Logical connections between some open problems concerning nil rings

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Introduction. As it is well known the question whether every nil ring is locally nilpotent has been negatively solved by Golod and Šafarevič [4], [5]. There are still many open problems concerning nil rings and radical rings (in Jacobson's sense). We shall formulate some of them and we shall investigate the interrelations among them.

Amitsur [2] has formulated the following problem.

P_1. If R is a nil ring, is the polynomial ring $R[x]$ a radical ring?

The next problem is that of Koethe [8].

P_2. If a ring R contains a one-sided nil ideal A, is A contained in a two-sided nil ideal of R?

If R is a ring, then by R_n we denote the ring of $n\times n$ matrices over R. Let us formulate the following problem:

P_3. If R is a nil ring, is R_2 also nil?

For algebraic algebras over a field this problem has been formulated by Jacobson [7].

Herstein [6] has asked the following question:

P_4. If for every element x and y of a ring R we have $(xy-yx)^n = 0$ for some n, do the nilpotent elements of R form an ideal in R?

To formulate the next problem, suggested by Professor S. A. Amitsur, we shall introduce the following definition.

Definition. We shall call a ring R an *absolutely nil ring* if for every $n \geqslant 0$ the ring $R[x_1, \ldots, x_n]$ of polynomials in commutative indeterminates $x_1, \ldots, x_n$ is a nil ring.

If R is an algebra over an infinite field F, then R is absolutely nil if and only if R is an LBI-algebra over F (Amitsur [1]).

Any locally nilpotent ring is of course absolutely nil. One can verify that the examples of non-locally nilpotent nil algebras constructed by Golod [4] are absolutely nil algebras.

P_5. If R is a nil ring, is R an absolutely nil ring?

This problem could be weakened as follows.

P_6. If R is a nil ring, is $R[x]$ also a nil ring?

Finally, we formulate the two last problems.

P_7. If R is a finitely generated ring, is the radical of R a nil ideal in R?

For algebras this problem has been mentioned in [7].

DEFINITION. We shall call a ring R a *weakly nil ring* if there is such a multiplicatively closed subset S of nilpotent elements from R that each element of R is a finite sum of elements from S.

One can verify that the semigrup algebra of the Morse semigrup [7] over any field is a finitely generated weakly nil and not radical algebra.

P_8. If R is a finitely generated radical weakly nil ring, is R a nil ring?

For algebras over a non-denumerable field Amitsur [1] has positively solved P_2, P_3, P_5, whence also P_6 and P_1, and P_7, whence also P_8. It will be shown that P_4 has also a positive solution for such algebras. Therefore it seems reasonable to ask the questions $P_1, \ldots, P_8$ for algebras over a field F. Problems thus formulated, which we investigate later, will be denoted by $P_1F, \ldots, P_8F$, where in the problems P_7F and P_8F, instead of finitely generated rings, we consider finitely generated algebras over the field F.

THEOREM 1. *Let R be a ring. Then the polynominal ring $R[x]$ is radical if and only if the matrix ring R_n is nil for every integer $n \geqslant 1$.*

Proof. Let $R[x]$ be a radical ring. It is well known [7] that the ring $(R[x])_n$ is also radical for any $n \geqslant 1$. But it is not difficult to check that $(R[x])_n = R_n[x]$, whence the ring $R_n[x]$ is radical. Now, applying Amitsur result [2], we find that R_n is nil.

Conversely, let us assume that the ring R_n is nil for any $n \geqslant 1$.

Let

$$p(x) = \sum_{i=1}^{m} a_i x^i \in R[x].$$

Let us take the formal power series

$$q(x) = \sum_{i=1}^{\infty} b_i x^i \in R[[x]],$$

where

$$(1)\qquad \begin{aligned} b_1 &= a_1, \\ b_i &= a_i + \sum_{j=1}^{i-1} a_{i-j} b_j \quad \text{for} \quad 2 \leqslant i \leqslant m, \\ b_i &= \sum_{j=1}^{m} a_{m+1-j} b_{i-m-1+j} \quad \text{for} \quad i > m. \end{aligned}$$

Then

$$(2)\qquad q(x) = p(x) + p(x)q(x).$$

Now we shall find such a matrix $C = (c_{ij}) \in R_m$ that for any $k \geqslant m$

$$(3)\qquad b_{k+i} = \sum_{j=1}^{m} c_{ij} b_{k-m+j}, \quad i = 1, \ldots, m.$$

The construction of C will proceed by induction on i. For $i = 1$ we define

$$c_{1j} = a_{m+1-j}, \quad j = 1, \ldots, m.$$

Then by (1)

$$b_{k+1} = \sum_{j=1}^{m} a_{m+1-j} b_{k-m+j} = \sum_{j=1}^{m} c_{1j} b_{k-m+1}.$$

Let us assume that we have already defined the elements c_{ij} for $i < l \leqslant m$, $j = 1, \ldots, m$. We put

$$c_{lj} = \sum_{i=1}^{l-1} a_{l-i} c_{ij} \qquad \text{for} \quad 1 \leqslant j < l,$$

$$c_{lj} = a_{m-j+l} + \sum_{i=1}^{l-1} a_{l-i} c_{ij} \qquad \text{for} \quad m \geqslant j \geqslant l.$$

Then

$$\begin{aligned} \sum_{j=1}^{m} c_{lj} b_{k-m+j} &= \sum_{j=1}^{l-1} \sum_{i=1}^{l-1} a_{l-i} c_{ij} b_{k-m+j} + \\ &\quad + \sum_{j=1}^{m} \sum_{i=1}^{l-1} a_{l-i} c_{ij} b_{k-m+j} + \sum_{j=l}^{m} a_{m-j+l} b_{k-m+j} \\ &= \sum_{i=1}^{l-1} a_{l-i} \sum_{j=1}^{m} c_{ij} b_{k-m+j} + \sum_{j=l}^{m} a_{m-j+l} b_{k-m+j}. \end{aligned}$$

But by the induction assumption we get

$$\sum_{i=1}^{l-1} a_{l-i} \sum_{j=1}^{m} c_{ij} b_{k-m+j} = \sum_{i=1}^{l-1} a_{l-i} b_{k+i} = \sum_{s=m-l+2}^{m} a_{m+1-s} b_{k+l-m-1+s}.$$

Moreover,

$$\sum_{j=l}^{m} a_{m-j+l} b_{k-m+j} = \sum_{s=1}^{m-l+1} a_{m+1-s} b_{k+l-m-1+s}.$$

Therefore, by (1),

$$\sum_{j=1}^{m} c_{lj} b_{k-m+j} = \sum_{s=1}^{m} a_{m+1-s} b_{k+l-m-1+s} = b_{k+l}.$$

Now let us consider the matrix $D = (d_{ij}) \in R_{m+1}$, where $d_{ij} = c_{ij}$ for $i, j = 1, \dots, m$, $d_{i,m+1} = b_i$ for $i = 1, \dots, m$ and $d_{m+1,j} = 0$ for $j = 1, \dots, m+1$. We shall prove that for any $l \geqslant 1$

$$f_{i,m+1} = b_{(l-1)m+i} \quad \text{for} \quad i = 1, \dots, m \tag{4}$$

and

$$f_{m+1,m+1} = 0,$$

where $F = (f_{ij}) = D^l$.

We shall proceed by induction on l. For $l = 1$ we have

$$f_{i,m+1} = d_{i,m+1} = b_i \quad \text{for} \quad i = 1, \dots, m$$

and

$$f_{m+1,m+1} = d_{m+1,m+1} = 0.$$

Let us put $G = (g_{ij}) = D^{l-1}$. Since $F = D^l = D \cdot D^{l-1}$, we have

$$f_{i,m+1} = \sum_{s=1}^{m+1} d_{is} g_{s,m+1}, \quad i = 1, \dots, m+1.$$

But by the induction assumption we have

$$g_{s,m+1} = b_{(l-2)m+s} \quad \text{for} \quad s = 1, \dots, m$$

and

$$g_{m+1,m+1} = 0.$$

Therefore

$$f_{i,m+1} = \sum_{s=1}^{m} d_{is} b_{(l-2)m+s} = \sum_{s=1}^{m} c_{is} b_{(l-1)m-m+s}.$$

Now, applying (3), we obtain

$$f_{i,m+1} = b_{(l-1)m+i} \quad \text{for} \quad i = 1, \dots, m.$$

Moreover,

$$f_{m+1,m+1} = \sum_{s=1}^{m+1} d_{m+1,s} g_{s,m+1} = 0,$$

since

$$d_{m+1,s} = 0 \quad \text{for} \quad s = 1, \dots, m+1.$$

Since R_n is nil for any n, we have $F = (f_{ij}) = D^l = 0$ for some integer l. Thus by (4) we obtain

$$f_{i,m+1} = b_{(l-1)m+i} = 0 \quad \text{for} \quad i = 1, \dots, m. \tag{5}$$

We shall prove that for any $r > p$, where $p = (l-1)m$, $b_r = 0$. We shall proceed by induction on r. Applying (5), we have $b_r = 0$ for $p < r \leqslant p+m$. Now let us assume that $b_r = 0$ for $r < s$, where $s > p+m$. Applying (1), we get

$$b_s = \sum_{j=1}^{m} a_{m+1-j} b_{s-m-1+j}.$$

But for $j = 1, \dots, m$ we have $s > s-m-1+j > p$. Then by the induction assumption $b_{s-m-1+j} = 0$, $j = 1, \dots, m$. Therefore $b_s = 0$. Thus we have proved that $q(x) \in R[x]$, which means that the polynomial $p(x)$ is quasi-regular in $R[x]$. Therefore the ideal $xR[x]$ is radical. Since R is nil, the ring $R[x]$ is also radical.

THEOREM 2. *For any $i = 1, 2, 3$ a positive solution of the problem P_i implies a positive solution of P_{i+1}, where $i+1$ is taken* mod 3.

Proof. Let us assume that P_1 has a positive solution and let A be a one-sided nil ideal of a ring R. Then, by assumption, $A[x]$ is a one-sided radical ideal of $R[x]$. Therefore $A[x] \subseteq J(R[x])$, where $J(R[x])$ is the radical of $R[x]$. Amitsur [2] has shown that $J(R[x]) = B[x]$, where B is a nil ideal of R. Thus $A[x] \subseteq B[x]$, whence $A \subseteq B$.

Now let us assume that P_2 has a positive solution and let R be a nil ring. Let A' be the set of all such matrices $(r_{ij}) \in R_2$ that $r_{2j} = 0$, $j = 1, 2$ and let A'' be the set of such $(r_{ij}) \in R_2$ that $r_{1j} = 0$, $j = 1, 2$. If C' is the set of such $(r_{ij}) \in A'$ that $r_{11} = 0$ and C'' the set of such $(r_{ij}) \in A''$ that $r_{22} = 0$, then $C'^2 = C''^2 = 0$. Since the rings A'/C' and A''/C'' are isomorphic to R, A' and A'' are right nil ideals of R_2. By assumption there are such two-sided nil ideals B' and B'' of R_2 that $A' \subseteq B'$, $A'' \subseteq B''$. Since $B' + B''$ is a nil ideal of R_2 and $A' + A'' = R_2$, R_2 is a nil ring.

Finally, let us assume that P_3 has a positive solution and let R be a nil ring. At first we shall prove by induction on n that the ring R_{2^n} is nil. This is true for $n = 1$. Since $R_{2^{n+1}} = (R_{2^n})_2$, $R_{2^{n+1}}$ is nil if R_{2^n} is. Since any matrix ring R_k is isomorphic to a subring of the ring R_{2^k}, R_k is nil for any k. Now, applying Theorem 1, we find that $R[x]$ is radical.

THEOREM 3. *A positive solution of P_2 implies a positive solution of P_4.*

Proof. Let us assume that for any elements x and y of a ring R there is such an integer n that $(xy-yx)^n = 0$. We shall adopt Herstein's idea [6] to prove that there are no nilpotent elements outside $K(R)$, where $K(R)$ is the maximal nil ideal (Koethe radical [3]) of R. Without loss of generality we can assume that R is K-semisimple. Let x be a nil element of R, and let m be the smallest positive integer such that $x^m = 0$. Let us suppose that $m \neq 1$. Then $a = x^{m-1} \neq 0$ and $a^2 = 0$. For any $r \in R$ there is such an n that $(ra - ar)^n = 0$. Multiplying this on the left by ra, we get $(ra)^{n+1} = 0$, whence Ra is a left nil ideal of R. If A is a left ideal of R generated by the element a, then $A^2 \subseteq Ra$. Thus A is a non-zero left nil ideal of R. Now, by assumption, A is contained in a two-sided nil ideal B of R, which is impossible since R is K-semisimple. Therefore $m = 1$, i.e. $x = x^1 = 0$.

PROPOSITION 1. *Problem* P_5 *has a positive solution if and only if* P_6 *has a positive solution.*

Proof. Let us assume that P_6 has a positive solution and let R be a nil ring. We shall prove by induction on n that $R[x_1, \ldots, x_n]$ is also nil. By assumption $R[x_1]$ is nil. Since the rings $R[x_1, \ldots, x_n]$ and $(R[x_1, \ldots, x_{n-1}])[x_n]$ are isomorphic and since by the induction assumption $R[x_1, \ldots, x_{n-1}]$ is nil, $R[x_1, \ldots, x_n]$ is also nil.

The converse implication is obvious.

THEOREM 4. *A positive solution of* P_8 *implies a positive solution of* P_6.

Proof. Let R be a nil ring. At first we shall prove that the polynomial ring $R[x]$ is radical. Let $a = (a_{ij}) \in R_n$. By A we denote the subring of R generated by the elements a_{ij}, $i, j = 1, \ldots, n$. Since A is nil, A_n is radical, as is well known [7]. Let S be the set of all matrices from A_n which have at most one non-zero entry. The set S defined in this way is of course a multiplicatively closed set of nilpotent elements. Therefore A_n is a weakly nil ring. But, on the other hand, A_n is finitely generated since A is finitely generated. Hence by assumption A_n is a nil ring. Since $a \in A_n$, a is nilpotent, which means that R_n is a nil ring. Now, applying Theorem 1, we find that the ring $R[x]$ is radical.

Now we shall show that every polynomial

$$f(x) = c_0 + c_1 x + \ldots + c_n x^n \in R[x], \quad n \geqslant 1$$

is nilpotent.

For any integer $i \geqslant 1$ we have

$$i - 1 = nq(i) + r, \quad 0 \leqslant r < n.$$

Besides, we put

$$q(0) = 0.$$

It is easy to see that for any non-negative integers i and j we have

$$q(i+j) \leqslant q(i) + q(j) + 1. \tag{1}$$

Let C be the subring of R generated by the elements $c_0, c_1, \ldots, c_n$. Now we define the set T of such polynomials $h(x) = d_0 + d_1 x + \ldots + d_k x^k \in C[x]$ that for any i, $0 \leqslant i \leqslant k$, $d_i \in C^{q(i)+1}$. Using (1), it is not difficult to check that T is a subring of $C[x]$ and $f(x) \in T$.

Now we shall show that the ring $T_1 = T \cap xR[x]$ is radical. Let $g(x) = a_1 x + \ldots + a_k x^k$ be a polynomial from T_1. Since, as we have already shown, $R[x]$ is radical, $xR[x]$ is also radical. Therefore there is such an

$h(x) = b_1 x + \ldots + b_l x^l \in xR[x]$ that $h(x) = g(x) + g(x)h(x)$. Then, as we have already mentioned in the proof of Theorem 1,

$$b_i = a_i + \sum_{j=1}^{i-1} a_{i-j} b_j \quad \text{for} \quad 1 \leqslant i \leqslant k, \tag{2}$$

$$b_i = \sum_{j=1}^{k} a_{k+1-j} b_{i-k-1+j} \quad \text{for} \quad i > k. \tag{3}$$

We shall show by induction on i that $b_i \in C^{q(i)+1}$, $i = 1, \ldots, l$. For $i = 1$ $b_1 = a_1 \in C^{q(1)+1}$. If $1 < i \leqslant k$, then $a_{i-j} \in C^{q(i-j)+1}$ for $j = 0, 1, \ldots, i-1$ and, by the induction assumption, $b_j \in C^{q(j)+1}$ for $j = 1, \ldots, i-1$. Thus $a_{i-j} b_j \in C^{q(i)+1}$ for $j = 1, \ldots, i-1$ since by (1) we have $q(i) + 1 \leqslant q(i-j) + q(j) + 2$. Therefore, by (2), $b_i \in C^{q(i)+1}$. If, however, $i > k$, then by the induction assumption $b_{i-k-1+j} \in C^{q(i-k-1+j)+1}$ for $j = 1, \ldots, k$. Thus $a_{k+1-j} b_{i-k-1+j} \in C^{q(i)+1}$, $j = 1, \ldots, k$, since, using (1) again, we have $q(i)+1 \leqslant q(k+1-j) + q(i-k-1+j) + 2$. Therefore, by (3), $b_i \in C^{q(i)+1}$. Thus we have proved that $h(x) \in T_1$, which means that the ring T_1 is radical.

By the isomorphism theorem the ring T/T_1 is isomorphic to a subring of the nil ring R. Since T_1 is radical, T is also radical.

Let S be the set of all monomials $ax^i \in T$, $i = 0, 1, \ldots$ The set S defined in this way is of course a multiplicatively closed set of nilpotent elements from T. Moreover, every polynomial from T is a finite sum of monomials from S. Therefore T is a weakly nil ring.

Now we shall show that T is a finitely generated ring. Let T' be the subring of $C[x]$ generated by the elements $c_i x^j$, $i, j = 0, 1, \ldots, n$. It is obvious that $T' \subseteq T$. To prove the converse inclusion it is enough to show that $S \subseteq T'$. Let $ax^i \in S$. We shall proceed by induction on $q(i)$. If $q(i) = 0$, then $a \in C$ since $i \leqslant n$. Hence a is a finite sum of elements of the form $mc_{k_1} \ldots c_{k_p}$, where m is an integer and $0 \leqslant k_j \leqslant n$ for $j = 1, \ldots, p$, $p \geqslant 1$. Then ax^i is also a finite sum of elements $mc_{k_1} \ldots c_{k_p} x^i$. Since $c_{k_1}, \ldots, c_{k_{p-1}} \in T'$ and $mc_{k_p} x^i \in T'$, we have $ax^i \in T'$. Now let us assume that $q(i) > 0$. Then $a \in C^{q(i)+1}$, whence $a = c_0 a_0 + \ldots + c_n a_n$, where $a_j \in C^{q(i)}$, $j = 0, 1, \ldots, n$. We have $ax^i = \sum_{j=0}^{n} c_j x^n a_j x^{i-n}$. Then $c_j x^n \in T'$ and by the induction assumption $a_j x^{i-n} \in T'$ since $q(i-n) = q(i) - 1$. Therefore, $ax^i \in T'$, which means that $S \subseteq T'$.

Thus we have proved that T is a finitely generated, radical, weakly nil ring. By assumption T is a nil ring. Since $f(x) \in T$, the polynomial $f(x)$ is nilpotent, which means that $R[x]$ is a nil ring.

Using all the theorems proved above, we obtain the following result.

THEOREM 5. *A positive solution of* P_8 *implies a positive solution of* P_i, $i = 1, \ldots, 6$.

Proof. By Theorem 4 we get a positive solution of P_6, which obviously implies a positive solution of P_1. Using Theorem 2, we arrive at a positive solution of P_2 and P_3. Then Theorem 3 provides a positive solution of P_4. Finally, by Proposition 1, we get a positive solution of P_5 from a positive solution of P_6.

Let F be a field. As we have already agreed, by P_iF we understand the problems P_i, $i = 1, \ldots, 8$, formulated for algebras over F. The Theorems 2, 3, 4 and 5 could therefore be interpreted as theorems on connections between the problems P_iF, $i = 1, \ldots, 8$. The connections between the problems P_iF and P_i, $i = 1, \ldots, 6$, will be studied in Theorem 6.

As we have already mentioned for a non-denumerable field F Amitsur [1] has positively solved P_7F, which obviously implies a positive solution of P_8F. Now, applying Theorem 5, we obtain positive solutions of $P_1F, \ldots, P_6F$, which has also been known to Amitsur [1].

THEOREM 6. *For any $i = 1, \ldots, 6$ the problem P_i has a positive solution if and only if the problem P_iF has a positive solution for every field F.*

At first we shall prove two lemmas. For convenience, by $K(R)$ we shall denote the maximal nil ideal (Koethe radical [3]) of a ring R.

LEMMA 1. *If $K(R[x]) \neq R[x]$, then there is such a prime ideal P of R that the ring $(R/P)[x]$ is K-semisimple.*

Proof. Let $f(x)$ be a non-nilpotent element of $R[x]$. By Zorn Lemma there is such a maximal ideal P of R that $f^n \notin P[x]$ for $n = 1, 2, \ldots$ We shall prove that P is prime. Let A and B be ideals of R properly containg P. Then $f^p \in A[x]$, $f^q \in B[x]$ for some p and q. Thus $f^{p+q} \in A[x]\cdot B[x] = (A\cdot B)[x]$, which means that $A\cdot B$ is not contained in P. Therefore P is prime.

Amitsur [2] has proved that $K\big((R/P)[x]\big) = (C/P)[x]$ for some ideal C/P of R/P. If $C \neq P$, then $f^k \in C[x]$ for some k. Therefore $(f^k)^l \in P[x]$ for some l, i.e. $f^{kl} \in P[x]$, which is impossible. Therefore $C = P$, i.e. $(R/P)[x]$ is K-semisimple.

LEMMA 2. *Let a be such an element of a ring R that $a \notin K(R)$. Then there is such a prime ideal P of R that $a \notin P$ and R/P is K-semisimple.*

Proof. The ideal A generated by a is not nil, therefore A contains a non-nilpotent element b. By the Zorn Lemma there is a maximal ideal P of R excluding b^n, $n = 1, 2, \ldots$, and containing $K(R)$. It is not difficult to check that P is prime and R/P is K-semisimple.

Proof of the theorem. We start with some general remarks. By $C(R)$ we denote the centroid of a ring R, i.e. the ring of such endomorphisms a of the additive group R^+ of R that $a(xy) = (ax)y = x(ay)$ for every $x, y \in R$. For any $a \in C(R)$ aR as well as $\ker a$ are ideals of R. Now let us assume that R is prime. Since for $a \in C(R)$ $\ker a \cdot aR = 0$, then either $\ker a = 0$ or $a = 0$. The ring $C(R)$ has no zero divisors since $aR\cdot\beta R = a\beta R^2$, $a, \beta \in C(R)$. Since $\ker(a\beta - \beta a) \supseteq R^2 \neq 0$, we have $a\beta - \beta a = 0$, i.e. $C(R)$ is commutative. Now let F be the field of quotients of $C(R)$. Then $R\otimes_C F$, where $C = C(R)$, is an algebra over F. Since $ax = 0$, $a \in C$, $x \in R$ implies $a = 0$ or $x = 0$, the ring R is isomorphic to the subring of $R\otimes_C F$ consisting of the elements which can be written in the form $x\otimes_C 1$, $x \in R$. Since F is the field of quotiens of C, each element of $R\otimes_C F$ can be written in the form $x\otimes f$, $f \in F$. If R is a nil ring, then $R\otimes_C F$ is a nil algebra over F.

Let us assume that the problem P_2F has a positive solution for any field F and let A be a one-sided nil ideal of a ring R. Let us suppose that A is not contained in $K(R)$. Then by Lemma 2 there is such a prime ideal P of R that A is not contained in P and the ring $R' = R/P$ is K-semisimple. The ring R' contains a non-zero one-sided nil ideal $A+P/P = A'$. Now let F be the field of quotiens of $C(R')$. Then the algebra $R'\otimes_C F$ over F contains a non-zero one-sided nil ideal $A'\otimes_C F$. By assumption $A'\otimes_C F$ is contained in a two-sided nil ideal B of $R'\otimes_C F$. Then $A' \subseteq B \cap R'$. Therefore $B \cap R'$ is a non-zero nil ideal of K-semisimple ring R', which is impossible.

Now let us assume that P_1F has a positive solution for any field F. By Theorem 2 the problem P_2F has a positive solution for any F, therefore — as we have just proved — P_2 has a positive solution. Now, applying Theorem 2 again, we get a positive solution of P_1. Applying verbatim the same arguments, we conclude that a positive solution of P_3F for any field F implies a positive solution of P_3.

Let us assume that P_4F has a positive solution for any field F, and let R be a ring in which for any $x, y \in R$ $(xy - yx)^n = 0$ for some n and the set N of nilpotent elements of R is not an ideal in R. Then there is a nilpotent element $a \notin K(R)$. Applying Lemma 2, we have such a prime ideal P of R that $a \notin P$ and the ring $R' = R/P$ is K-semisimple. Now we shall consider the algebra $R'\otimes_C F$ over F, where $C = C(R')$ and F is the field of quotiens of C. One can easily verify that the commutator of any two elements from $R'\otimes_C F$ is nilpotent. Therefore by assumption the set B of all nilpotent elements from $R'\otimes_C F$ is an ideal. Then $B \cap R'$ is the set of all nilpotent elements from R'. Since $0 \neq a + P \in R'$, $B \cap R'$ is a non-zero nil ideal of the K-semisimple ring R', which is impossible.

Now let us assume that P_6F has a positive solution for any field F and let R be a nil ring. Suppose that $R[x]$ is not nil. Then by Lemma 1 there is such a prime ideal P of R that the ring $R'[x]$ is K-semisimple where $R' = R/P$. Now we can consider again the algebra $R'\otimes_C F$, where

$C = C(R')$ and F is the field of quotiens of C. The algebra $R' \otimes_C F$ is nil, whence by assumption $(R' \otimes_C F)[x]$ is also nil. Since, however, $R'[x]$ is contained in $(R' \otimes_C F)[x]$, $R'[x]$ is nil, which is impossible.

Finally, let us assume that $P_5 F$ has a positive solution for any field F. Then obviously $P_6 F$ has a positive solution, whence — as we have just proved above — P_6 has a positive solution. Now, applying Proposition 1, we obtain a positive solution of P_5.

The converse implications are obvious.

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On interpretability in theories containing arithmetic

by

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0. Introduction. In [3], a ZF-formula φ was constructed such that (ZF, φ) is relatively interpretable in ZF but (GB, φ) is not relatively interpretable in GB, provided ZF is ω-consistent. (ZF denotes the Zermelo–Fraenkel set theory, and GB the Gödel–Bernays set theory.) This result is generalized in the present paper in two ways: first, we replace the assumption of ω-consistency by the assumption of (usual) consistency and, secondly, we replace ZF and GB by an arbitrary couple of theories related similarly as ZF and GB and containing arithmetic. Similarly as in [3], our result is an immediate consequence of a general theorem (Theorem 1) concerning reflexive theories containing arithmetic. A technical lemma (Lemma 1) concerning "nice" numerations of recursively enumerable sets, which is the key device of removing the assumption of ω-consistency, is — in a certain sense — a generalization of the result of [1] and might be useful also in other connections. Some other consequences of Theorem 1 are listed at the end of the paper. The knowledge of [3] is not necessary to understand this paper, but the reader is supposed to be familiar with [2] and with some topics of the recursion theory.

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1. Preliminaries. Theories are assumed to be formalized in the predicate calculus with equality, denumerably many predicates and functions of each finite arity, denumerably many constants and denumerably many sorts of variables (there are denumerably many variables of each sort). A *theory* is a pair consisting of a language and of a set of formulas of that language (special axioms), a language being a list of predicates, functions, constants and sorts of variables. A sort s is *subordinated* to a sort t in a theory T if $\mathrm{T} \vdash (\forall \mathrm{x}^s)(\exists \mathrm{y}^t)(\mathrm{x}^s = \mathrm{y}^t)$ (where x^s is a variable of the sort s, etc.). A sort s is *universal* in T if each sort of the language of T is subordinated to s in T. We restrict ourselves to theories having a universal