\emptyset$.*

When f is single-valued compact field defined on the unit ball of a Banach space, see Granas [10].

§ 8. Extension of Brouwer's invariance of domains.

(8.1) LEMMA. *Let Y be a subset of a separated locally convex space E and $\mathcal{K}E$ the family of all non-empty compact convex subsets of E. Then $p\in Y$ is an interior point of Y iff there exists an open convex symmetric neighbourhood A of $o\in E$ and a singular set-valued compact field f in $C(\partial A, \bar{A}, p)$ such that $f(\bar{A})\subset Y$.*

Proof. If p is an interior point of Y, there exists an open convex symmetric neighbourhood A of $o\in E$ such that $p+\bar{A}\subset Y$. Define $f(x) = x+p$ for $x\in\bar{A}$. Then clearly f is a compact field in $C(\partial A, \bar{A}, p)$ and $f(\bar{A})\subset Y$. Let $\psi(x)=x$ for all $x\in\bar{A}$. By (5.1), since $o\in A$, ψ is singular in $C(\partial A, \bar{A}, o)$. By (4.1), f is singular in $C(\partial A, \bar{A}, o)$. Conversely suppose that A is an open convex symmetric neighbourhood of $o\in E$ and f a singular set-valued compact field in $C(\partial A, \bar{A}, p)$ with $f(\bar{A})\subset Y$. Let V be a convex neighbourhood of $o\in E$ such that $(p+V)\cap f(\partial A)=\emptyset$. Let $g(x)=f(x)-v$ for all $x\in\bar{A}$ and $v\in V$. Then $f\simeq g$ in $C(\partial A, \bar{A}, p)$ and hence g is singular in $C(\partial A, \bar{A}, p)$. By (3.1), we have $p\in g(A)$ i.e. $p+v\in f(A)\subset Y$. Since v is an arbitrary point of V, we have $p+V\subset Y$ and p is an interior point of Y.

Let B be an open subset of a separated locally convex space E and $\mathcal{K}E$ the family of all non-empty compact convex subsets of E. Following [16], a set-valued map $g\colon B\to\mathcal{K}E$ is called a *local boundary map* if for each $b\in B$, there exists an open convex symmetric neighbourhood A of $o\in E$ such that $b+\bar{A}\subset B$ and for every $x_1, x_2\in b+\bar{A}$ with $g(x_1)\cap g(x_2)\neq\emptyset$, we have $x_1-x_2\notin\partial A$. Note that every single-valued injection is a local boundary map. Following [16], a set-valued map $g\colon B\to\mathcal{K}E$ is called a *locally non-opposite map* if for each $b\in B$, there exists an open convex symmetric neighbourhood A of $o\in E$ such that $b+\bar{A}\subset B$ and for all $\lambda\geqslant 0$ and $x\in b+\partial A$, we have $\lambda(b-x)\notin g(x)-g(b)$. The following theorem can be proved without using degree theory.

(8.2) THEOREM ([16]; 19.2, 19.3). *Let B be an open set in a separated locally convex space E, $\mathcal{K}E$ the family of all non-empty compact convex subsets of E and $g\colon B\to\mathcal{K}E$ a set-valued compact field. If g is a local boundary map or a locally non-opposite map, then $g(B)$ is open in E.*

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