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Uniformly movable compact spaces and their algebraic properties

by

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Contents

Let us consider a pointed compact Hausdorff space (X, x_0) and an ANR-system $(\underline{X}, \boldsymbol{x}_0)$ associated with (X, x_0) (see [3]). Let $\pi_n(\boldsymbol{X}, \boldsymbol{x}_0)$ be an inverse system of nth homotopy groups. Its inverse limit does not depend on the choice of $(\boldsymbol{X}, \boldsymbol{x}_0)$ (see 6.3) and here is referred to as the nth limit homotopy group of (X, x_0) (in symbols $\pi_n^*(X, x_0)$).

Consider two pairs, (X, x_0) and (Y, y_0), and the associated inverse systems $(\boldsymbol{X}, \boldsymbol{x}_0)$ and $(\boldsymbol{Y}, \boldsymbol{y}_0)$. To any map $\boldsymbol{f}$: $(\boldsymbol{X}, \boldsymbol{x}_0) \to (\boldsymbol{Y}, \boldsymbol{y}_0)$ (in the sense of [3]) and a natural number n we can assign the induced morphism $\boldsymbol{f}_n$: $\pi_n(\boldsymbol{X}, \boldsymbol{x}_0) \to \pi_n(\boldsymbol{Y}, \boldsymbol{y}_0)$ and its inverse limit, $\varprojlim \boldsymbol{f}_n = f_n^*$: $\pi_n^*(X, x_0) \to \pi_n^*(Y, y_0)$.

In general, the algebraic properties of $\varprojlim \boldsymbol{f}_n$ do not determine the algebraic properties of $\boldsymbol{f}_n$. For instance, the implication

$$(*) \qquad \varprojlim \boldsymbol{f}_n \text{ is a bimorphism} \Rightarrow \boldsymbol{f}_n \text{ is a bimorphism}$$

in general fails (see § 6).

The purpose of this paper is to distinguish a class of spaces which satisfies (*). This leads to the notion of uniform movability of an inverse system in an arbitrary category, in particular in the category of ANR's or in the category of groups (§§ 3, 4). A uniformly movable inverse system of spaces has the uniformly movable system of homotopy groups. This makes our proofs purely algebraic (§§ 1–4).

Some results of Mardešić and Segal enable us to define the uniform movability of a compact Hausdorff space (a pointed space) by means of inverse systems (§ 5). The resulting class of spaces is a shape invariant and is contained in the class of all movable compact Hausdorff spaces ([2], [5]). As an example, any plane continuum is proved to be uniformly movable.

For the notions of category theory, see [7].

Introduction. We start by recalling some definitions.

Consider the arbitrary category $\mathcal{K}$ and take $f \in \mathrm{Mor}_{\mathcal{K}}(X, Y)$.

$$f \text{ is a } \textit{monomorphism} \underset{\mathrm{Df}}{\Longleftrightarrow} \bigwedge_{v,v' \in \mathrm{Mor}_{\mathcal{K}}(Z,X)} fv = fv' \Rightarrow v = v',$$

$$f \text{ is an } \textit{epimorphism} \underset{\mathrm{Df}}{\Longleftrightarrow} \bigwedge_{u,u' \in \mathrm{Mor}_{\mathcal{K}}(Y,Z)} uf = u'f \Rightarrow u = u',$$

$$f \text{ is a } \textit{bimorphism} \underset{\mathrm{Df}}{\Longleftrightarrow} f \text{ is a monomorphism} \wedge f \text{ is an epimorphism},$$

$$f \text{ is an } \textit{r-morphism} \underset{\mathrm{Df}}{\Longleftrightarrow} \bigvee_{g \in \mathrm{Mor}_{\mathcal{K}}(Y,X)} fg = 1_Y,$$

$$f \text{ is an } \textit{isomorphism} \underset{\mathrm{Df}}{\Longleftrightarrow} \bigvee_{g \in \mathrm{Mor}_{\mathcal{K}}(Y,X)} fg = 1_Y \wedge gf = 1_X.$$

Obviously any isomorphism is a bimorphism.

We say that the object Y is *r-dominated* by X in $\mathcal{K}$ (in symbols $X \underset{r}{\geqslant} Y$) if there exists an r-morphism $f \in \mathrm{Mor}_{\mathcal{K}}(X, Y)$.

The set A is said to be *directed* with respect to the relation $\geqslant$, whenever

$$\alpha \geqslant \alpha,$$
$$\alpha'' \geqslant \alpha' \wedge \alpha' \geqslant \alpha \Rightarrow \alpha'' \geqslant \alpha,$$
$$\bigwedge_{\alpha,\alpha'} \bigvee_{\alpha''} \alpha'' \geqslant \alpha \wedge \alpha'' \geqslant \alpha'.$$

If, additionally,

$$\bigwedge_{\alpha} \bigvee_{n \in N} \overline{\overline{\{\alpha' \colon \alpha \geqslant \alpha'\}}} = n,$$

N being the set of natural numbers, then A is a *closure-finite directed set* (see [3]).

The system $X = (X_\alpha, p_\alpha^{\alpha'}, A)$ is said to be an *inverse system in a category* $\mathcal{K}$ whenever

A is a (closure-finite) directed set (1),

$X_\alpha \in \mathrm{Ob}_{\mathcal{K}}$ for any $\alpha \in A$,

(1) In §§ 1–4, A is assumed to be a directed set, but subsequently, with regard to compact spaces, A must be additionally assumed to be closure-finite.

$p_\alpha^{\alpha'} \in \mathrm{Mor}_{\mathcal{K}}(X_{\alpha'}, X_\alpha)$ for any $\alpha' \geqslant \alpha$,

$p_\alpha^\alpha = 1_{X_\alpha}$ and $p_\alpha^{\alpha'} p_{\alpha'}^{\alpha''} = p_\alpha^{\alpha''}$ for $\alpha'' \geqslant \alpha' \geqslant \alpha$.

Given two inverse systems $X = (X_\alpha, p_\alpha^{\alpha'}, A)$, $Y = (Y_\beta, q_\beta^{\beta'}, B)$, the system $f = (\varphi, f_\beta)$ is said to be a *map of* X *into* Y *in* $\mathcal{K}$ whenever $\varphi \colon B \to A$ is an increasing function,

$$f_\beta \in \mathrm{Mor}_{\mathcal{K}}(X_{\varphi(\beta)}, Y_\beta),$$

and all the diagrams

$$\begin{tikzcd} X_{\varphi(\beta)} \arrow[d, "f_\beta"'] & X_{\varphi(\beta')} \arrow[l, "p_{\varphi(\beta)}^{\varphi(\beta')}"'] \arrow[d, "f_{\beta'}"] \\ Y_\beta & Y_{\beta'} \arrow[l, "q_\beta^{\beta'}"] \end{tikzcd} \qquad \text{for} \quad \beta' \geqslant \beta$$

are commutative, i.e. $f_\beta p_{\varphi(\beta)}^{\varphi(\beta')} = q_\beta^{\beta'} f_{\beta'}$.

The composition of maps of inverse systems is defined as follows: if $f = (\varphi, f_\beta)\colon X \to Y$, $g = (\psi, g_\gamma)\colon Y \to Z$, then

$$gf \underset{\mathrm{Df}}{=} (\varphi\psi, g_\gamma f_{\psi(\gamma)}).$$

We thus obtain a new category $\mathcal{K}^*$ of inverse systems in $\mathcal{K}$.

Any object in $\mathcal{K}$ can be treated as a constant inverse system. More precisely, we can define a covariant functor

$$\mathrm{Const}\colon \mathcal{K} \to \mathcal{K}^*$$

as follows:

$$\mathrm{Const}\, X \underset{\mathrm{Df}}{=} (X, 1_X, A),$$

A being an arbitrary (closure-finite) directed set.

If $f \in \mathrm{Mor}_{\mathcal{K}}(X, Y)$, then

$$\mathrm{Const} f \underset{\mathrm{Df}}{=} (1_A, f).$$

The object X of $\mathcal{K}$ is said to be an *inverse limit of* X (in symbols $X = \varprojlim X$) whenever there exists a map $p\colon X \to X$ such that

$$\bigwedge_{Z \in \mathrm{Ob}_{\mathcal{K}}} \bigwedge_{h\colon Z \to X} \dot{\bigvee}_{h \in \mathrm{Mor}_{\mathcal{K}}(Z,X)} ph = h \ (2),$$

(2) $\dot{\bigvee}_h$ — there exists a unique h.

i.e. the diagram

$$\begin{tikzcd} X & \mathbf{X} \arrow[l, "p"'] \\ & Z \arrow[ul, "h"] \arrow[u, "\mathbf{h}"'] \end{tikzcd}$$

is commutative.

Let us refer to p as a *projection*.

Obviously, if $X = \varprojlim \mathbf{X}$, $Y = \varprojlim \mathbf{Y}$, and $p\colon \mathbf{X} \to X$, $q\colon \mathbf{Y} \to Y$ are projections, then for every $\mathbf{f}\colon \mathbf{X} \to \mathbf{Y}$ there is a unique morphism $f \in \mathrm{Mor}_{\mathcal{K}}(X, Y)$ such that

$$fp = q\mathbf{f}.$$

(in symbols $f = \varprojlim \mathbf{f}$).

It is known that

$$\varprojlim \mathbf{gf} = \varprojlim \mathbf{g} \varprojlim \mathbf{f},$$

and thus $\varprojlim\colon \mathcal{K}^* \to \mathcal{K}$ is a covariant functor.

The values of this functor are determined uniquely up to an isomorphism.

1. Categories of inverse systems. An ordinary category of inverse systems (see the Introduction) is a particular case of the category $\mathcal{K}^*_{\sim}$ defined as follows.

Consider the pair $(\mathcal{K}, \sim)$, $\mathcal{K}$ being a category and $\sim$ — an equivalence relation in $\mathrm{Mor}_{\mathcal{K}}$ satisfying the following condition:

for any $f, f' \in \mathrm{Mor}_{\mathcal{K}}(X, Y)$; $g, g' \in \mathrm{Mor}_{\mathcal{K}}(Y, Z)$

$$f \sim f' \wedge g \sim g' \Rightarrow gf \sim g'f'.$$

Let the objects of $\mathcal{K}^*_{\sim}$ be inverse systems in $\mathcal{K}$, i.e. $\mathrm{Ob}_{\mathcal{K}^*_{\sim}} = \mathrm{Ob}_{\mathcal{K}^*}$.

Take $\mathbf{X}, \mathbf{Y} \in \mathrm{Ob}_{\mathcal{K}^*}$, $\mathbf{X} = (X_\alpha, p^{\alpha'}_\alpha, A)$, $\mathbf{Y} = (Y_\beta, q^{\beta'}_\beta, B)$ and let $\mathbf{f} = (\varphi, f_\beta)$, where $\varphi\colon B \to A$ is an increasing function and $f_\beta \in \mathrm{Mor}_{\mathcal{K}}(X_{\varphi(\beta)}, Y_\beta)$; then

$$\mathbf{f} \in \mathrm{Mor}_{\mathcal{K}^*_{\sim}}(\mathbf{X}, \mathbf{Y}) \underset{\mathrm{Df}}{\Longleftrightarrow} q^{\beta'}_\beta f_{\beta'} \sim f_\beta p^{\varphi(\beta')}_{\varphi(\beta)} \quad \text{for any } \beta' \geqslant \beta,$$

i.e. the diagram

$$\begin{tikzcd} X_{\varphi(\beta)} \arrow[d, "f_\beta"'] & X_{\varphi(\beta')} \arrow[l, "p^{\varphi(\beta')}_{\varphi(\beta)}"'] \arrow[d, "f_{\beta'}"] \\ Y_\beta & Y_{\beta'} \arrow[l, "q^{\beta'}_\beta"] \end{tikzcd}$$

commutes up to $\sim$.

The composition of morphisms in $\mathcal{K}^*_{\sim}$ is defined as in the ordinary case. We thus have

$$\mathcal{K}^* = \mathcal{K}^*_{=}.$$

The relation $\sim$ generates the following equivalence relation $\approx$ in the set $\mathrm{Mor}_{\mathcal{K}^*_{\sim}}$: let

$$\mathbf{X}, \mathbf{Y} \in \mathrm{Ob}_{\mathcal{K}^*_{\sim}}, \quad \mathbf{f}, \mathbf{f}' \in \mathrm{Mor}_{\mathcal{K}^*_{\sim}}(\mathbf{X}, \mathbf{Y}),$$
$$\mathbf{X} = (X_\alpha, p^{\alpha'}_\alpha, A), \quad \mathbf{Y} = (Y_\beta, q^{\beta'}_\beta, B),$$
$$\mathbf{f} = (\varphi, f_\beta), \quad \mathbf{f}' = (\varphi', f_{\beta'});$$

then

$$\mathbf{f} \approx \mathbf{f}' \underset{\mathrm{Df}}{\Longleftrightarrow} \bigwedge_{\beta \in B} \bigvee_{\alpha \geqslant \varphi(\beta), \varphi'(\beta)} f_\beta p^\alpha_{\varphi(\beta)} \sim f'_\beta p^\alpha_{\varphi'(\beta)}.$$

One can easily verify

1.1. *For any* $\mathbf{f}, \mathbf{f}' \in \mathrm{Mor}_{\mathcal{K}^*_{\sim}}(\mathbf{X}, \mathbf{Y})$, $\mathbf{g}, \mathbf{g}' \in \mathrm{Mor}_{\mathcal{K}^*_{\sim}}(\mathbf{Y}, \mathbf{Z})$

$$\mathbf{f} \approx \mathbf{f}' \wedge \mathbf{g} \approx \mathbf{g}' \Rightarrow \mathbf{gf} \approx \mathbf{g}'\mathbf{f}'.$$

We can thus define a new category $\hat{\mathcal{K}}^*_{\sim}$ with the same objects and with morphisms being the equivalence classes of morphisms in $\mathcal{K}^*_{\sim}$ with respect to the relation $\approx$.

Statement 1.1 enables us to define the composition in $\mathrm{Mor}_{\hat{\mathcal{K}}^*_{\sim}}$:

$$[\mathbf{g}][\mathbf{f}] \underset{\mathrm{Df}}{=} [\mathbf{gf}].$$

Thus, given the pair $(\mathcal{K}, \sim)$, we have defined two categories of inverse systems in $\mathcal{K}$: the category $\mathcal{K}^*_{\sim}$ and its quotient category $\hat{\mathcal{K}}^*_{\sim}$.

In the case of $\sim$ being the identity relation, we write simply $\hat{\mathcal{K}}^*$ instead of $\hat{\mathcal{K}}^*_{=}$. In this case, we use the symbol $\simeq$ to denote the equivalence relation generated by $=$.

Example 1. Let $\mathcal{T}$ be the category of topological spaces with continuous maps as morphisms and $\sim$ — the relation of homotopy ($\simeq$). Then $\mathcal{T}^*_{\sim}$, $\hat{\mathcal{T}}^*_{\sim}$ are the categories studied by S. Mardešić and J. Segal in [3].

Example 2. Let $\mathcal{T}$ be the same category of topological spaces and $\sim$ — the relation of identity ($=$). Then $\mathcal{T}^*_{\sim}$, $\hat{\mathcal{T}}^*_{\sim}$ are the categories studied in [6], $\approx$ being the relation of similarity.

2. Limit morphisms in $\mathcal{K}^*$. Given any category $\mathcal{K}$, let us consider two categories $\mathcal{K}^*$ and $\hat{\mathcal{K}}^*$ as defined in § 1.

One can easily show that

2.1. $\mathbf{f} \simeq \mathbf{f}' \Rightarrow \varprojlim \mathbf{f} = \varprojlim \mathbf{f}'$ (3).

(3) The proof is quite analogous to that for the special case of the category of topological spaces — see [6].

Some properties of morphisms are preserved by $\varprojlim$. For instance, as a consequence of 2.1, we get

2.2. *f is an isomorphism in $\hat{\mathcal{K}}^* \Rightarrow \varprojlim f$ is an isomorphism in $\mathcal{K}$.*

We have also

2.3. *f is a monomorphism in $\hat{\mathcal{K}}^* \Rightarrow \varprojlim f$ is a monomorphism in $\mathcal{K}$.*

Proof. Let f be a monomorphism in $\hat{\mathcal{K}}^*$, $f \in \mathrm{Mor}_{\hat{\mathcal{K}}^*}(X, Y)$. Let $X = \varprojlim X$, $Y = \varprojlim Y$, $f = \varprojlim f$.

We have

$$fp = qf, \tag{1}$$

$p\colon X \to X$, $q\colon Y \to Y$ being the projections.

Take $Z \in \mathrm{Ob}_{\mathcal{K}}$, $v, v' \in \mathrm{Mor}_{\mathcal{K}}(Z, X)$ and assume

$$fv = fv'. \tag{2}$$

We have to show that $v = v'$.

Put

$$\boldsymbol{v} \underset{\mathrm{Df}}{=} \boldsymbol{p}v, \qquad \boldsymbol{v}' \underset{\mathrm{Df}}{=} \boldsymbol{p}v'; \tag{3}$$

we have

$$v = \varprojlim \boldsymbol{v}, \qquad v' = \varprojlim \boldsymbol{v}'. \tag{4}$$

By (1), (3), we get

$$\boldsymbol{f}\boldsymbol{v} = \boldsymbol{f}\boldsymbol{p}v = \boldsymbol{q}fv \quad \text{and} \quad \boldsymbol{f}\boldsymbol{v}' = \boldsymbol{f}\boldsymbol{p}v' = \boldsymbol{q}fv';$$

thus, by (2),

$$\boldsymbol{f}\boldsymbol{v} = \boldsymbol{f}\boldsymbol{v}'.$$

Since $\boldsymbol{f}$ is a monomorphism, it follows that $\boldsymbol{v} = \boldsymbol{v}'$ and hence, by 2.1 and (4), $v = v'$. ∎

Let us notice that a similar implication for epimorphisms fails. In fact, we have

Example 1. Let $\mathcal{K} = \mathcal{G}$-the category of groups. Take an inverse sequence $X = (X_n, p_n^{n+1})$, where $X_n = \mathfrak{N}$ — a cyclic infinite group, $p_n^{n+1}(x) = 2x$ for $n = 1, 2, \ldots$; take a constant sequence $Y = (Y_n, q_n^{n+1})$, where $Y_n = \mathfrak{N}$, $q_n^{n+1}(y) = y$ and let $f = (1, f_n)$, $f_n(x) = 2^n x$.

$$\begin{tikzcd}
\mathfrak{N} \arrow[drrr, "f_1"'] & \mathfrak{N} \arrow[l, "p_1^2"'] \arrow[drr, "f_2"] & \mathfrak{N} \arrow[l, "p_2^3"'] \arrow[dr, "f_3"] & 0 \arrow[l, dotted] \arrow[d, "f"] \\
 & & & \mathfrak{N}
\end{tikzcd}$$

It is easy to show that f is an epimorphism in $\hat{\mathcal{G}}^*$; however, since $\varprojlim X = (0)$, $\varprojlim f$ is not an epimorphism in $\mathcal{G}$. ∎

All of the converse implications in general fail.

Example 2 (4). Let $\mathcal{K} = \mathcal{G}$. Take $X = (X_n, p_n^{n+1})$, $Y = (Y_n, q_n^{n+1})$, $X_n = Y_n = \mathfrak{N}$, $p_n^{n+1}(x) = 6x$, $q_n^{n+1}(y) = 3y$, and let $f = (1, f_n)$, $f_n(x) = 2^n x$. Since $\varprojlim X = (0) = \varprojlim Y$, $\varprojlim f$ is an isomorphism. On the other hand, f is not an isomorphism. ∎

Example 3. Take the category $\mathcal{G}$ and X, Y as in Example 2. Now, let $f\colon X \to Y$ be the trivial morphism, i.e. $f_n(x) = 0$ for $n = 1, 2, \ldots$ Of course f is not a monomorphism, but $\varprojlim f$ is a monomorphism. ∎

Example 4. Take a trivial sequence $X \in \mathrm{Ob}_{\mathcal{G}^*}$ and take Y as in Examples 2, 3. The trivial morphism $f\colon X \to Y$ is not an epimorphism but $\varprojlim f$ is an epimorphism. ∎

We are interested in the case of bimorphisms. The question arises what assumptions on inverse systems are to be made in order to obtain the implication:

$$\varprojlim f \text{ is a bimorphism in } \mathcal{K} \Rightarrow f \text{ is a bimorphism in } \hat{\mathcal{K}}^*.$$

The answer is given in § 4. There the above implication is proved for uniformly movable systems as defined in § 3.

3. Movable and uniformly movable inverse systems. The notion of movability introduced by K. Borsuk in [2] for compacta and expressed by S. Mardešić and J. Segal in [5] by means of inverse systems, can easily be generalized to an arbitrary category.

Let $X = (X_\alpha, p_\alpha^{\alpha'}, A) \in \mathrm{Ob}_{\mathcal{K}^*_\sim}$.

X is said to be *movable in* $\mathcal{K}^*_\sim$ whenever

$$\bigwedge_{\alpha_0 \in A} \bigvee_{\hat{\alpha}_0 \geqslant \alpha_0} \bigwedge_{\alpha \geqslant \alpha_0} \bigvee_{h_{\hat{\alpha}_0\alpha} \in \mathrm{Mor}_{\mathcal{K}}(X_{\hat{\alpha}_0}, X_\alpha)} p_{\alpha_0}^{\alpha} h_{\hat{\alpha}_0\alpha} \sim p_{\alpha_0}^{\hat{\alpha}_0}.$$

Notice that there is no connection between two maps $h_{\hat{\alpha}_0\alpha}$, $h_{\hat{\alpha}_0\alpha'}$ for $\alpha \neq \alpha'$.

Now, for any (closure-finite) directed set A, let

$$A^{(\alpha_0)} \underset{\mathrm{Df}}{=} \{\alpha \in A\colon \alpha \geqslant \alpha_0\}, \qquad \alpha_0 \in A.$$

Then, to any inverse system $X = (X_\alpha, p_\alpha^{\alpha'}, A)$ in $\mathcal{K}$ the following collection of inverse systems can be assigned:

$$X^{(\alpha_0)} \underset{\mathrm{Df}}{=} (X_\alpha, p_\alpha^{\alpha'}, A^{(\alpha_0)}), \qquad \alpha_0 \in A.$$

We shall refer to $X^{(\alpha_0)}$ as a *partial system of* X.

Obviously

(4) This example is due to W. Holsztyński.

3.1. *$X^{(\alpha_0)}$ is cofinal to X for every $\alpha_0 \in A$.*

X is said to be *uniformly movable in* $\mathcal{K}_\sim$ whenever there is a collection

$$\{\chi^{(\alpha_0)}\colon A^{(\alpha_0)} \to A\}_{\alpha_0 \in A}$$

of constant functions and a collection

$$\{\boldsymbol{h}^{(\alpha_0)}\colon X_{\hat{\alpha}_0} \to X^{(\alpha_0)}\}_{\alpha_0 \in A}$$

of morphisms in $\mathcal{K}^*_\sim$ such that

$$\chi^{(\alpha_0)} \text{ is increasing with respect to } \alpha_0$$

(i) $$\Big(\text{i.e. } \alpha_0' \geqslant \alpha_0 \Rightarrow \bigwedge_{\alpha \geqslant \alpha_0} \chi^{(\alpha_0')}(\alpha) \geqslant \chi^{(\alpha_0)}(\alpha)\Big)$$

$$\chi^{(\alpha_0)}(\alpha) = \hat{\alpha}_0 \geqslant \alpha_0 \quad \text{for every } \alpha \in A^{(\alpha_0)},$$

$$\boldsymbol{h}^{(\alpha_0)} = (\chi^{(\alpha_0)}, h_\alpha^{(\alpha_0)}),$$

(ii) $$\bigwedge_{\alpha_0} \bigwedge_{\alpha \in A^{(\alpha_0)}} p^\alpha_{\alpha_0} h^{(\alpha_0)}_\alpha \sim p^{\hat{\alpha}_0}_{\alpha_0}.$$

We have

3.2. *Let $\{\chi^{(\alpha_0)}\}_{\alpha_0 \in A}$ and $\{\boldsymbol{h}^{(\alpha_0)}\}_{\alpha_0 \in A}$ satisfy the conditions* (i), (ii). *Consider a collection $\{\chi'^{(\alpha_0)}\}_{\alpha_0 \in A}$ of constant functions such that*

(i′) $$\chi'^{(\alpha_0)} \textit{ is increasing with respect to } \alpha_0,$$
$$\chi'^{(\alpha_0)}(\alpha) = \hat{\alpha}_0' \geqslant \hat{\alpha}_0 \quad \textit{for every } \alpha \in A^{(\alpha_0)}.$$

Then there exists a collection $\{\boldsymbol{h}'^{(\alpha_0)}\}_{\alpha_0 \in A}$ such that

$$\boldsymbol{h}'^{(\alpha_0)} = (\chi'^{(\alpha_0)}, h'^{(\alpha_0)}_\alpha),$$

(ii′) $$\bigwedge_{\alpha_0} \bigwedge_{\alpha \in A^{(\alpha_0)}} p^\alpha_{\alpha_0} h'^{(\alpha_0)}_\alpha \sim p^{\hat{\alpha}_0'}_{\alpha_0}.$$

Proof. Take $\{\chi^{(\alpha_0)}\}_{\alpha_0 \in A}$ and $\{\boldsymbol{h}^{(\alpha_0)}\}_{\alpha_0 \in A}$ satisfying (i), (ii) and let $\{\chi'^{(\alpha_0)}\}_{\alpha_0 \in A}$ satisfy (i′). We have to find $\{\boldsymbol{h}'^{(\alpha_0)}\}_{\alpha_0 \in A}$ satisfying (ii′).

$$\begin{tikzcd}
X_{\alpha_0} & X_\alpha \arrow[l, "p^\alpha_{\alpha_0}"'] \\
X_{\hat{\alpha}_0} \arrow[u, "p^{\hat{\alpha}_0}_{\alpha_0}"] \arrow[ur, "h^{(\alpha_0)}_\alpha"] & X_{\hat{\alpha}_0'} \arrow[l, "p^{\hat{\alpha}_0'}_{\hat{\alpha}_0}"] \arrow[u, "h'^{(\alpha_0)}_\alpha"']
\end{tikzcd}$$

Let

$$\boldsymbol{h}'^{(\alpha_0)} = (\chi'^{(\alpha_0)}, h'^{(\alpha_0)}_\alpha), \quad \text{where} \quad h'^{(\alpha_0)}_\alpha \underset{\mathrm{Df}}{=} h^{(\alpha_0)}_\alpha p^{\hat{\alpha}_0'}_{\hat{\alpha}_0} \quad \text{for} \quad \alpha \in A^{(\alpha_0)}.$$

Then, by (ii), we have

$$p^\alpha_{\alpha_0} h'^{(\alpha_0)}_\alpha = p^\alpha_{\alpha_0} h^{(\alpha_0)}_\alpha p^{\hat{\alpha}_0'}_{\hat{\alpha}_0} \sim p^{\hat{\alpha}_0}_{\alpha_0} p^{\hat{\alpha}_0'}_{\hat{\alpha}_0} = p^{\hat{\alpha}_0'}_{\alpha_0}. \ \blacksquare$$

Let us notice that

3.3. *X is uniformly movable in $\mathcal{K}^*_\sim$ iff there is a collection of maps of X into its partial systems,*

$$\{\boldsymbol{h}^{(\alpha_0)}\colon X \to X^{(\alpha_0)}\}_{\alpha_0 \in A},$$

such that

$$\boldsymbol{h}^{(\alpha_0)}(\chi^{(\alpha_0)}, h^{(\alpha_0)}_\alpha),$$

where

(1) $$\chi^{(\alpha_0)}\colon A^{(\alpha_0)} \to A \quad (\chi^{(\alpha_0)}(\alpha) = \hat{\alpha}_0 \quad \textit{for every } \alpha \in A)$$

are constant functions satisfying (i), *and*

(2) $$h^{(\alpha_0)}_{\alpha_0} \sim p^{\hat{\alpha}_0}_{\alpha_0}.$$

Proof. If $\chi^{(\alpha_0)}$ is a constant function, then any map $\boldsymbol{h}^{(\alpha_0)} = (\chi^{(\alpha_0)}, h^{(\alpha_0)}_\alpha)$ can be treated either as a map of X into $X^{(\alpha_0)}$ or as a map of $X_{\hat{\alpha}_0}$ into $X^{(\alpha_0)}$ (where $\hat{\alpha}_0 = \chi^{(\alpha_0)}(\alpha)$).

Since $\boldsymbol{h}^{(\alpha_0)} \in \mathrm{Mor}_{\mathcal{K}^*_\sim}$, we have

$$p^\alpha_{\alpha_0} h^{(\alpha_0)}_\alpha \sim h^{(\alpha_0)}_{\alpha_0} \quad \text{for every } \alpha \geqslant \alpha_0,$$

i.e. the diagram

$$\begin{tikzcd}
X_{\hat{\alpha}_0} \arrow[d, "h^{(\alpha_0)}_{\alpha_0}"'] & X_{\hat{\alpha}_0} \arrow[l, "1"'] \arrow[d, "h^{(\alpha_0)}_\alpha"] \\
X_{\alpha_0} & X_\alpha \arrow[l, "p^\alpha_{\alpha_0}"]
\end{tikzcd}$$

commutes up to $\sim$.

Hence, the two conditions

$$p^\alpha_{\alpha_0} h^{(\alpha_0)}_\alpha \sim p^{\hat{\alpha}_0}_{\alpha_0} \quad \text{and} \quad h^{(\alpha_0)}_{\alpha\alpha} \sim p^{\hat{\alpha}_0}_{\alpha_0}$$

are equivalent to each other. $\blacksquare$

One can easily prove that

3.4. *If X is uniformly movable in $\mathcal{K}^*_\sim$, then X is movable as well.*

We have

3.5. *Every constant system in $\mathcal{K}$ is uniformly movable in $\mathcal{K}^*_\sim$.*

Proof. Take $X = (X, 1_X, A)$. Let $a_0 \in A$. Define $\chi^{(a_0)}(a) = a_0$ for $a \geqslant a_0$, $h_a^{(a_0)} \underset{\mathrm{Df}}{=} 1_X$. We get $h^{(a_0)} = (\chi^{(a_0)}, h_a^{(a_0)})$: $X_{\hat{a}_0} \to X^{(a_0)}$ satisfying the condition (i), (ii). ■

Now, let us consider two pairs, $(\mathcal{K}, \sim_{\mathcal{K}})$ and $(\mathcal{L}, \sim_{\mathcal{L}})$ (see § 1), and the corresponding categories $\mathcal{K}^*_\sim$, $\mathcal{L}^*_\sim$. Let π: $(\mathcal{K}, \sim_{\mathcal{K}}) \to (\mathcal{L}, \sim_{\mathcal{L}})$ be a covariant functor preserving the equivalence relation, i.e. for any $X, Y \in \mathrm{Ob}_{\mathcal{K}}$ and $f_0, f_1 \in \mathrm{Mor}_{\mathcal{K}}(X, Y)$

$$f_0 \sim_{\mathcal{K}} f_1 \Rightarrow \pi(f_0) \sim_{\mathcal{L}} \pi(f_1).$$

It is easily seen that π carries any inverse system in $\mathcal{K}$ onto an inverse system in $\mathcal{L}$ and any map of systems onto a map of systems, i.e., π generates the following functor:

$$\pi\colon\ \mathcal{K}^*_\sim \to \mathcal{L}^*_\sim:$$

if $X = (X_a, p_a^{a'}, A)$, then

$$\pi(X) = (\pi(X_a), \pi(p_a^{a'}), A);$$

if $f = (\varphi, f_\beta)$: $X = (X_a, p_a^{a'}, A) \to (Y_\beta, q_\beta^{\beta'}, B) = Y$, then

$$\pi(f) = (\varphi, \pi(f_\beta))\colon\ \pi(X) \to \pi(Y).$$

Obviously the generated functor is also covariant.

Let us notice that

3.6. *If X and Y are isomorphic in $\hat{\mathcal{K}}^*_\sim$, then $\pi(X)$ and $\pi(Y)$ are isomorphic in $\hat{\mathcal{L}}^*_\sim$.*

It follows by the definition of uniform movability that

3.7. *X is uniformly movable in $\mathcal{K}^*_\sim \Rightarrow \pi(X)$ is uniformly movable in $\mathcal{L}^*_\sim$.*

Now, let us assume all the objects of $\mathcal{K}^*_\sim$ to be inverse systems over a fixed directed set A.

We are going to prove that the uniform movability in $K^*_\sim$ is an invariant of any isomorphism in $\hat{\mathcal{K}}^*_\sim$; moreover — that it is an r-invariant in $\hat{\mathcal{K}}^*_\sim$ (statements 3.9, 3.10).

First let us notice that

3.8. *Given two inverse systems X, Y over A, for every $f \in \mathrm{Mor}_{\mathcal{K}^*}(X, Y)$ there exists an $f' = (\varphi', f'_\beta)$ such that $f' \cong f$ and $\varphi'(\beta) \geqslant \beta$ for any $\beta \in A$.*

Proof. Take $X = (X_a, p_a^{a'}, A)$, $Y = (Y_\beta, q_\beta^{\beta'}, A)$ and $f = (\varphi, f_\beta)$: $X \to Y$.

For every $\beta \in A$ choose $a \in A$ such that $a \geqslant \beta, \varphi(\beta)$ and put

$$\varphi'(\beta) \underset{\mathrm{Df}}{=} a.$$

Since $\varphi'(\beta) \geqslant \varphi(\beta)$, we can define f'_β: $X_{\varphi'(\beta)} \to Y_\beta$ as follows:

$$f'_\beta \underset{\mathrm{Df}}{=} f_\beta p_{\varphi(\beta)}^{\varphi'(\beta)}.$$

The morphism $f' = (\varphi', f'_\beta)$ satisfies the required conditions:

1° by definition, $\varphi'(\beta) \geqslant \beta$ for every $\beta \in A$,

2° for every $\beta \in A$ there is an $a \underset{\mathrm{Df}}{=} \varphi'(\beta)$ such that $a \geqslant \varphi(\beta), \varphi'(\beta)$ and

$$f_\beta p^a_{\varphi(\beta)} = f'_\beta = f'_\beta p^a_{\varphi'(\beta)};$$

thus $f' \cong f$. ■

Let us prove

3.9. Theorem. *Let X, Y be two inverse systems in $\mathcal{K}$ over a directed set A. If $X \underset{r}{\geqslant} Y$ in $\hat{\mathcal{K}}^*_\sim$, then*

X is uniformly movable in $\mathcal{K}^*_\sim \Rightarrow Y$ is uniformly movable in $\mathcal{K}^*_\sim$.

Proof. Take $X = (X_a, p_a^{a'}, A)$, $Y = (Y_\beta, q_\beta^{\beta'}, A)$. By hypothesis, there exist

$$f = (\varphi, f_\beta)\colon X \to Y \quad \text{and} \quad g = (\psi, g_a)\colon Y \to X$$

such that $fg \approx 1_Y$, i.e.

(1) $$\bigwedge_\beta \bigvee_{\beta' \geqslant \beta, \psi\varphi(\beta)} \Psi(\beta, \beta'),$$

the formula Ψ being defined as follows:

(2) $$\Psi(\beta, \beta') \underset{\mathrm{Df}}{\Leftrightarrow} f_\beta g_{\varphi(\beta)} q^{\beta'}_{\psi\varphi(\beta)} \sim q^{\beta'}_\beta.$$

Notice that

(3) $$\bar{\beta}' \geqslant \beta' \Rightarrow \bigwedge_\beta [\Psi(\beta, \beta') \Rightarrow \Psi(\beta, \bar{\beta}')].$$

By 3.8, we can assume

(4) $$\varphi(\beta) \geqslant \beta \quad \text{and} \quad \psi(a) \geqslant a \quad \text{for every } a, \beta \in A;$$

thus condition (1) can be replaced by

(1′) $$\bigwedge_a \bigvee_{\beta' \geqslant \psi\varphi(\beta)} \Psi(\beta, \beta').$$

Let X be uniformly movable in $\mathcal{K}^*_\sim$. Prove Y to be uniformly movable.

Fix $\beta_0 \in A$ and put

(5) $$a_0 \underset{\mathrm{Df}}{=} \varphi(\beta_0).$$

By (1′), there is a $\beta'_0 \geqslant \psi(a_0)$ such that

(6) $$\Psi(\beta_0, \beta'_0).$$

Let

$$A^{(a_0)} = \{a\colon a \geqslant a_0\}, \quad A^{(\beta_0)} = \{\beta\colon \beta \geqslant \beta_0\}.$$

Since X is uniformly movable, there is a constant function $\chi^{(\alpha_0)}\colon A^{(\alpha_0)} \to A$ and a map $h^{(\alpha_0)}\colon X_{\hat{\alpha}_0} \to X^{(\alpha_0)}$ such that

$$\begin{gathered}\chi^{(\alpha_0)}(\alpha) = \hat{\alpha}_0\,, \quad \chi^{(\alpha_0)} \text{ is increasing with respect to } \alpha_0\,, \\ \chi^{(\alpha_0)}_{(\alpha)} \geqslant \alpha_0 \quad \text{and} \quad p^{\alpha}_{\alpha_0} h^{(\alpha_0)}_{\alpha} \sim p^{\hat{\alpha}_0}_{\alpha_0} \quad \text{for} \quad \alpha \geqslant \alpha_0\,.\end{gathered} \tag{7}$$

By 3.2, we can assume that

$$\hat{\alpha}_0 \geqslant \beta_0'\,. \tag{8}$$

By (6), (8), (3) and (4) we get

$$\Psi\big(\beta_0, \psi(\hat{\alpha}_0)\big)\,. \tag{9}$$

Put

$$\hat{\beta}_0 \underset{\mathrm{Df}}{=} \psi(\hat{\alpha}_0) \tag{10}$$

and define

$$\varkappa^{(\beta_0)}\colon A^{(\beta_0)} \to A \quad \text{and} \quad k^{(\beta_0)}\colon X_{\hat{\beta}_0} \to X$$

as follows:

$$\begin{gathered}\varkappa^{(\beta_0)}(\beta) \underset{\mathrm{Df}}{=} \hat{\beta}_0 \quad \text{for any } \beta \in A^{(\beta_0)}, \\ k^{(\beta_0)} \underset{\mathrm{Df}}{=} (\varkappa^{(\beta_0)}, k^{(\beta_0)}_{\beta})\,, \qquad k^{(\beta_0)}_{\beta} \underset{\mathrm{Df}}{=} f_\beta h^{(\alpha_0)}_{\varphi(\beta)} g_{\hat{\alpha}_0}\colon\ Y_{\hat{\beta}_0} \to Y_\beta\,.\end{gathered} \tag{11}$$

Let us notice that, by 3.3, $h^{(\alpha_0)}$ can be treated as a morphism of X into $X^{(\alpha_0)}$, $k^{(\beta_0)}$ — as a morphism of Y into $Y^{(\beta_0)}$. By (5), we can define $f'\colon X^{(\alpha_0)} \to Y^{(\beta_0)}$:

$$f' \underset{\mathrm{Df}}{=} (\varphi', f_\beta)\,, \quad \text{where} \quad \varphi'\colon A^{(\beta_0)} \to A^{(\alpha_0)},\ \varphi'(\beta) = \varphi(\beta) \text{ for } \beta \geqslant \beta_0\,,$$

and then we get the diagram

$$Y \xrightarrow{g} X \xrightarrow{h^{(\alpha_0)}} X^{(\alpha_0)} \xrightarrow{f'} Y^{(\beta_0)}\,.$$

Thus $k^{(\beta_0)} = f' h^{(\alpha_0)} g$, and therefore $k^{(\beta_0)} \in \mathrm{Mor}_{\mathcal{K}^*_{\sim}}(Y, Y^{(\beta_0)})$.

It remains to verify the conditions (i), (ii) for $\{\varkappa^{(\beta_0)}\}_{\beta_0 \in A}$, $\{k^{(\beta_0)}\}_{\beta_0 \in A}$.

By (10), (11), (5), $\varkappa^{(\beta_0)} = \psi\chi^{(\varphi(\beta_0))}$. Thus, $\varkappa^{(\beta_0)}$ is increasing with respect to β_0, since φ, ψ are both increasing and $\chi^{(\alpha_0)}$ is increasing with respect to α_0. Moreover, by (4) and (7),

$$\varkappa^{(\beta_0)}(\beta) \geqslant \chi^{(\varphi(\beta_0))}(\beta) \geqslant \varphi(\beta_0) \geqslant \beta_0 \quad \text{for every } \beta \geqslant \beta_0\,;$$

hence $\{\varkappa^{(\beta_0)}\}_{\beta_0}$ satisfies (i). Let us prove that $\{k^{(\beta_0)}\}_{\beta_0}$ satisfies (ii), i.e.

$$q^{\beta}_{\beta_0} k^{(\beta_0)}_{\beta} \sim q^{\hat{\beta}_0}_{\beta_0} \quad \text{for every} \quad \beta \geqslant \beta_0$$

$$\begin{tikzcd} Y_{\beta_0} & Y_\beta \arrow[l, "q^{\beta}_{\beta_0}"'] \\ Y_{\hat{\beta}_0} \arrow[u, "q^{\hat{\beta}_0}_{\beta_0}"] \arrow[ur, "k^{(\beta_0)}_{\beta}"'] & \end{tikzcd}$$

Since $f, g \in \mathrm{Mor}_{\mathcal{K}^*_{\sim}}$, we have

$$q^{\beta}_{\beta_0} f_\beta \sim f_{\beta_0} p^{\varphi(\beta)}_{\varphi(\beta_0)} \quad \text{for} \quad \beta \geqslant \beta_0 \tag{12}$$

$$\begin{tikzcd} X_{\varphi(\beta_0)} \arrow[d, "f_{\beta_0}"'] & X_{\varphi(\beta)} \arrow[l, "p^{\varphi(\beta)}_{\varphi(\beta_0)}"'] \arrow[d, "f_\beta"] \\ Y_{\beta_0} & Y_\beta \arrow[l, "q^{\beta}_{\beta_0}"] \end{tikzcd}$$

and

$$p^{\hat{\alpha}_0}_{\alpha_0} g_{\hat{\alpha}_0} \sim g_{\alpha_0} q^{\psi(\hat{\alpha}_0)}_{\psi(\alpha_0)} \tag{13}$$

$$\begin{tikzcd} Y_{\psi(\alpha_0)} \arrow[d, "g_{\alpha_0}"'] & Y_{\psi(\hat{\alpha}_0)} \arrow[l, "q^{\psi(\hat{\alpha}_0)}_{\psi(\alpha_0)}"'] \arrow[d, "g_{\hat{\alpha}_0}"] \\ X_{\alpha_0} & X_{\hat{\alpha}_0} \arrow[l, "p^{\hat{\alpha}_0}_{\alpha_0}"] \end{tikzcd}$$

Applying in turn (5), (11), (12), (7), (13), (5), (10), we obtain

$$\begin{aligned} q^{\beta}_{\beta_0} k^{(\beta_0)}_{\beta} &= q^{\beta}_{\beta_0} f_\beta h^{\varphi(\beta_0)}_{\varphi(\beta)} g_{\hat{\alpha}_0} \sim f_{\beta_0} p^{\varphi(\beta)}_{\varphi(\beta_0)} h^{\varphi(\beta_0)}_{\varphi(\beta)} g_{\hat{\alpha}_0} \sim f_{\beta_0} p^{\hat{\alpha}_0}_{\varphi(\beta_0)} g_{\hat{\alpha}_0} \\ &= f_{\beta_0} p^{\hat{\alpha}_0}_{\alpha_0} g_{\hat{\alpha}_0} \sim f_{\beta_0} g_{\alpha_0} q^{\psi(\hat{\alpha}_0)}_{\psi(\alpha_0)} \\ &= f_{\beta_0} g_{\varphi(\beta_0)} q^{\hat{\beta}_0}_{\psi\varphi(\beta_0)}\,. \end{aligned}$$

Thus, by (2) and (9), we get $q^{\beta}_{\beta_0} k^{(\beta_0)}_{\beta} \sim q^{\hat{\beta}_0}_{\beta}$, i.e. $\{k^{(\beta_0)}\}_{\beta_0}$ satisfies (ii). Hence Y is proved to be uniformly movable as well as X. ■

As an immediate consequence of 3.9 we obtain

3.10. COROLLARY. *In an arbitrary category* $\mathcal{K}^*_{\sim}$, *every isomorphism preserves uniform movability.*

4. Algebraic properties of uniformly movable systems. In this section we are concerned with the category $\mathcal{K}^*$, as defined in the Introduction [5].

[5] As has been noticed in § 1, $K^* = K^*_{=}$.

We start with proving the following statement:

4.1. *If the inverse system* $\boldsymbol{X} = (X_\alpha, p_\alpha^{\alpha'}, A)$ *is uniformly movable in* $\mathcal{K}^*$, $X = \varprojlim \boldsymbol{X}$ *and* $\boldsymbol{p} = (p_\alpha)\colon X \to \boldsymbol{X}$ *is a projection, then there exist an increasing function* $\chi\colon A \to A$ *and a collection* $\{h^{(\alpha_0)}\colon X_{\chi(\alpha_0)} \to X\}_{\alpha_0 \in A}$ *such that* $\chi(\alpha_0) \geqslant \alpha_0$ *and*

$$\bigwedge_{\alpha'} \bigwedge_{\alpha_0 \geqslant \alpha'} p_{\alpha'} h^{(\alpha_0)} = p_{\alpha'}^{\chi(\alpha_0)},$$

i.e. the diagram

$$\begin{tikzcd} X_{\alpha'} & X \arrow[l, "p_{\alpha'}"'] \\ X_{\chi(\alpha_0)} \arrow[u, "p_{\alpha'}^{\chi(\alpha_0)}"] \arrow[ur, "h^{(\alpha_0)}"'] & \end{tikzcd}$$

is commutative.

Proof. Assume $\boldsymbol{X}$ to be uniformly movable in $\mathcal{K}^*$. Take

$$\{\chi^{(\alpha_0)}\colon A^{(\alpha_0)} \to A\}_{\alpha_0 \in A} \quad \text{and} \quad \{\boldsymbol{h}^{(\alpha_0)}\colon X_{\hat{\alpha}_0} \to \boldsymbol{X}^{(\alpha_0)}\}_{\alpha_0 \in A}$$

satisfying (i), (ii) (see § 3).

Define $\chi\colon A \to A$ as follows:

(1) $$\chi(\alpha_0) \underset{\mathrm{Df}}{=} \hat{\alpha}_0$$

(i.e. $\chi(\alpha_0) = \chi^{(\alpha_0)}(\alpha)$ for any α). Then $\boldsymbol{h}^{(\alpha_0)} \in \mathrm{Mor}_{\mathcal{K}^*}(X_{\chi(\alpha_0)}, \boldsymbol{X}^{(\alpha_0)})$.

Since, by 3.1, $\boldsymbol{X}^{(\alpha_0)}$ is cofinal to $\boldsymbol{X}$, we have

$$\varprojlim \boldsymbol{X}^{(\alpha_0)} = \varprojlim \boldsymbol{X} = X.$$

Put

(2) $$h^{(\alpha_0)} \underset{\mathrm{Df}}{=} \varprojlim \boldsymbol{h}^{(\alpha_0)}\colon X_{\chi(\alpha_0)} \to X.$$

Let us show that the formulae (1), (2) determine the desired function χ and collection $\{h^{(\alpha_0)}\}$.

By the properties of a limit map, we have

(3) $$\boldsymbol{p} h^{(\alpha_0)} = \boldsymbol{h}^{(\alpha_0)}, \quad \text{i.e.} \quad p_\alpha h^{(\alpha_0)} = h_\alpha^{(\alpha_0)} \quad \text{for any } \alpha \geqslant \alpha_0.$$

By (i), χ is an increasing function and $\chi(\alpha_0) \geqslant \alpha_0$. Take $\alpha' \geqslant \alpha_0$; applying (3) and putting $\chi(\alpha_0)$ for α in (ii), we get

$$\begin{aligned} p_{\alpha'} h^{(\alpha_0)} &= p_{\alpha'}^{\chi(\alpha_0)} p_{\chi(\alpha_0)} h^{(\alpha_0)} = p_{\alpha'}^{\chi(\alpha_0)} h_{\chi(\alpha_0)}^{(\alpha_0)} = p_{\alpha'}^{\alpha_0} [p_{\alpha_0}^{\chi(\alpha_0)} h_{\chi(\alpha_0)}^{(\alpha_0)}] \\ &= p_{\alpha'}^{\alpha_0} p_{\alpha_0}^{\chi(\alpha_0)} = p_{\alpha'}^{\chi(\alpha_0)}. \ \blacksquare \end{aligned}$$

Proposition 4.2, which we are going to prove now, states an implication converse to 2.1 in the case of $\boldsymbol{X}$ being uniformly movable in $\mathcal{K}^*$.

4.2. *Let* $\boldsymbol{X}$ *be uniformly movable in* $\mathcal{K}^*$. *Then, for an arbitrary* $\boldsymbol{Y} \in \mathrm{Ob}_{\mathcal{K}^*}$ *and for any two maps* $\boldsymbol{f}, \boldsymbol{f}'\colon \boldsymbol{X} \to \boldsymbol{Y}$,

$$\varprojlim \boldsymbol{f} = \varprojlim \boldsymbol{f}' \Rightarrow \boldsymbol{f} \simeq \boldsymbol{f}'.$$

Proof. Take a uniformly movable system $\boldsymbol{X} = (X_\alpha, p_\alpha^{\alpha'}, A)$, an arbitrary $\boldsymbol{Y} = (Y_\beta, q_\beta^{\beta'}, B)$, and two maps $\boldsymbol{f} = (\varphi, f_\beta)$, $\boldsymbol{f}' = (\varphi', f_\beta')$ of $\boldsymbol{X}$ into $\boldsymbol{Y}$.

Let $X = \varprojlim \boldsymbol{X}$, $Y = \varprojlim \boldsymbol{Y}$. Assume $\varprojlim \boldsymbol{f} = f = \varprojlim \boldsymbol{f}'$, i.e.

(1) $$f_\beta p_{\varphi(\beta)} = f_\beta' p_{\varphi'(\beta)} \quad \text{for every } \beta \in B.$$

We are going to show that $\boldsymbol{f} \simeq \boldsymbol{f}'$, i.e.

(2) $$\bigwedge_{\beta} \bigvee_{\alpha \geqslant \varphi(\beta), \varphi'(\beta)} f_\beta p_{\varphi(\beta)}^\alpha = f_\beta' p_{\varphi'(\beta)}^\alpha.$$

By 4.1, there exist $\chi\colon A \to A$ and a system

$$\{h^{(\alpha_0)}\colon X_{\chi(\alpha_0)} \to X\}_{\alpha_0 \in A} \quad \text{such that} \quad \chi(\alpha_0) \geqslant \alpha_0 \quad \text{for} \quad \alpha_0 \in A$$

and

(3) $$\bigwedge_{\alpha'} \bigwedge_{\alpha_0 \geqslant \alpha'} p_{\alpha'} h^{(\alpha_0)} = p_{\alpha'}^{\chi(\alpha_0)}.$$

Take $\beta \in B$ and $\alpha_0 \geqslant \varphi(\beta), \varphi'(\beta)$; let

$$\alpha \underset{\mathrm{Df}}{=} \chi(\alpha_0).$$

We have $\alpha \geqslant \varphi(\beta), \varphi'(\beta)$.

Putting $\varphi(\beta)$ for α' in (3), we get

(4) $$p_{\varphi(\beta)}^\alpha = p_{\varphi(\beta)} h^{(\alpha_0)};$$

putting $\varphi'(\beta)$, we get

(5) $$p_{\varphi'(\beta)}^\alpha = p_{\varphi'(\beta)} h^{(\alpha_0)}.$$

Applying in turn (4), (1), (5), we obtain

$$f_\beta p_{\varphi(\beta)}^\alpha = f_\beta p_{\varphi(\beta)} h^{(\alpha_0)} = f_\beta' p_{\varphi'(\beta)} h^{(\alpha_0)} = f_\beta' p_{\varphi'(\beta)}^\alpha.$$

Thus condition (2) is proved. ■

Let $\hat{\mathcal{K}}_0^*$ be the subcategory of $\hat{\mathcal{K}}^*$ with uniformly movable inverse systems in $\mathcal{K}^*$ as objects.

As a consequence of 4.2 we obtain the following two corollaries concerning any morphism of uniformly movable inverse systems in $\mathcal{K}$.

4.3. Corollary. $\varprojlim \boldsymbol{f}$ *is an epimorphism in* $\mathcal{K} \Rightarrow \boldsymbol{f}$ *is an epimorphism in* $\hat{\mathcal{K}}_0^*$.

4.4. COROLLARY. $\varprojlim f$ *is a monomorphism in* $\mathcal{K} \Rightarrow f$ *is a monomorphism in* $\hat{\mathcal{K}}_0^*$.

Proof of 4.3. Take $f\colon X \to Y$ and let $f = \varprojlim \mathbf{f}$. Assume f to be an epimorphism in $\mathcal{K}$.

Take any $\mathbf{Z}$ and $\mathbf{u}, \mathbf{u}'\colon \mathbf{Y} \to \mathbf{Z}$, and let $u = \varprojlim \mathbf{u}$, $u' = \varprojlim \mathbf{u}'$. It follows by 2.1 and 4.2 that

$$\mathbf{u}\mathbf{f} \cong \mathbf{u}'\mathbf{f} \Rightarrow uf = u'f \Rightarrow u = u' \Rightarrow \mathbf{u} \cong \mathbf{u}';$$

thus $\mathbf{f}$ is an epimorphism in $\hat{\mathcal{K}}_0^*$. ∎

Proof of 4.4. Take $\mathbf{f}\colon \mathbf{X} \to \mathbf{Y}$ and let $f = \varprojlim \mathbf{f}$. Assume f to be a monomorphism in $\mathcal{K}$. Take any $\mathbf{Z}$ and $\mathbf{v}, \mathbf{v}'\colon \mathbf{Z} \to \mathbf{X}$, and let $v = \varprojlim \mathbf{v}$, $v' = \varprojlim \mathbf{v}'$. It follows by 2.1 and 4.2 that

$$\mathbf{f}\mathbf{v} \cong \mathbf{f}\mathbf{v}' \Rightarrow fv = fv' \Rightarrow v = v' \Rightarrow \mathbf{v} \cong \mathbf{v}';$$

thus $\mathbf{f}$ is a monomorphism in $\hat{\mathcal{K}}_0^*$. ∎

By 4.3 and 4.4 we get

4.5. COROLLARY. $\varprojlim \mathbf{f}$ *is a bimorphism in* $\mathcal{K} \Rightarrow \mathbf{f}$ *is a bimorphism in* $\hat{\mathcal{K}}_0^*$.

5. Uniformly movable compact spaces. Let us consider the category $\mathfrak{R}$ of compact ANR's as objects and continuous mappings as morphisms. Obviously, $\mathfrak{R}$ is a subcategory of the category $\mathcal{C}$ (see § 1, Examples 1, 2).

By a theorem of Mardešić and Segal ([3], Th. 12, Cor. 1), two ANR-systems associated with the same compact Hausdorff space X are isomorphic in category $\hat{\mathfrak{R}}^*_{\simeq}$ ($\simeq$ being the homotopy relation). Thus, Corollary 3.10 enables us to define the uniform movability of any compact Hausdorff space by means of inverse systems (in a similar way as has been followed by Mardešić and Segal in [5] for movability).

A compact Hausdorff space X is said to be *uniformly movable* whenever there is an inverse system $\mathbf{X}$ in $\mathfrak{R}$ such that $X = \varprojlim \mathbf{X}$ and $\mathbf{X}$ is uniformly movable in $\mathfrak{R}^*_{\simeq}$.

By the Mardešić and Segal Th. 10 of [3], *shape domination* for compact Hausdorff spaces can be defined as follows:

$\mathrm{Sh}(X) \geqslant \mathrm{Sh}(Y)$ iff there exist ANR-systems $\mathbf{X}, \mathbf{Y}$ associated with X, Y and such that $\mathbf{Y}$ is r-dominated by $\mathbf{X}$ in the category $\hat{\mathfrak{R}}^*_{\simeq}$ (see Introduction).

By the results of [4], in the case of metric compact spaces shape domination coincides with fundamental domination in the sense of Borsuk (see [1]).

By the statements 3.9, 3.10 we obtain

5.1. COROLLARY. *Let X, Y be two compact Hausdorff spaces. If* $\mathrm{Sh}(X) \geqslant \mathrm{Sh}(Y)$ *and X is uniformly movable, then Y is uniformly movable as well.*

5.2. COROLLARY. *The uniform movability is a shape invariant.*

Since every ANR-space is an inverse limit of a constant ANR-system in $\mathfrak{R}$, 3.4 implies

5.3. $X \in \mathrm{ANR} \Rightarrow X$ *is uniformly movable.*

Let us prove

5.4. THEOREM. *Any plane continuum is uniformly movable.*

Proof. By a theorem of Borsuk ([1], p. 235) any plane continuum is fundamentally dominated by the continuum X defined as follows: Take

$$J_0 = \{(x_1, x_2)\colon\ 0 \leqslant x_1 \leqslant 1,\ x_2 = 0\},$$

$$J_1 = \{(x_1, x_2)\colon\ 0 \leqslant x_1 \leqslant 1,\ x_2 = x_1\},$$

$$I_k = \left\{(x_1, x_2)\colon\ x_1 = \frac{1}{2^{k-1}},\ 0 \leqslant x_2 \leqslant \frac{1}{2^{k-1}}\right\},$$

$$A_n = \left\{(x_1, x_2)\colon\ 0 \leqslant x_1 \leqslant \frac{1}{2^{n-1}},\ 0 \leqslant x_2 \leqslant x_1\right\} \quad \text{for} \quad n = 1, 2, \ldots$$

and put

$$X_1 \underset{\mathrm{Df}}{=} A_1, \qquad X_n \underset{\mathrm{Df}}{=} A_n \cup J_0 \cup J_1 \cup \bigcup_{k=1}^{n-1} I_k \quad \text{for} \quad n \geqslant 2.$$

Obviously $X_n \in \mathrm{ANR}$ and $X_{n+1} \subset X_n$ for $n = 1, 2, \ldots$

Let

$$X \underset{\mathrm{Df}}{=} \bigcap_{n=1}^{\infty} X_n.$$

Since all X_n are continua, X is a continuum as well. By 5.1, it suffices to prove the uniform movability of X.

Take the inclusion ANR-sequence $\mathbf{X} = (X_n, p_n^{n'}, N)$. We have

$$X = \varprojlim \mathbf{X}.$$

For every $n \geqslant n_0$ consider the set

$$X_n^{n_0} = A_n \cup J_0 \cup J_1 \cup \bigcup_{k=1}^{n_0-1} I_k.$$

Obviously $X_n^{n_0}$ is a subset of X_n homeomorphic to X_{n_0}. Take $n_0 \in N$ and define $\mathbf{h}^{(n_0)} = (\chi^{(n_0)}, h_n^{(n_0)})\colon \mathbf{X}_{n_0} \to \mathbf{X}^{(n_0)}$ as follows:

$$\chi^{(n_0)}(n) = n_0 \quad \text{for every } n \geqslant n_0,$$

$h_n^{(n_0)}\colon X_{n_0} \to X_n$ — a topological imbedding satisfying the condition

$$h_n^{(n_0)}(X_{n_0}) = X_n^{n_0}.$$

It is easily seen that

$$p_n^{n'} h_{n'}^{(n_0)} \simeq h_n^{(n_0)} \quad \text{for} \quad n' \geqslant n \geqslant n_0.$$

i. e. the diagram

$$\begin{tikzcd} X_n & & X_{n'} \arrow[ll, "p_n^{n'}"'] \\ & & \\ X_{n_0} \arrow[uu, "h_n^{(n_0)}"] \arrow[uurr, "h_{n'}^{(n_0)}"'] & & \end{tikzcd}$$

commutes up to $\simeq$;

thus $h^{(n_0)}$ is a map of inverse sequences. Moreover,

$$p_{n_0}^n h_n^{(n_0)} \simeq 1_{X_{n_0}} = p_{n_0}^{n^0} = p_{n_0}^{\hat{n}_0}.$$

Hence X is uniformly movable and thus the proof is complete. ■

The following problem remains open:

5.5. Does there exist a compactum which is movable but not uniformly movable (6) ?

6. Homomorphisms of limit homotopy groups for uniformly movable compact spaces. We are interested in the category $\dot{\mathfrak{K}}$ of pointed ANR's (7) and the category $\mathfrak{G}$ of groups.

Let us consider two pairs $(\dot{\mathfrak{K}}, \simeq)$ and $(\dot{\mathfrak{G}}, =)$, a natural number n and the covariant functor

$$\pi_n\colon (\dot{\mathfrak{K}}, \simeq) \to (\dot{\mathfrak{G}}, =)$$

defined as usually, i.e.

$$\pi_n(X, x_0) \text{ is the } n\text{th homotopy group of } (X, x_0),$$

for any $f\colon (X, x_0) \to (Y, y_0)$

$$\pi_n(f) = f_n\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$$

the induced homomorphism.

As was noticed in § 3, the functor π_n generates a covariant functor

$$\pi_n\colon \dot{\mathfrak{K}}^*_{\simeq} \to \mathfrak{G}^*.$$

By 3.7, the functor π_n preserves uniform movability, thus we have

6.1. *If the* ANR*-system* $(\boldsymbol{X}, \boldsymbol{x}_0)$ *is uniformly movable in* $\dot{\mathfrak{K}}^*_{\simeq}$, *then the homotopy system* $\pi_n(\boldsymbol{X}, \boldsymbol{x}_0)$ *is uniformly movable in* $\mathfrak{G}^*$.

Let (X, x_0) be a pointed compact Hausdorff space. By Theorem 7 of [3], there is an ANR-system $(\boldsymbol{X}, \boldsymbol{x}_0)$ such that

$$(X, x_0) = \underleftarrow{\lim}(\boldsymbol{X}, \boldsymbol{x}_0).$$

(6) See Remark 6.7.

(7) By a pointed space we mean the pair (X, x_0), x_0 being any point of the space X.

Let us consider the group $\underleftarrow{\lim}\,\pi_n(\boldsymbol{X}, \boldsymbol{x}_0)$ and prove that it does not depend on the choice of the inverse system $(\boldsymbol{X}, \boldsymbol{x}_0)$ (the statement 6.3).

6.2. *If* $(\boldsymbol{X}, \boldsymbol{x}_0), (\boldsymbol{Y}, \boldsymbol{y}_0)$ *are homotopicaly equivalent, then the groups* $\underleftarrow{\lim}\,\pi_n(\boldsymbol{X}, \boldsymbol{x}_0)$, $\underleftarrow{\lim}\,\pi_n(\boldsymbol{Y}, \boldsymbol{y}_0)$ *are isomorphic.*

Proof. Let $(\boldsymbol{X}, \boldsymbol{x}_0), (\boldsymbol{Y}, \boldsymbol{y}_0)$ be homotopicaly equivalent, i.e. isomorphic in the category $\hat{\dot{\mathfrak{K}}}^*_{\simeq}$. Then, by 3.6, the two inverse systems $\pi_n(\boldsymbol{X}, \boldsymbol{x}_0)$ and $\pi_n(\boldsymbol{Y}, \boldsymbol{y}_0)$ are isomorphic in the category $\hat{\mathfrak{G}}^*$. Thus, by 2.2, their inverse limits are isomorphic in $\mathfrak{G}$. ■

By Theorem 12, Corollary 1 of [3], the statement 6.2 implies that

6.3. *If two* ANR*-systems,* $(\boldsymbol{X}, \boldsymbol{x}_0)$ *and* $(\boldsymbol{X}', \boldsymbol{x}_0')$, *have the same inverse limit* (X, x_0), *then the groups* $\underleftarrow{\lim}\,\pi_n(\boldsymbol{X}, \boldsymbol{x}_0)$ *and* $\underleftarrow{\lim}\,\pi_n(\boldsymbol{X}', \boldsymbol{x}_0')$ *are isomorphic (in* $\mathfrak{G}$).

The last proposition enables us to define the group $\pi_n^*(X, x_0)$ as follows:

$$\pi_n^*(X, x_0) \underset{\mathrm{Df}}{=} \underleftarrow{\lim}\,\pi_n(\boldsymbol{X}, \boldsymbol{x}_0),$$

the system $(\boldsymbol{X}, \boldsymbol{x}_0)$ being any ANR-system associated with (X, x_0). We shall refer to the group $\pi_n^*(X, x_0)$ as the n*-th limit homotopy group of* (X, x_0). By 6.2, π_n^* is a shape invariant.

This group can be proved to be isomorphic to the fundamental group $\underline{\pi}_n(X, x_0)$ as defined by K. Borsuk in [1].

Every map $\boldsymbol{f}\colon (\boldsymbol{X}, \boldsymbol{x}_0) \to (\boldsymbol{Y}, \boldsymbol{y}_0)$ induces a homomorphism of nth limit homotopy groups,

$$f_n^*\colon \pi_n^*(X, x_0) \to \pi_n^*(Y, y_0),$$

$$f_n^* \underset{\mathrm{Df}}{=} \underleftarrow{\lim}\,\boldsymbol{f}_n, \quad \text{where} \quad \boldsymbol{f}_n = \pi_n(\boldsymbol{f}).$$

By 2.2, we have

6.4. $\boldsymbol{f}_n$ *is an isomorphism in* $\hat{\mathfrak{G}}^* \Rightarrow f_n^*$ *is an isomorphism in* $\mathfrak{G}$.

By 2.3, we get

6.5. $\boldsymbol{f}_n$ *is a monomorphism in* $\hat{\mathfrak{G}}^* \Rightarrow f_n^*$ *is a monomorphism in* $\mathfrak{G}$.

By Example 1 (§ 2), we infer that a similar implication for epimorphisms fails. In fact, let X be a Van Danzig solenoid, $X = \underleftarrow{\lim}\,\boldsymbol{X}$, where $\boldsymbol{X} = (S^1, p_m^{m+1})$, $p_m^{m+1}(z) = z^2$, and let $\boldsymbol{Y} = S^1$ — a constant system and $n = 1$. Take $\boldsymbol{f} = (1, f_m)$, $f_m(z) = z^{2^m}$. Then, by Example 1, $\boldsymbol{f}_1$ is an epimorphism in $\hat{\mathfrak{G}}^*$ but $\underleftarrow{\lim}\,\boldsymbol{f}_1$ is not an epimorphism in $\mathfrak{G}$.

In a similar way, applying Examples 2, 3, 4 of § 2, we infer that the converse implications in general fail.

However, for uniformly movable spaces some positive results can be obtained. Indeed, by 6.1 and 4.3–4.5, we get the following

6.6. Corollary. *If (X, x_0), (Y, y_0) are uniformly movable pointed compact Hausdorff spaces, and $(\boldsymbol{X}, \boldsymbol{x}_0)$, $(\boldsymbol{Y}, \boldsymbol{y}_0)$ — the associated* ANR-*systems, then for any map $\boldsymbol{f}$: $(\boldsymbol{X}, \boldsymbol{x}_0) \to (\boldsymbol{Y}, \boldsymbol{y}_0)$*

(1) f_n^* *is a monomorphism in* $\mathcal{G} \Rightarrow \boldsymbol{f}_n$ *is a monomorphism in* $\hat{\mathcal{G}}^*$,

(2) f_n^* *is an epimorphism in* $\mathcal{G} \Rightarrow \boldsymbol{f}_n$ *is an epimorphism in* $\hat{\mathcal{G}}^*$,

(3) f_n^* *is a bimorphism in* $\mathcal{G} \Rightarrow \boldsymbol{f}_n$ *is a bimorphism in* $\hat{\mathcal{G}}^*$.

6.7. Remark. When the paper was in press, the question 5.5 was answered by S. Spież [8]. He proved that every movable compactum is uniformly movable.

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An atomic map onto an arbitrary metric continuum

by

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A continuum means in this paper a compact connected Hausdorff space. A continuous map $f\colon X \xrightarrow{\text{onto}} Y$ is said to be *atomic* if for each subcontinuum K of X such that $f(K)$ is non-degenerate we have $f^{-1}(f(K)) = K$. The notion of an atomic (continuous and open) map was originally introduced by Anderson [2] and was applied by Anderson and Choquet [3], and then by Cook [4], to the constructions of some singular continua. In 1966 Mahavier [8] and Thomas [10] showed independently that there is no atomic map from an irreducible, metric continuum onto an arc such that the preimage of each point is a non-degenerate, hereditarily decomposable, chainable continuum. In 1970 Mahavier [9] showed that if K is a metric continuum, then there is an atomic map from a separable, first countable, irreducible continuum onto an arc such that the preimage of each point is homeomorphic to K. In this note we show that if X is a metric continuum and K_x, $x \in X$, are metric continua, then there is an atomic map f from a separable, first countable Hausdorff continuum onto X such that the preimage under f of any point x of X is homeomorphic to K_x. If, in addition, X is irreducible, then the continuum in question proves to be irreducible, and so the construction given here is a generalization of that of Mahavier. A similar construction is given also in a paper of Fedorčuk [6], who applied it to the proof of the existence of a compact Hausdorff space having the dimension dim less than the dimension ind. However, Fedorčuk's construction is incomparable with that of the present paper: although it satisfies some special conditions, the map is not atomic, and X and K_x, for $x \in X$, are rather special spaces, such as an n-sphere or an n-torus, and are locally connected continua in the most general case.

Let X be an arbitrary metric continuum. For each $x \in X$, let M_x be a metric continuum and let $T_x\colon M_x \xrightarrow{\text{onto}} X$ be a continuous map. Let $S = \bigcup \{\{x\} \times T_x^{-1}(x)\colon x \in X\}$. For each $x \in X$ and an open subset U of M_x which intersects $T_x^{-1}(x)$, let $R(x, U)$ denote the subset of S to which