$(A, B) \notin \langle \mathfrak{U}, \mathfrak{V} \rangle$. Suppose that $(a, b) \in W(A) \cap W(B)$. Since $(a, b) \in W(A)$ there exists $(x, y) \in A$ such that $((a, b), (x, y)) \in W$. Hence $(a, x) \in U$ and $(b, y) \in K$. Similarly there exists $(x', y') \in B$ such that $(a, x') \in U$ and $(b, y') \in K$. Now since $(b, y), (b, y') \in K$ there exists $a, a' < \mathfrak{n}$ such that $(y, y_a), (y', y_{a'}) \in K$. Thus we have $(y', b), (b', y), (y, y_a) \in K$ so $(y', y) \in K \circ K \circ K \subset V$. Since $(x, y) \in A$ and $(x', y') \in B$ we have $(x, y) \in A^a$ and $(x', y') \in B^a$. Now $x \in A^a$, $x' \in B^a$ and $(a, x), (a, x') \in U$ so $a \in U(A(a, 1))$ and $a \in U(B(a, 1))$. But $U \subset U_a$ so we have $a \in U_a(A(a, 1)) \cap U_a(B(a, 1))$. This contradicts the hypothesis that $U_a(A(a, 1)) \cap U_a(B(a, 1)) = \emptyset$ and so we obtain $W(A) \cap W(B) = \emptyset$.

THEOREM 3.12. *If δ_i is a family of $\mathfrak{m}$-proximities on X_i respectively (where $i \in I$) such that at least two admit non-$\mathfrak{m}$-bounded $\mathfrak{m}$-uniformities, then the weak $\mathfrak{m}$-product proximity $\delta^{\mathfrak{m}}(3)$ is not an $\mathfrak{m}$-total proximity.*

Proof. Let δ_i and δ_j be two proximities which admit non-$\mathfrak{m}$-bounded uniformities. Let $\langle \mathfrak{U}_i: i \in I \rangle^{\mathfrak{m}} = \mathfrak{U}^j$, where $\mathfrak{U}_i$ is the $\mathfrak{m}$-bounded $\mathfrak{m}$-uniformity compatible with δ_i for all $i \neq j$ and $\mathfrak{U}_j = \mathfrak{V}_j$. Let $\mathfrak{U}^k$ be similarly defined with j replaced by k. It easily follows from Theorems 3.9 and 3.11 that $\langle \mathfrak{U}^j \rangle = \delta^{\mathfrak{m}}(3) = \langle \mathfrak{U}^k \rangle$. Suppose that $\mathfrak{U}'$ is compatible with $\delta^{\mathfrak{m}}(3)$. Now $\mathfrak{U}'$ is the $\mathfrak{m}$-product of $\{\mathfrak{U}'_i: i \in I\}$ and it follows from 3.10 that $\mathfrak{U}'_i$ must be $\mathfrak{m}$-bounded for all but at most one $i \in I$. If $i \neq k$, then $\mathfrak{U}' \not> \mathfrak{U}^k$; if $i = k$, then $\mathfrak{U}' \not> \mathfrak{U}^j$. Thus $\mathfrak{U}'$ is not an $\mathfrak{m}$-total proximity. We conclude that $\delta^{\mathfrak{m}}(3)$ is not an $\mathfrak{m}$-total proximity.

If $\mathfrak{m} = \aleph_0$ we have the following corollary.

COROLLARY 3.13. *If δ_i is a family of proximities on X_i respectively such that at least two admit more than one compatible uniformity, then the weak product proximity does not admit a strongest compatible uniformity.*

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On the topology of curves IV

by

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As is well-known, each arc is an acyclic and atriodic curve, and these two properties characterize arcs within some considerably large classes of curves, for instance the class of locally connected curves. The second author has proved that all acyclic Suslinian curves possess a decomposition property (see [5], Theorem 2.2). An analogue for atriodic curves is established in this paper (see § 1). Actually, we show the decomposition property to be possessed by all Suslinian curves which are locally atriodic in a weak sense, and we derive a stronger decomposition property for all atriodic Suslinian curves (see § 3). The latter property, however, is not necessarily possessed by all acyclic Suslinian curves (see § 4). Although the general question remains unsolved (see [6], Problem 10), it seems now to be answered almost completely for the class of atriodic curves, which comprises some interesting cases: a classical example of a plane curve constructed by G. T. Whyburn [10] as well as other curves obtained by means of the method of R. D. Anderson and Gustave Choquet [1]. The topological structure of atriodic hereditarily decomposable curves is essential in our approach (see § 2). Also, at the end of the paper, we provide an example of a chainable Suslinian curve that is not rational.

§ 1. Hereditarily discontinuous subsets. A space is called *hereditarily discontinuous* provided each continuum contained in it is degenerate [1]. A curve X is called *atriodic* provided, for each three subcurves C_1, C_2, C_3 of X such that

$$C_0 = C_1 \cap C_2 = C_2 \cap C_3 = C_1 \cap C_3 \neq \emptyset$$

is connected, C_0 coincides with at least one of the curves C_1, C_2, C_3. We follow [3] to mean by a *clump* any non-degenerate collection $\boldsymbol{C}$ of continua whose union is a continuum and for which there exists a non-empty continuum C_0, called the *core* of $\boldsymbol{C}$, such that C_0 is a proper subset of every

[1] Hereditarily discontinuous spaces were called "ponctiform" in [5] but now we adopt the terminology of [4] which seems to be more suitable for this paper.

element of $\boldsymbol{C}$ and C_0 is the common part of each two elements of $\boldsymbol{C}$. Thus no clump having more than two elements can exist in an atriodic curve. We recall that a curve X is called *Suslinian* provided each collection of pairwise disjoint subcurves of X is countable (see [5] and [6], for a discussion).

1.1. THEOREM. *If X is a Suslinian curve such that $X \neq P \cup Q$, where P is hereditarily discontinuous and Q is countable, then there exists a point $p \in X$ and two infinite sequences of continua $C_n \subset X$ and $K_n \subset X$ such that*

$$p \in C_n, \quad C_{n+1} \subset C_n, \quad \operatorname{diam} C_n < n^{-1},$$

$$p \notin K_n, \quad K_n \cap K_m = \emptyset, \quad \operatorname{diam} K_n < n^{-1},$$

and $C_n \cap K_n \neq \emptyset \neq K_n \setminus C_1$ for all positive integers n and m $(m \neq n)$.

Proof. Let $\{G_1, G_2, \ldots\}$ be a countable open basis in X and let $\boldsymbol{C}_i$ denote the collection of all non-degenerate components of $\operatorname{cl} G_i$ $(i = 1, 2, \ldots)$. For $C \in \boldsymbol{C}_i$, let $q(C)$ be a point belonging to C. Since X is Suslinian, each $\boldsymbol{C}_i$ is countable. Therefore the set

$$Q = \bigcup_{i=1}^{\infty} \{q(C)\colon\ C \in \boldsymbol{C}_i\}$$

is countable and its complement $X \setminus Q$ is not hereditarily discontinuous. Let $Y \subset X \setminus Q$ be a non-degenerate continuum such that $\operatorname{diam} Y < 1$. Since X is hereditarily decomposable (see [5], p. 131), we have $Y = C_1 \cup K_1$ where C_1 and K_1 are proper subcontinua of Y. Take any point $p \in C_1 \setminus K_1$ and assume that continua $C_1, \ldots, C_n$ and $K_1, \ldots, K_n$ are already defined in such a way that they fulfill conditions of 1.1. We define continua C_{n+1} and K_{n+1} as follows. Let C_{n+1} be any continuum such that

$$p \in C_{n+1} \subset C_n, \quad \operatorname{diam} C_{n+1} < (n+1)^{-1},$$

and let $q \in C_{n+1}$ be a point such that $p \neq q \notin K_m$ for $m = 1, \ldots, n$. Then there exists a positive integer j such that

$$q \in G_j \subset \operatorname{cl} G_j \subset X \setminus (\{p\} \cup K_1 \cup \ldots \cup K_n),$$

and $\operatorname{diam} G_j < (n+1)^{-1}$. Let K_{n+1} be the component of $\operatorname{cl} G_j$ which contains q. Thus $q \in C_{n+1} \cap K_{n+1}$ and K_{n+1} is non-degenerate, whence $K_{n+1} \in \boldsymbol{C}_j$. Consequently, we also have

$$q(K_{n+1}) \in K_{n+1} \cap Q \subset K_{n+1} \setminus Y \subset K_{n+1} \setminus C_1,$$

and the proof of 1.1 is complete.

1.2. *If X is a Suslinian curve such that each point of X has a neighborhood in which no infinite clump exists, then X admits a decomposition $X = P \cup Q$, where P is hereditarily discontinuous and Q is countable.*

Proof. Suppose C_n and K_n are continua satisfying 1.1. Then the collection

$$\boldsymbol{C}_n = \{C_n \cup K_{n+i}\colon\ i = 1, 2, \ldots\}$$

is an infinite clump whose core is C_n $(n = 1, 2, \ldots)$. Moreover, the elements of $\boldsymbol{C}_n$ lie in a $(2/n)$-neighborhood of the point p, and 1.2 follows from 1.1.

1.3. *If X is an atriodic Suslinian curve, then X admits a decomposition $X = P \cup Q$, where P is hereditarily discontinuous and Q is countable.*

Proof. Since no infinite clump exists in an atriodic curve, 1.2 implies 1.3.

Remark. A result stronger than 1.3 is obtained in 3.2 below; but our argument leading to 3.2 utilizes 1.3.

§ 2. Atriodic hereditarily decomposable curves. We denote by S the standard 1-sphere, i.e. the unit circle composed of complex numbers with module one and furnished with the natural topology.

2.1. THEOREM. *If X is an atriodic hereditarily decomposable curve, then there exists a monotone continuous mapping $g\colon X \to S$ such that $g^{-1}(z)$ has void interior for $z \in S$.*

Proof. We distinguish two cases.

Case 1: X is an irreducible continuum. Then there exists a monotone continuous mapping $\varphi\colon X \to I$ of X onto the unit segment I of the real line such that $\varphi^{-1}(t)$ has void interior in X for $t \in I$ (see [4], pp. 200 and 216); let $f\colon I \to S$ be an embedding and let us put $g = f\varphi$.

Case 2: X is not an irreducible continuum. Since X is atriodic, it follows that X is not unicoherent (see [9], p. 456). Thus there exists a decomposition $X = X_1 \cup X_2$ of X into subcurves X_1 and X_2 such that the common part $X_1 \cap X_2$ is not connected. Let $X_1 \cap X_2 = A \cup B$, where A and B are non-empty disjoint closed subsets. There exists a closed set $Y_1 \subset X_1$ which is irreducibly connected between A and B (see [4], p. 222). Consequently, we have

$$A \cap Y_1 \neq \emptyset \neq B \cap Y_1$$

and, for each two points $a \in A \cap Y_1$ and $b \in B \cap Y_1$, the set Y_1 is an irreducible continuum between a and b (ibidem). As in Case 1, there exists a monotone continuous mapping $\psi_1\colon Y_1 \to I$ of Y_1 onto I such that $\psi_1^{-1}(t)$ has void interior in Y_1 for $t \in I$. Therefore the sets $\psi_1^{-1}(t)$ also have void interiors in X and we can assume that

$$A \cap Y_1 \subset \psi_1^{-1}(0), \quad B \cap Y_1 \subset \psi_1^{-1}(1). \tag{1}$$

Similarly, there exists a closed set $Y_2 \subset X_2$ which is irreducibly connected between $A \cap Y_1$ and $B \cap Y_1$, whence

$$A \cap Y_1 \cap Y_2 \neq \emptyset \neq B \cap Y_1 \cap Y_2$$

and Y_2 is an irreducible continuum. Moreover, there exists a monotone continuous mapping $\psi_2\colon Y_2 \to I$ such that $\psi_2^{-1}(t)$ has void interior for $t \in I$, and

$$(2) \qquad A \cap Y_1 \cap Y_2 \subset \psi_2^{-1}(0)\,, \qquad B \cap Y_1 \cap Y_2 \subset \psi_2^{-1}(1)\,.$$

We shall prove that $X = Y_1 \cup Y_2$. Indeed, suppose on the contrary that $X_i \setminus (Y_1 \cup Y_2) \neq \emptyset$ for $i = 1$ or 2. Assume $i = 1$ and observe that Y_2 is a continuum which meets two disjoint closed sets A and B. Thus Y_2 is not contained in $A \cup B$ and, consequently, the continuum Y_2 meets the open set $X \setminus X_1$. But since all the sets $\psi_2^{-1}(t)$ have void interiors in Y_2, we conclude that there exist numbers $t_0, t_1 \in I$ such that

$$\psi_2^{-1}(t_0) \setminus X_1 \neq \emptyset \neq \psi_2^{-1}(t_1) \setminus X_1$$

and $0 < t_0 < t_1 < 1$. Then the sets

$$\begin{aligned} C_1 &= X_1\,, \\ C_2 &= Y_1 \cup \{y \in Y_2\colon\ 0 \leqslant \psi_2(y) \leqslant t_0\}\,, \\ C_3 &= Y_1 \cup \{y \in Y_2\colon\ t_1 \leqslant \psi_2(y) \leqslant 1\} \end{aligned}$$

are continua, by (2), which have a point in common and no one of them is a subset of the union of the other two. The latter statement follows from the inclusions

$$\begin{aligned} \emptyset &\neq X_1 \setminus (Y_1 \cup Y_2) \subset C_1 \setminus (C_2 \cup C_3)\,, \\ \emptyset &\neq \psi_2^{-1}(t_0) \setminus X_1 \subset C_2 \setminus (C_1 \cup C_3)\,, \\ \emptyset &\neq \psi_2^{-1}(t_1) \setminus X_1 \subset C_3 \setminus (C_1 \cup C_2)\,, \end{aligned}$$

and it contradicts the condition that X is atriodic (see [9], p. 443). The argument for $i = 2$ being quite analogous, we conclude $X = Y_1 \cup Y_2$ must be true.

Now, since we also have

$$\begin{aligned} Y_1 \cap Y_2 = X_1 \cap X_2 \cap Y_1 \cap Y_2 &= (A \cap Y_1 \cap Y_2) \cup (B \cap Y_1 \cap Y_2) \\ &\subset [\psi_1^{-1}(0) \cap \psi_2^{-1}(0)] \cup [\psi_1^{-1}(1) \cap \psi_2^{-1}(1)] \end{aligned}$$

according to (1) and (2), a continuous mapping $g\colon X \to S$ can be defined by the formula

$$g(x) = \begin{cases} e^{\pi i \psi_1(x)} & \text{for} \quad x \in Y_1\,, \\ e^{-\pi i \psi_2(x)} & \text{for} \quad x \in Y_2\,; \end{cases}$$

and then $z \in S \setminus \{-1, 1\}$ implies that $g^{-1}(z)$ is either one of the sets $\psi_1^{-1}(t)$ or one of the sets $\psi_2^{-1}(t)$. But the equalities

$$g^{-1}(1) = \psi_1^{-1}(0) \cup \psi_2^{-1}(0)\,, \qquad g^{-1}(-1) = \psi_1^{-1}(1) \cup \psi_2^{-1}(1)$$

imply that the sets $g^{-1}(1)$ and $g^{-1}(-1)$ are continua too, by (1) and (2). Moreover, all these sets have void interiors which completes the proof of 2.1.

2.2. *If X is an atriodic curve and $f, g\colon X \to S$ are monotone continuous mappings such that $f^{-1}(z)$ and $g^{-1}(z)$ have void interiors for $z \in S$, then $f^{-1}f(x) = g^{-1}g(x)$ for $x \in X$.*

Proof. Suppose there exist two points $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$ and $g(x_1) \neq g(x_2)$. Then $C_0 = f^{-1}f(x_1)$ is a continuum and there is an arc $J \subset g(C_0)$ with end points $g(x_1)$ and $g(x_2)$. Consequently, the interior G of $g^{-1}(J)$ is a non-empty open subset of X, and so $G \setminus C_0 \neq \emptyset$ because C_0 has void interior. But since all the sets $g^{-1}(z)$ have void interiors, we conclude that there exist three points $z_1, z_2, z_3 \in J$ such that

$$g^{-1}(z_i) \cap (G \setminus C_0) \neq \emptyset$$

for $i = 1, 2, 3$. Then the sets

$$C_i = C_0 \cup g^{-1}(z_i)$$

are continua which differ from C_0 and $C_0 = C_1 \cap C_2 = C_2 \cap C_3 = C_1 \cap C_3$, contrary to the assumption that X is atriodic. We have thus proved that $f(x_1) = f(x_2)$ implies $g(x_1) = g(x_2)$, and a symmetric argument yields the reverse implication completing the proof of 2.2.

2.3. *If X is an atriodic hereditarily decomposable curve, then there exists exactly one upper semi-continuous decomposition of X into frontier continua such that the decomposition space is either an arc or a simple closed curve.*

Proof. The existence of such an upper semi-continuous decomposition follows from 2.1 and its uniqueness is a consequence of 2.2; so 2.3 is proved.

Given an atriodic hereditarily decomposable curve X, we call *tranches* of X the frontier continua which appear in 2.3. The curve X is the union of its tranches and each tranche is a proper subcontinuum of X. However, some tranches may be degenerate and let us slightly extend our definition to mean by a tranche of a degenerate set the set itself. For an atriodic hereditarily decomposable curve X (or X degenerate) and a point $x \in X$, we denote by $T(X, x)$ the tranche of X that contains x. We write $T^0(X, x) = X$, and we use a transfinite induction to define $T^\alpha(X, x)$ for each ordinal α, namely

$$T^{\alpha+1}(X, x) = T\big(T^\alpha(X, x), x\big)$$

and

$$T^{\lambda}(X, x) = \bigcap_{\alpha<\lambda} T^{\alpha}(X, x)$$

for limit λ. The set $T^{\alpha}(X, x)$ is said to be the *tranche of order* α of the atriodic hereditarily decomposable curve X at the point x. Observe that $T^{\alpha}(X, x)$ is a subcontinuum of X, and therefore the decreasing transfinite sequence

$$T^0(X, x) \supset T^1(X, x) \supset \ldots \supset T^{\alpha}(X, x) \supset \ldots$$

must have a term $T^{\alpha}(X, x)$ which is equal to the next term $T^{\alpha+1}(X, x)$. This means $T^{\alpha}(X, x)$ is degenerate, and $T^{\alpha}(X, x) = \{x\} = T^{\beta}(X, x)$ for all ordinals $\beta > \alpha$. Moreover, the ordinal

$$\mathrm{nd}(X, x) = \mathrm{Min}\{\alpha\colon\ T^{\alpha}(X, x) = T^{\alpha+1}(X, x)\}$$

is countable for $x \in X$, the space X being separable metric. Thus the countable ordinal $\mathrm{nd}(X, x)$ indicates the level on which the tranches of higher orders of X at x become no longer non-degenerate. Let us point out that the collection of all tranches of a fixed order $\alpha < \Omega$ of X constitutes a decomposition of X into pairwise disjoint continua ([2]). In general, however, this decomposition is not upper semi-continuous for $\alpha > 1$.

2.4. *If X is an atriodic hereditarily decomposable curve and $A \subset X$ is a hereditarily discontinuous set, then*

$$\mathrm{Sup}\{\mathrm{nd}(X, x)\colon\ x \in X\} = \mathrm{Sup}\{\mathrm{nd}(X, x)\colon\ x \in X\setminus A\}.$$

Proof. Denote by σ_1 and σ_2 the suprema which stay on the left and the right sides, respectively. Clearly, we have $\sigma_1 \geqslant \sigma_2$. If $\alpha < \sigma_1$, then there exists a point $x_{\alpha} \in X$ such that $\mathrm{nd}(X, x_{\alpha}) > \alpha$ which means that

$$T^{\alpha}(X, x_{\alpha}) \neq T^{\alpha+1}(X, x_{\alpha}),$$

whence $T^{\alpha}(X, x_{\alpha})$ is non-degenerate. The set A being hereditarily discontinuous, there exists a point $y_{\alpha} \in X\setminus A$ such that $y_{\alpha} \in T^{\alpha}(X, x_{\alpha})$. Consequently, we have

$$T^{\alpha}(X, x_{\alpha}) = T^{\alpha}(X, y_{\alpha}),$$

whence $T^{\alpha}(X, y_{\alpha})$ is non-degenerate. It follows that

$$\alpha < \mathrm{nd}(X, y_{\alpha}) \leqslant \sigma_2,$$

and we see $\alpha < \sigma_1$ implies $\alpha < \sigma_2$. Thus $\sigma_1 \leqslant \sigma_2$ which completes the proof of 2.4.

§ 3. Hereditarily disconnected subsets. A space is called *hereditarily disconnected* provided each connected set contained in it is degenerate; each hereditarily disconnected space is hereditarily discontinuous.

([2]) We denote by Ω the minimum uncountable ordinal.

3.1. *If X is an atriodic Suslinian curve, then*

$$\mathrm{Sup}\{\mathrm{nd}(X, x)\colon\ x \in X\} < \Omega.$$

Proof. By 1.3, there exists a subset $P \subset X$ such that P is hereditarily discontinuous and $X\setminus P$ is countable. By 2.4, we obtain

$$\mathrm{Sup}\{\mathrm{nd}(X, x)\colon\ x \in X\} = \mathrm{Sup}\{\mathrm{nd}(X, x)\colon\ x \in X\setminus P\} < \Omega.$$

3.2. Theorem. *If X is an atriodic Suslinian curve, then X admits a decomposition $X = P \cup Q$, where P is hereditarily disconnected and Q is countable.*

Proof. According to 3.1, the ordinal

$$\sigma = \mathrm{Sup}\{\mathrm{nd}(X, x)\colon\ x \in X\}$$

is countable, and all tranches of orders $\alpha \geqslant \sigma$ of X are degenerate. Since X is a Suslinian curve and, for $\alpha < \sigma$, the collection $\boldsymbol{C}_{\alpha}$ of all non-degenerate tranches of order α of X consists of pairwise disjoint subcurves of X, we conclude $\boldsymbol{C}_{\alpha}$ is countable. For $C \in \boldsymbol{C}_{\alpha}$, let $g_C\colon C \to S$ be a monotone continuous mapping such that the sets $g_C^{-1}(z)$ are tranches of C as given by 2.1 and 2.3. The latter sets are frontier continua and those of them which are non-degenerate belong to $\boldsymbol{C}_{\alpha+1}$. Thus the mapping g_C has only countably many non-degenerate point-inverses, and there exists a countable set $D_C \subset S$ such that D_C is dense in $g_C(C)$ and $g_C^{-1}(z)$ is degenerate for $z \in D_C$. Consequently, the sets $g_C^{-1}(D_C)$ are countable for $C \in \boldsymbol{C}_{\alpha}$ and $\alpha < \sigma$. We define

$$Q = \bigcup_{\alpha<\sigma} \bigcup_{C \in \boldsymbol{C}_{\alpha}} g_C^{-1}(D_C)$$

so that Q is countable and it remains to show the set $P = X\setminus Q$ is hereditarily disconnected.

Let us suppose on the contrary that there exists a non-degenerate connected set $A \subset P$. Take a point $a \in A$. First, let us prove by induction on α that

$$A \subset T^{\alpha}(X, a) \tag{3}$$

for $\alpha \leqslant \sigma$. Inclusion (3) is trivially true for $\alpha = 0$. We assume that $\beta \leqslant \sigma$ and that (3) holds for each ordinal $\alpha < \beta$. It will be proved that (3) holds for $\alpha = \beta$. Indeed, if β is a limit ordinal, the inclusion follows directly from the definition of the tranche of a limit order. On the other hand, if $\beta = \alpha+1$, we have $\alpha < \sigma$ and $A \subset C \in \boldsymbol{C}_{\alpha}$, where $C = T^{\alpha}(X, a)$. But since

$$A \subset C \cap P = C\setminus Q \subset C\setminus g_C^{-1}(D_C),$$

the sets D_C and $g_C(A)$ are disjoint. Thus the connected set $g_C(A)$ has void interior in $g_C(C) \subset S$ which means $g_C(A)$ is degenerate. In other words, the set A is contained in a tranche of C, i.e. we have

$$A \subset T(C, a) = T(T^{\alpha}(X, a), a) = T^{\beta}(X, a).$$

Applying inclusion (3) for $\alpha = \sigma$, we see that the tranche $T^{\sigma}(X, a)$ contains a non-degenerate set A, so it is itself non-degenerate. This contradicts the definition of σ and 3.2 is proved.

Remark. There exists an example of a locally connected Suslinian curve due to S. Mazurkiewicz [7] which shows that the condition of being atriodic cannot be removed from 3.2. In the next section we describe another example to the same effect. Our curve, as constructed in 4.1, is not locally connected but it has an advantage over the Mazurkiewicz curve by being acyclic. Both curves lie on the plane. We also show in 4.2 that 3.2 cannot be strengthened by requiring that P is totally disconnected rather than hereditarily disconnected.

§ 4. Some examples of Suslinian curves. A curve X is said to be a *λ-dendroid* provided X is hereditarily unicoherent and hereditarily decomposable. A curve X is called *acyclic* provided each continuous mapping of X into S is homotopic to a constant mapping. It is known that a hereditarily decomposable curve is acyclic if and only if it is hereditarily unicoherent. Since all Suslinian curves are hereditarily decomposable (see [5], p. 131), we note that the Suslinian λ-dendroids are acyclic curves.

4.1. Example. *There exists a Suslinian λ-dendroid X such that X lies on the plane and, for each countable subset $Q \subset X$, the set $X \setminus Q$ is connected.*

Proof. Let $\pi = (D, A)$ be an ordered pair composed of a disk D and an arc A which is contained in the boundary $\mathrm{bd}D$ of D. We take points $p_i = (i, (-1)^i)$ on the plane and triangles T_i with vertexes p_i, p_{i+1}, p_{i+2} for $i = 1, 2, \ldots$ Then the union of the straight segments $\overline{p_i p_{i+1}}$ and $\overline{p_{i+1} p_{i+2}}$ is an arc contained in $\mathrm{bd}\,T_i$. Let us denote

$$K = \bigcup_{i=1}^{\infty} T_i, \qquad L = \bigcup_{i=1}^{\infty} \overline{p_i p_{i+1}}$$

and, for a given positive integer n, let us take an embedding $h\colon K \to D$ such that

$$h(K) \cap \mathrm{bd}D = \emptyset, \qquad A = \mathrm{cl}\,h(K) \setminus h(K) = \mathrm{cl}\,h(L) \setminus h(L),$$

and $\mathrm{diam}\,h(T_i) < n^{-1}$ for $i = 1, 2, \ldots$ Moreover, we assume that the diameters of the disks $h(T_i)$ converge to zero when i tends to the infinity. It follows that, for any selection of points $q_i \in h(T_i)$ $(i = 1, 2, \ldots)$, each point of the arc A is the limit point of a subsequence of the sequence $q_1, q_2, \ldots$ We put

$$\mathfrak{T}(\pi, n) = \{(h(T_i), h(\overline{p_i p_{i+1}} \cup \overline{p_{i+1} p_{i+2}}))\colon i = 1, 2, \ldots\}.$$

Given a pair π as above, we define collections $\mathfrak{T}_n$ of pairs inductively by setting

$$\mathfrak{T}_0 = \{\pi\}, \qquad \mathfrak{T}_{n+1} = \bigcup_{\pi \in \mathfrak{T}_n} \mathfrak{T}(\pi, n+1),$$

for $n = 0, 1, \ldots$ Finally, we denote by X_n the union of all disks in pairs from $\mathfrak{T}_n$, and define

$$X = \bigcap_{n=0}^{\infty} \mathrm{cl}\,X_n.$$

The sets X_n form a decreasing sequence and so do their closures. Moreover, each of these closures is a subcontinuum of the disk, and so is X. For $n = 0, 1, \ldots$, let us select a countable dense subset in the boundary of each disk which appears in a pair belonging to $\mathfrak{T}_n$. Let Q_n be the union of all these countable sets. Since $\mathfrak{T}_n$ clearly are countable collections, the sets Q_n are countable, and we are going to show that the set

$$P = X \setminus \bigcup_{n=0}^{\infty} Q_n$$

is hereditarily discontinuous. Suppose on the contrary that there exists a non-degenerate continuum $C \subset P$. Since C is disjoint with Q_n, the common part of C and the boundary of each disk from $\mathfrak{T}_n$ is a compact 0-dimensional set $(n = 0, 1, \ldots)$. Observe that the diameters of disks from $\mathfrak{T}_n$ converge to zero when n tends to the infinity. Thus there exists a non-negative integer j such that no disk from $\mathfrak{T}_j$ contains C. If $C \cap X_{j+1} = \emptyset$, then C would be contained in the set $\mathrm{cl}\,X_{j+1} \setminus X_{j+1}$ which is contained in the union of the boundaries of disks from $\mathfrak{T}_i$, where $i \leqslant j$. Consequently, the continuum C would be the union of all intersections of C with these boundaries, i.e. the union of countably many compact 0-dimensional sets. Therefore C would be 0-dimensional, i.e. degenerate, which is not the case. We infer that $C \cap X_{j+1} \neq \emptyset$ and let D' be a disk from $\mathfrak{T}_j$ which contains a disk D'' from $\mathfrak{T}_{j+1}$ such that $C \cap D'' \neq \emptyset$. Then D'' is the image of a triangle T_{i_0} under an embedding $h'\colon K \to D'$ and the interior

$$\mathrm{int}\,D' = D' \setminus \mathrm{bd}\,D'$$

does not meet any disk from $\mathfrak{T}_j$ but D'. It follows that

$$\mathrm{cl}\,X_{j+1} \cap \mathrm{int}\,D' \subset h'(K),$$

whence $C \cap \mathrm{int}\,D' \subset h'(K)$. By the definition of j, the continuum C is not contained in D'. Thus if we had $C \cap h'(T_{i_1}) = \emptyset$ for an integer $i_1 > i_0$, the set

$$C \cap \bigcup_{i=1}^{i_1-1} h'(T_i) \supset C \cap D'' \neq \emptyset$$

would be a proper closed-open subset of C, which is impossible. Consequently, for each integer $i > i_0$, there exists a point $q'_i \in C \cap h'(T_i)$, and each point of an arc $A' \subset \operatorname{bd} D'$ is the limit point of a subsequence of the sequence $q'_{i_0+1}, q'_{i_0+2}, \ldots$ We obtain $A' \subset C$, which contradicts the fact that $C \cap \operatorname{bd} D'$ is 0-dimensional. This proves that P is indeed hereditarily discontinuous. But $X \backslash P$ being countable, we conclude that X is a Suslinian curve (see [5], p. 131).

The continua $\operatorname{cl} X_n$ do not cut the plane $(n = 0, 1, \ldots)$ what one can readily check by using their definition. Hence the curve X does not cut the plane too, and it follows that X is acyclic (see [4], p. 470). Since X is also Suslinian, X is hereditarily decomposable. Thus X is a Suslinian λ-dendroid.

It will be proved that $X \backslash Q$ is connected for each countable subset $Q \subset X$, if we show that each compact 0-dimensional set $Z \subset X$ with non-connected complement $X \backslash Z$ contains a Cantor set. Let us consider such a set Z. Then $X \backslash Z = G \cup H$, where G, H are disjoint non-empty open subsets of X. Given any pair $\pi = (D, A)$ belonging to $\mathfrak{T}_n$, we have an embedding $h\colon K \to D$ such that $A = \operatorname{cl} h(L) \backslash h(L)$ and $h(L)$ is a topological copy of the closed half-line. Let $\boldsymbol{L}_n$ denote the collection of all these half-lines $h(L)$, where $\pi \in \mathfrak{T}_n$ $(n = 0, 1, \ldots)$. Observe that each point of $h(L)$ belongs to at least one arc $h(\overline{p_i p_{i+1}}) \subset A'$, where

$$\big(h(T_i), A'\big) \in \mathfrak{T}(\pi, n+1) \subset \mathfrak{T}_{n+1},$$

and $A' = \operatorname{cl} h'(L) \backslash h'(L)$, where $h'(L) \in \boldsymbol{L}_{n+1}$. As a result we get $|\boldsymbol{L}_n| \subset \operatorname{cl} |\boldsymbol{L}_{n+1}|$ (³). But clearly $|\boldsymbol{L}_n| \subset X_n$ whence $|\boldsymbol{L}_n| \subset \operatorname{cl} X_m$ for $m \geqslant n$, and $|\boldsymbol{L}_n| \subset X$ for $n = 0, 1, \ldots$ Since the disks from $\mathfrak{T}_n$ meet $|\boldsymbol{L}_n|$ and their diameters converge to zero, the union $|\boldsymbol{L}_0| \cup |\boldsymbol{L}_1| \cup \ldots$ is a dense subset of X. We have only one half-line in $\boldsymbol{L}_0$ and the closure of each half-line belonging to $\boldsymbol{L}_{n+1}$ intersects a half-line from $\boldsymbol{L}_n$. Thus, if all the half-lines from $\boldsymbol{L}_n$ $(n = 0, 1, \ldots)$ were contained in one of the sets G, H, say in G, the set G would be dense in X whence $H = \emptyset$ which is impossible. Consequently, there exists a positive integer k and a half-line $L_{-1} \in \boldsymbol{L}_{k-1}$ such that $L_{-1} \cap G \neq \emptyset \neq L_{-1} \cap H$. But the set

$$L_{-1} \backslash (G \cup H) = L_{-1} \cap Z$$

being 0-dimensional, there exists a point $q_* \in L_{-1} \cap Z$ such that q_* belongs to the closures of both sets $L_{-1} \cap G$, $L_{-1} \cap H$. Moreover, we have $L_{-1} = h_{-1}(L)$, where h_{-1} is a homeomorphism, and there exists a positive integer j_* such that the arc

$$A_* = h_{-1}(\overline{p_{j_*} p_{j_*+1}} \cup \overline{p_{j_*+1} p_{j_*+2}})$$

(³) By $|\boldsymbol{L}_n|$ we denote the union of all half-lines belonging to $\boldsymbol{L}_n$.

contains q_* as an interior point. Hence $A_* \cap G \neq \emptyset \neq A_* \cap H$ and $\pi_* = (D_*, A_*) \in \mathfrak{T}_k$, where $D_* = h_{-1}(T_{j_*})$. The collection $\mathfrak{T}_{k+1}$ contains the collection $\mathfrak{T}(\pi_*, k+1)$ which is obtained by taking images of triangles T_i under an embedding $h_*\colon K \to D_*$. The half-line $L_* = h_*(L)$ is an element of the collection $\boldsymbol{L}_k$ and $A_* = \operatorname{cl} L_* \backslash L_*$. It follows that $L_* \cap G \neq \emptyset \neq L_* \cap H$ and there exists a point $q_0 \in L_* \cap Z$ such that q_0 belongs to the closures of both sets $L_* \cap G$, $L_* \cap H$. Accordingly, there exists a positive integer j_0 such that the arc

$$A_0 = h_*(\overline{p_{j_0} p_{j_0+1}} \cup \overline{p_{j_0+1} p_{j_0+2}})$$

contains q_0 as an interior point. Hence $A_0 \cap G \neq \emptyset \neq A_0 \cap H$ and $\pi_0 = (D_0, A_0) \in \mathfrak{T}_{k+1}$, where $D_0 = h_*(T_{j_0})$. What is more, in the last three sentences we can, in lieu of L_*, take as well the half-line

$$L'_* = h_*\Big(\bigcup_{i=j_0+3}^{\infty} \overline{p_i p_{i+1}}\Big)$$

and then we obtain a point $q_1 \in L'_* \cap Z$ belonging to the closures of both sets $L'_* \cap G$, $L'_* \cap H$ and being an interior point of an arc

$$A_1 = h_*(\overline{p_{j_1} p_{j_1+1}} \cup \overline{p_{j_1+1} p_{j_1+2}}),$$

where $j_1 > j_0 + 2$. Hence also $A_1 \cap G \neq \emptyset \neq A_1 \cap H$ and $\pi_1 = (D_1, A_1) \in \mathfrak{T}_{k+1}$, where $D_1 = h_*(T_{j_1})$. Since $j_1 > j_0 + 2$, the triangles T_{j_0}, T_{j_1} are disjoint, and so are the disks D_0, D_1. Let us notice that D_0, D_1 contain the points q_0, q_1 of Z, respectively. Repeating the same procedure, we find pairwise disjoint disks $D_{00}, D_{01}, D_{10}, D_{11}$ such that each of them meets Z. The disks D_{00}, D_{01} and the disks D_{10}, D_{11} appear in some pairs belonging to $\mathfrak{T}(\pi_0, k+2)$ and $\mathfrak{T}(\pi_1, k+2)$, respectively. Thus $D_{00}, D_{01} \subset D_0$ and $D_{10}, D_{11} \subset D_1$. Continuing this procedure, one gets 2^m pairwise disjoint disks from $\mathfrak{T}_{k+m}$ such that each of them meets Z $(m = 1, 2, \ldots)$. Let U_m denote the union of all these 2^m disks. We have $U_1 \supset U_2 \supset \ldots$ Since Z is compact and the diameters of disks from $\mathfrak{T}_{k+m}$ converge to zero when m tends to the infinity, it follows that the intersection $U_1 \cap U_2 \cap \ldots$ is a Cantor set contained in Z. The combination of properties of the curve X as stated in 4.1 is now verified.

Remark. The curve described in 4.1 resembles to some extent a curve constructed by J. J. Charatonik [2] which, however, has admitted countable cuttings. An example of a curve given in 4.2 below is a modification of a Suslinian curve whose construction has also been published earlier (see [5], p. 135), and the key role in both these examples is played by an idea essentially due to W. Sierpiński [8].

A space is called *totally disconnected* provided each of its quasi-coimponents is degenerate. Each totally disconnected space is hereditarlydisconnected. It is known that a curve X is rational if and only if X admits a decomposition $X = P \cup Q$, where P is totally disconnected and Q is countable (see [6], p. 95).

4.2. Example. *There exists a chainable Suslinian curve X such that $X \neq P \cup Q$, where P is totally disconnected and Q is countable.*

Proof. Let $\varrho = (R, v)$ be a pair composed of a right triangle R and a vertex v of an acute angle in R. We denote by S the side of R opposite to v, and let us write $S = \overline{rr'}$, where r is the vertex of the right angle in R.

Take points $p_1, p_2 \in S$ and $q_1, q_2 \in \overline{vr'}$ such that

$$\operatorname{dist}(p_1, r) = \tfrac{1}{3}\operatorname{dist}(r, r'), \quad \operatorname{dist}(p_2, r) = \tfrac{2}{3}\operatorname{dist}(r, r'),$$
$$\operatorname{dist}(q_1, v) = \tfrac{1}{3}\operatorname{dist}(v, r'), \quad \operatorname{dist}(q_2, v) = \tfrac{2}{3}\operatorname{dist}(v, r'),$$

and points $x_i \in \overline{vr}$, $y_i \in \overline{vr'}$ such that

$$\operatorname{dist}(x_i, r) = i^{-1}\operatorname{dist}(v, r), \quad \operatorname{dist}(y_i, r') = i^{-1}\operatorname{dist}(v, r')$$

for $i = 1, 2, \ldots$ Let z_{ij} be the intersection point of the segment $\overline{x_{3i}y_{3i}}$ with the segment $\overline{p_jq_j}$ for $i = 1, 2, \ldots$ and $j = 1, 2$. We can find points $r_i \in \overline{z_{i1}z_{i2}}$ such that $r_1 = z_{11}$ and each point of the segment $\overline{p_1p_2}$ is the limit point of a subsequence of the sequence $r_1, r_2, \ldots$ Let $v_i \in \overline{x_{3i-1}y_{3i-1}}$, $v_i' \in \overline{x_{3i+1}y_{3i+1}}$ be points such that $r_i \in \overline{v_iv_i'}$ and the segment $\overline{v_iv_i'}$ is parallel to the side $\overline{vr}$. Denote by R_i' the right triangle with vertexes v_i, r_i, x_{3i}, by R_i'' the right triangle with vertexes v_i', r_i, y_{3i}, and put

$$\mathcal{R}(\varrho) = \{(R_i', v_i)\colon i = 1, 2, \ldots\} \cup \{(R_i'', v_i')\colon i = 1, 2, \ldots\}.$$

Moreover, let $\boldsymbol{S}(\varrho)$ denote the collection consisting of the segment $\overline{vv_1}$ and of all the segments $\overline{v_i'v_{i+1}}$ for $i = 1, 2, \ldots$ Observe that the segments belonging to $\boldsymbol{S}(\varrho)$ join the triangles occurring in pairs from $\mathcal{R}(\varrho)$ so that their union is a connected set.

Given a pair ϱ as above, we define collections $\mathcal{R}_n$ of pairs inductively by setting

$$\mathcal{R}_0 = \{\varrho\}, \quad \mathcal{R}_{n+1} = \bigcup_{\varrho \in \mathcal{R}_n} \mathcal{R}(\varrho)$$

for $n = 0, 1, \ldots$ Finally, we denote by X_n the union of all triangles in pairs from $\mathcal{R}_n$ and of all segments belonging to $\boldsymbol{S}(\varrho)$, where $\varrho \in \mathcal{R}_0 \cup \ldots \cup \mathcal{R}_n$, and define X to be the intersection of the decreasing sequence of the continua $\operatorname{cl} X_n$ $(n = 0, 1, \ldots)$.

One has to use only a standard technique in showing that X is a chainable curve. The argument for the decomposition property of X is a replica of an argument for same property of a dendroid (see [5], p. 136). Actually, one shows that each compact 0-dimensional set cutting X between some points of the segment S is uncountable (ibidem, Lemma 3.2). The short proof of the fact that same dendroid is Suslinian can also be applied here almost without change to show that X is Suslinian.

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