identity map on X_n. We apply Lemma 4 for spaces P_i, E_j and isomorphisms $\varphi_k^{(i,j)}$ except for P_{n_1}, E_{n_2} and $\varphi_{n_3(n_1,n_2)}^{(n_1,n_2)}$. Thus we obtain the space $\tilde{B} \supset B_n$. Since $B_n \supset X_n$ and X_n is isometric to l_n^∞ the projection $\pi_n\colon B_n \to X_n$, $\|\pi_n\| = 1$ can be extended to a projection $\tilde{\pi}\colon \tilde{B} \to X_n$ of norm one. Thus we apply Lemma 5 to obtain the space B_{n+1} which contains X_{n+1} and there is a projection π_{n+1} of norm one from B_{n+1} onto X_{n+1} and $\pi_{n+1}|B_n = \pi_n$. The space B_{n+1} satisfies (iv) in view of Lemma 2. This completes the proof.

Remark. By the same method one may establish the following statement:

For any finite set of separable preduals of L_1, $X_1, \ldots, X_k$ there exists a space of universal disposition $\Gamma_{X_1,\ldots,X_k}$ such that $X_i \subset \Gamma_{X_1,\ldots,X_k}$, $i = 1, 2, \ldots, k$, and there are projections of norm one from $\Gamma_{X_1,\ldots,X_k}$ onto X_i for $i = 1, 2, \ldots, k$.

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Construction of an orthonormal basis in $C^m(I^d)$ and $W_p^m(I^d)$

by

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Abstract. The space $C^m(I^d)$ is equipped in the natural scalar product induced from $L_2(I^d)$. A special orthonormal set of functions in $C^m(I^d)$ is constructed. This set of functions turns out to be a basis for the Banach spaces $C^m(I^d)$ and $W_p^m(I^d)$.

1. Introduction. The sequence $(x_n, n = 1, 2, \ldots)$ of elements of a given Banach space X is called a *basis* whenever each $x \in X$ has unique expansion

$$x = \sum_{n=1}^{\infty} a_n x_n$$

convergent in the norm. It is known that the coefficients $a_n = a_n(x)$ are linear functionals over X.

In this paper we shall deal mainly with the following two real Banach spaces:

The space $C^m(I^d)$, $m \geqslant 0$, $d \geqslant 1$, of m times continuously differentiable functions on I^d, $I = \langle 0, 1\rangle$, with the norm

$$\|f\|^{(m)} = \max_{|\boldsymbol{k}| \leqslant m} \max_{t \in I} |D^{\boldsymbol{k}} f(t)|,$$

where $\boldsymbol{k} = (k_1, \ldots, k_d)$, k_f, and $1 \leqslant f \leqslant d$, being non-negative integers, $|\boldsymbol{k}| = k_1 + \ldots + k_d$ and $D^{\boldsymbol{k}}$ is the differential operator corresponding to $\boldsymbol{k}$, i.e.

$$D^{\boldsymbol{k}} = \frac{\partial^{|\boldsymbol{k}|}}{\partial t_1^{k_1} \ldots \partial t_p^{k_d}}.$$

The Sobolev space $W_p^m(I^d)$ with $m \geqslant 0$, $d \geqslant 1$ and $1 \leqslant p < \infty$, which is the set of all $f \in L_p(I^d)$ such that the generalized derivatives $D^{\boldsymbol{k}} f$ are functions and belong to $L_p(I^d)$ for each $\boldsymbol{k}$, $|\boldsymbol{k}| \leqslant m$. The norm is defined as follows

$$\|f\|_p^{(m)} = \Big(\sum_{|\boldsymbol{k}| \leqslant m} \|D^{\boldsymbol{k}} f\|_p\Big)^{1/p},$$

where $\| \ \|_p$ is the usual L_p-norm over I^d.

In the space $C^m(I^d)$ the scalar product

(1) $$(f, g) = \int_{I^d} f(t) g(t)\, dt$$

is continuous in both variables.

We are going to construct a basis in $C^m(I^d)$ with $m \geqslant 0$ and $d \geqslant 1$. It will be shown that this basis is also a basis in the corresponding Sobolev space $W_p^m(I^d)$, where $1 \leqslant p < \infty$.

The basis constructed in this paper is orthogonal with respect to the scalar product (1). On the other hand the algebraic polynomials are dense in $C^m(I^d)$. Therefore by theorems of Olevskii [8] there exists an orthogonal basis of polynomials in $C^m(I^d)$.

The question of existence of a basis in $C^1(I^2)$ was raised already by Banach in [1], p. 238. Only recently one of the authors [5] and independently Schonefeld [14] exhibited a basis in $C^1(I^2)$. The extension to the case of $C^1(I^d)$ with $d > 1$ is immediate. The solution of the problem for arbitrary $m \geqslant 0$ and $d \geqslant 1$ requires a new approach. The solution proposed here is suggested by the investigations of the Franklin orthonormal system in [3] and [4]. Different construction of a basis in $C^m(I^d)$ with $d \geqslant 1$ and $0 \leqslant m \leqslant 4$, and also in $C^m(T^d)$ for $m \geqslant 0$ and $d \geqslant 1$, where T^d is the d-dimensional torus, was communicated to the authors by S. Schonefeld [15].

The relation between Schonefeld's construction and the one presented here is more or less like the relation between Schauder and orthonormal Franklin bases in $C^0(I)$. Our system is a basis in $W_p^m(I^d)$ like the Franklin system is a basis in $L_p(I)$. The Schonefeld functions do not form a basis in $W_p^m(I^d)$ like the Schauder functions do not form a basis in $L_p(I)$.

The idea of the construction presented here was announced at the Conferences on Constructive Function Theory in Budapest, August–September 1969, and in Varna, May 1970. There will appear notes in the Proceedings of these Conferences.

All the proofs given here depend essentially on the result established by one of the authors in [7].

2. The B-splines. Let S denote the partition of $(-\infty, +\infty)$ given by the sequence

(2) $$\ldots < s_{-1} < s_0 < s_1 < \ldots,$$

and let $I_j = \langle s_{j-1}, s_j \rangle$, $j = 0, \pm 1, \ldots$ The function $f \in C^m(-\infty, +\infty)$ is called a *spline of degree* $m+1$ with respect to the partition (2) if it is in each interval I_j a polynomial of degree at most $m+1$. We accept this definition for $m = 0, 1, \ldots$

The divided difference of the function f taken at the points $s_{i-1}, \ldots, {}_{i+m+1}$ is denoted by $[s_{i-1}, \ldots, s_{i+m+1}; f(s)]$. Now, let $t_+ = \max(0, t)$.

It is clear that the functions $(s_j - t)_+^{m+1}$ are splines of degree $m+1$. Linear combinations of splines are again splines. Thus, the functions

(3) $$N_i^{(m)}(t) = (s_{i+m+1} - s_{i-1}) [s_{i-1}, \ldots, s_{i+m+1}; (s-t)_+^{m+1}],$$

where t is the independent variable, are splines of degree $m+1$. They are called *B-splines* (basic).

The B-splines were introduced by Schoenberg [11] and they have the following properties (cf. [6], [12] and [13]):

(P.1) $N_i^{(m)}(t) \geqslant 0$ for $t \in (-\infty, +\infty)$, $i = 0, \pm 1, \ldots$

(P.2) $\operatorname{supp} N_i^{(m)}(t) = \langle s_{i-1}, s_{i+m+1} \rangle$ for $i = 0, \pm 1, \ldots$

(P.3) We have the following identities

$$\sum_{i=-\infty}^{\infty} N_i^{(m)}(t) = 1 \quad \text{for} \quad t \in (-\infty, +\infty)$$

and

$$\int_{-\infty}^{+\infty} N_i^{(m)}(t)\, dt = \frac{s_{i+m+1} - s_{i-1}}{m+2} \quad \text{for} \quad i = 0, \pm 1, \ldots$$

(P.4) For $m \geqslant 1$ and for arbitrary reals ξ_j we have

$$D \sum_{i=-\infty}^{\infty} \xi_i N_i^{(m)} = (m+1) \sum_{i=-\infty}^{\infty} \frac{\xi_i - \xi_{i-1}}{s_{i+m} - s_{i-1}} N_i^{(m-1)},$$

where Df is the derivative of f.

(P.5) The system of functions $\{N_i^{(m)},\ \operatorname{supp} N_i^{(m)} \cap (a, b) \neq \emptyset\}$ is a basis in the finite dimensional space of splines restricted to $\langle a, b \rangle$, $-\infty < a < b < +\infty$. In particular if $s_0 = a$ and $s_n = b$, then the system $\{N_i^{(m)}, i = -m, \ldots, n\}$ is a basis in the space $\mathscr{S}_n^m(S)$ of splines of degree $m+1$ corresponding to the partition S and restricted to $\langle s_0, s_n \rangle$.

3. The sequence of special partitions. For $n = 1$ we define $s_{n,i} = i$, $i = 0, \pm 1, \ldots$ Now, every $n > 1$ can be written in unique way in the form $n = 2^\mu + \nu$, where μ and ν are integers such that $\mu \geqslant 0$ and $1 \leqslant \nu \leqslant 2^\mu$. For $n > 1$ the partition is defined as follows

(4) $$s_{n,i} = \begin{cases} \dfrac{i}{2^{\mu+1}} & \text{for} \quad i = \ldots, -1, 0, 1, \ldots, 2\nu, \\[2ex] \dfrac{i-\nu}{2^\mu} & \text{for} \quad i = 2\nu+1, 2\nu+2, \ldots \end{cases}$$

Notice that $s_{n,0} = 0$ and $s_{n,n} = 1$ for all n. Thus for each $n \geqslant 1$ we have the partition

(5) $$S_n = \{s_{n,i},\ i = 0, \pm 1, \ldots\}$$

such that $\ldots < s_{n,-1} < s_{n,0} = 0 < s_{n,1} < \ldots < s_{n,n} = 1 < s_{n,n+1} < \ldots$

4. The sequence of finite dimensional spaces. Let $[x_1, \ldots, x_r]$ denote the linear subspace spanned by the elements $x_1, \ldots, x_r$ of $C^0(I)$. For any $n \geqslant -m$, $m \geqslant 0$, we define the subspace $C_n^m(I)$ of $C^0(I)$ as follows

(6) $$C_n^m \equiv C_n^m(I) = \begin{cases} [1, t, \ldots, t^{m+n}] & \text{for} \quad -m \leqslant n \leqslant 0, \\ [N_{n,i}^{(m)},\ i = -m, \ldots, n] & \text{for} \quad n > 0; \end{cases}$$

where $N_{n,i}^{(m)}$, $i = 0, \pm 1, \ldots$, are the B-splines of degree $m+1$ corresponding to the partition S_n defined in (4) and (5). Notice that $C_n^m(I) = \mathscr{S}_n^m(S_n)$ for $n > 0$ and $\dim C_n^m = m+n+1$ for $n \geqslant -m$, $m \geqslant 0$. It is clear that $C_n^m \subset C_{n+1}^m$. Moreover, for each $m \geqslant 0$ the set

(7) $$\mathscr{S}^m(I) = \bigcup_{n=1}^{\infty} C_n^m(I)$$

is dense in $C^0(I)$.

It is convenient at this place to introduce the following notation

(8) $$J_n^m = \{-m, -m+1, \ldots, n\}.$$

5. Special orthonormal sets in $C^0(I)$. We define for each $m \geqslant 0$ the sequence $(f_j^{(m)},\ j = -m, -m+1, \ldots)$ in the following way: $f_{-m}^{(m)} = 1$, and for $j > -m$, $f_j^{(m)}$ is one of the different from zero elements in C_j^m which are orthogonal with respect to the scalar product (1), $d = 1$, to the space C_{j-1}^m. The functions $f_j^{(m)}$ are normalized in such a way that

(9) $$(f_i^{(m)}, f_j^{(m)}) = \delta_{i,j} \quad \text{for} \quad i, j = -m, -m+1, \ldots$$

In the case of $m = 0$ this orthonormal set is known, it is called Franklin system, and it is a basis in $C^0(I)$ (for a simple proof of this fact we refer to [3]).

6. The main inequality. According to (P. 5) for each $n > 0$ the system $(N_{n,i}^{(m)},\ i \in J_n^m)$ is a basis in the space $C_n^m(I)$. Now, let

(10) $$G_{n;i,j}^{(m)} = (N_{n,i}^{(m)}, N_{n,j}^{(m)}) \quad \text{for} \quad i, j \in J_n^m,\ m \geqslant 0,\ n > 0.$$

The matrix $\boldsymbol{G}_n^{(m)} = (G_{n;i,j}^{(m)})_{i,j=-m,\ldots,n}$ is the *Gram* matrix for this basis. There exists the inverse

(11) $$\boldsymbol{A}_n^{(m)} = (A_{n;i,j}^{(m)})_{i,j=-m,\ldots,n} = (\boldsymbol{G}_n^{(m)})^{-1}.$$

The Dirichlet kernel of the orthonormal set $(f_j^{(m)}, j = -m, -m+1 \ldots)$ can be written as follows

(12) $$K_n^{(m)}(t, s) = \sum_{j=-m}^{n} f_j^{(m)}(t) f_j^{(m)}(s) = \sum_{i,j=-m}^{n} A_{n;i,j}^{(m)} N_{n,i}^{(m)}(t) N_{n,j}^{(m)}(s).$$

To see this notice that $f_j^{(m)} \in C_n^m(I)$ for $j \in J_n^m$. Consequently, the kernel $K_n^{(m)}(t, s)$ can be written in the form (12). The integral operator with the kernel $K_n^{(m)}(t, s)$ is an orthogonal projection of $C^0(I)$ onto $C_n^{(m)}(I)$ and therefore

$$\int_I K_n^{(m)}(t, s) N_{n,k}^{(m)}(s)\,ds = N_{n,k}^{(m)}(t) \quad \text{for} \quad t \in I,\ k \in J_n^m$$

whence by (12)

$$\sum_{i,j=-m}^{n} A_{n;i,j}^{(m)} G_{n;j,k}^{(m)} N_{n,i}^{(m)}(t) = N_{n,k}^{(m)}(t) \quad \text{for} \quad t \in I,\ k \in J_n^m.$$

Using the linear independence of the B-splines we get (11).

Now we are ready to state the basic result which was proved for $m = 0$ in [4] and for arbitrary $m \geqslant 0$ in [7].

THEOREM 1. *For each $m \geqslant 0$ there exist constants C_m and q_m, $0 < q_m < 1$, independent of n, such that*

$$|A_{n;i,j}^{(m)}| \leqslant n C_m q_m^{|i-j|}$$

holds for $n > 0$ and $i, j = -m, \ldots, n$.

For the sake of completeness we are going to prove the following well-known lemma.

Let $\langle E_1, \sigma_1, M_1 \rangle$ and $\langle E_2, \sigma_2, M_2 \rangle$ be measure spaces with σ-algebras σ_1, σ_2 and measures M_1, M_2 respectively. Let $L_p(E_i)$ denote the space of real functions integrable over E_i with respect to M_i with the exponent p and with the norm

(13) $$\|f\|_p = \begin{cases} \left(\int_{E_i} |f(t)|^p\,dM_i(t)\right)^{1/p} & \text{for} \quad 1 \leqslant p < \infty, \\ \operatorname{ess\,sup}\{|f(t)|:\ t \in E_i\} & \text{for} \quad p = \infty, \end{cases}$$

for $i = 1, 2$.

LEMMA 1. *Let the integral operator $W\colon L_p(E_2) \to L_p(E_1)$ with the measurable kernel*

$$(Wf)(t_1) = \int_{E_2} w(t_1, t_2) f(t_2)\,dM_2(t_2), \quad t_1 \in E_1,$$

satisfy the inequalities

(14) $$\int_{E_2} |w(t_1, t_2)|\,dM_2(t_2) \leqslant N \quad \text{a.e. on} \quad E_1$$

and

(15) $$\int_{E_1} |w(t_1, t_2)|\,dM_1(t_1) \leqslant N \quad \text{a.e. on} \quad E_2.$$

Then W is bounded and

(16) $$\|W\|_p \leqslant N \quad \text{for} \quad 1 \leqslant p \leqslant \infty.$$

Proof. We use the argument given e.g. in [9], pp. 12–13. Let q denote the conjugate to p, i.e. $q = p/(p-1)$. Applying Hölder's inequality we obtain

$$\begin{aligned}
\|Wf\|_p^p &= \int_{E_1} \Big| \int_{E_2} w(t_1, t_2) f(t_2)\, dM_2(t_2) \Big|^p dM_1(t_1) \\
&\leqslant \int_{E_1} \Big(\int_{E_2} |w(t_1, t_2)|^{1/q} |w(t_1, t_2)|^{1/p} |f(t_2)|\, dM_2(t_2) \Big)^p dM_1(t_1) \\
&\leqslant \int_{E_1} N^{p/q} \int_{E_2} |w(t_1, t_2)|\, |f(t_2)|^p\, dM_2(t_2)\, dM_1(t_1) \\
&\leqslant N^{p/q} \int_{E_2} |f(t_2)|^p \int_{E_1} |w(t_1, t_2)|\, dM_1(t_1)\, dM_2(t_2) \\
&\leqslant N^p \|f\|_p^p,
\end{aligned}$$

and

$$\begin{aligned}
\|Wf\|_\infty &= \operatorname{ess\,sup}\Big\{ \Big| \int_{E_1} w(t_1, t_2) f(t_2)\, dM_2(t_2) \Big| :\ t_1 \in E_1 \Big\} \\
&\leqslant \operatorname{ess\,sup}\Big\{ \int_{E_2} |w(t_1, t_2)|\, |f(t_2)|\, dM_2(t_2) :\ t_1 \in E_1 \Big\} \\
&\leqslant \|f\|_\infty N.
\end{aligned}$$

7. Uniform isomorphism of some finite dimensional spaces. Let us denote by $(\underline{N}_{n,j}^{(m)}, j \in J_n^m)$ the dual basis in $C_n(I)$ to the basis $(N_{n,i}^{(m)}, i \in J_n^m)$ – dual with respect to the scalar product (1) with $d = 1$. It is easy to see that

$$\underline{N}_{n,j}^{(m)} = \sum_{i=-m}^{n} A_{n;j,i}^{(m)} N_{n,i}^{(m)} \quad \text{for} \quad j \in J_n^m, \tag{17}$$

where $A_n^{(m)}$ is defined in (11). Let $\langle t \rangle$ denote the unique integer solution of the inequalities

$$s_{\langle t \rangle - 1} \leqslant t < s_{\langle t \rangle}, \qquad t \in I. \tag{18}$$

Now, the dual functions can be estimated with the help of properties (P.1)–(P.3) and Theorem 1 in the following way

$$|\underline{N}_{n,j}^{(m)}(t)| \leqslant \sum_{i=-m}^{n} C_m n q_m^{|i-j|} N_{n,i}^{(m)}(t) \leqslant \sum_{i=\langle t \rangle - m - 1}^{\langle t \rangle} C_m n q_m^{|i-j|} \leqslant C_m n q_m^{l} \quad \text{for} \quad j \in J_n^m,\ t \in I, \tag{19}$$

where

$$\begin{aligned}
l = \min\{|i-j| :\ i = \langle t \rangle - m - 1, \ldots, \langle t \rangle\} &\geqslant |\langle t \rangle - j| - m - 1 \\
&\geqslant \frac{n}{2} |t - s_j| - m - 2.
\end{aligned} \tag{20}$$

Combining (19) and (20) we obtain

$$|\underline{N}_{n,j}^{(m)}(t)| \leqslant D_m n q_m^{n|t-s_j|/2} \quad \text{for} \quad j \in J_n^m,\ t \in I \quad \text{and} \quad n > 0, \tag{21}$$

where D_m is a suitable constant depending on m only. Using (21) we check easily that for some constant N_m we have

$$\int_I |\underline{N}_{n,j}^{(m)}(t)|\, dt \leqslant N_m \quad \text{for} \quad j \in J_n^m \quad \text{and} \quad n > 0, \tag{22}$$

and

$$\frac{1}{n+m+1} \sum_{j=-m}^{n} |\underline{N}_{n,j}^{(m)}(t)| \leqslant N_m \quad \text{for} \quad t \in I \quad \text{and} \quad n > 0. \tag{23}$$

Now, let us consider the spaces $L_p(E_1)$ with $E_1 = J_n^m$ and with the corresponding measure M_1 equal to the uniform mass distribution on J_n^m i.e. $M_1(j) = 1/(m+n+1)$ for all $j \in J_n^m$, and $L_p(E_2)$ with $E_2 = I$ and M_2 equal to the Lebesgue measure. In order to apply Lemma 1 we put $w(j, t) = \underline{N}_{n,j}^{(m)}(t)$. Notice that (22) and (23) correspond to (14) and (15), respectively. Consequently, Lemma 1 gives

$$\Big(\frac{1}{n+m+1} \sum_{j=-m}^{n} |(\underline{N}_{n,j}^{(m)}, f)|^p \Big)^{1/p} \leqslant N_m \|f\|_p \tag{24}$$

for $f \in L_p(I)$ and $1 \leqslant p \leqslant \infty$.

It is convenient to introduce the following notation

$$P_n^m f = \sum_{j=-m}^{n} (\underline{N}_{n,j}^{(m)}, f) N_{n,j}^{(m)}.$$

Since $(N_{n,j}^{(m)}, j \in J_n^m)$ is a partition of unity (cf. (P.3)) it follows from Jensen's inequality that for suitable constant M_m, dependent on m only, we have

$$\begin{aligned}
\|P_n^m f\|_p^p &= \int_I \Big| \sum_{j=-m}^{n} (\underline{N}_{n,j}^{(m)}, f) N_{n,j}^{(m)}(t) \Big|^p dt \\
&\leqslant \sum_{j=-m}^{n} |(\underline{N}_{n,j}^{(m)}, f)|^p \int_I N_{n,j}^{(m)}(t)\, dt \\
&\leqslant M_m^p \frac{1}{m+n+1} \sum_{j=-m}^{n} |(\underline{N}_{n,j}^{(m)}, f)|^p.
\end{aligned} \tag{25}$$

Using (24) and (25) we get

$$\|P_n^m f\|_p \leqslant (M_m N_m) \|f\|_p \quad \text{for} \quad f \in L_p(I),\ 1 \leqslant p \leqslant \infty \quad \text{and} \quad n > 0. \tag{26}$$

LEMMA 2. *Let $n > 0$, $m \geqslant 0$ and $f \in C_n^m(I)$ and let*

$$f = \sum_{i=-m}^{n} \xi_i N_{n,i}^{(m)}.$$

Then there exist constants A_m and B_m which depend on m only such that

$$B_m \|\xi\|_p \leqslant \|f\|_p \leqslant A_m \|\xi\|_p$$

holds for $1 \leqslant p \leqslant \infty$, where

$$\|f\|_p^p = \int_I |f(t)|^p dt, \qquad \|\xi\|_p^p = \frac{1}{m+n+1} \sum_{i=-m}^{n} |\xi_i|^p.$$

For the proof notice that for $f \in C_n^m(I)$ we have $P_n^m f = f$ and therefore the lemma is a consequence of (24) and (25).

8. Bernstein's inequality.

LEMMA 3. *Let $m > 0$. Then there exists constant K_m independent of n and p such that for each $f \in C_n^m(I)$ the inequality*

(27) $\|D^k f\|_p \leqslant K_m \cdot \delta_n^{-k} \cdot \|f\|_p, \quad k = 0, \ldots, m+1, n > 0, 1 \leqslant p \leqslant \infty,$

holds with $\delta_n = \inf\{(s_{n,i} - s_{n,i-1})\colon\ i = 1, \ldots, n\} = 2^{-(\mu+1)}$ for which $1/(2n) < \delta_n \leqslant 1/n$.

Proof. It is sufficient to show the existence of such constant L_m that

(28) $\|Df\|_p \leqslant L_m \delta_n^{-1} \|f\|_p, \quad f \in C_n^m(I), n > 0, 1 \leqslant p \leqslant \infty.$

To see this let

$$f = \sum_{i=-m}^{n} \xi_i N_{n,i}^{(m)}.$$

Applying the formula given in (P.4) and Lemma 2 we obtain (28).

9. Exponential estimates for the Dirichlet kernel. To state the main result of this section we need the following notation

(29) $$(Hf)(t) = \int_t^1 f(s)\,ds \quad \text{for} \quad t \in I,$$

(30) $$(Gf)(t) = \int_0^t f(s)\,ds \quad \text{for} \quad t \in I.$$

It follows immediately that

(31) $$\frac{(s-t)^{l-1}}{(l-1)!} = \int_I \frac{(s-u)^{l-1}}{(l-1)!} K_n^{(m)}(t, u)\,du = (H^{(0,l)} K_n^{(m)})(t, s) - (-1)^l G^{(0,l)} K_n^{(m)}(t, s)$$

holds for $l = 1, \ldots, m$. The upper index $(0, l)$ means that the l-th power of the corresponding operator acts on $K_n^{(m)}(t, s)$ as on a functions of the second variable. This convention applies to the operator D too. Thus, applying $D^{(k,0)}$ to both sides of (31) we obtain

(32) $$(D^{(k,0)} H^{(0,l)} K_n^{(m)})(t, s) = (-1)^l (D^{(k,0)} G^{(0,l)} K_n^{(m)})(t, s)$$

for $t, s \in I$, $0 \leqslant l \leqslant k \leqslant m$.

LEMMA 4. *For each $m \geqslant 0$ there exist constants L_m and r_m independent of n, $0 < r_m < 1$, such that*

(33) $$|(D^{(k,0)} K_n^{(m)})(t, s)| \leqslant L_m n^{k+1} r_m^{n|t-s|}$$

holds for $n > 0$, $0 \leqslant k \leqslant m$ and $t, s \in I$.

Proof. The Bernstein's inequality (cf. Lemma 3) gives

(34) $|D^k N_{n,i}^{(m)}(t)| < K_m (2n)^k \quad \text{for} \quad t \in I, i \in J_n^m, 0 \leqslant k \leqslant m, n > 0.$

Notice that (11), (12) and (17) give

$$(D^{(k,0)} K_n^{(m)})(t, s) = \sum_{i=-m}^{n} D^k N_{n,i}^{(m)}(t) \underline{N}_{n,i}^{m)}(s) = \sum_{i=\langle t\rangle-m-1}^{\langle t\rangle} D^k N_{n,i}^{(m)}(t) \underline{N}_{n,i}^{(m)}(s),$$

whence by (21) and (34) we obtain (33).

THEOREM 2. *Let m be given. Then there exist constants M_m and r_m, $0 < r_m < 1$, such that*

(35) $$|(H^{(0,l)} D^{(k,0)} K_n^{(m)})(t, s)| \leqslant M_m n^{k+1-l} r_m^{n|t-s|}$$

holds for $n > 0$, $m \geqslant k \geqslant l \geqslant 0$ and $t, s \in I$.

Proof. Denoting by $R_n^{(m)}(t, s)$ the right-hand side of (33) and assuming that $1 \geqslant s \geqslant t \geqslant 0$ we obtain by Lemma 4

$$|H^{(0,l)} D^{(k,0)} K_n^{(m)}(t, s)| \leqslant (H^{(0,l)} R_n^{(m)})(t, s) \leqslant (n|\log r_m|)^{-l} R_n^{(m)}(t, s).$$

Similarly, for $0 \leqslant s \leqslant t \leqslant 1$ we have

$$|(G^{(0,l)} D^{(k,0)} K_n^{(m)})(t, s)| \leqslant (n|\log r_m|)^{-l} R_n^{(m)}(t, s).$$

Combining these inequalities with (32) we obtain (35).

It can be proved with the help of Theorem 2 and Lemma 1 the following

LEMMA 5. *Let us define for given m and k, $0 \leqslant k \leqslant m$, the operator $P_n^{(m,k)}$ as follows*

(36) $$(P_n^{(m,k)} f)(t) = \int_I K_n^{(m,k)}(t, s) f(s)\,ds \quad \textit{for} \quad t \in I, n \geqslant k - m,$$

where

(37) $$K_n^{(m,k)} = H^{(0,k)} D^{(k,0)} K_n^{(m)}.$$

Then there exist constants P_m depending on m only such that

$$\|P_n^{(m,k)}\|_p \leqslant P_m \quad \text{for} \quad 0 \leqslant k \leqslant m,\ n \geqslant k-m \quad \text{and} \quad 1 \leqslant p \leqslant \infty,$$

where $\|P_n^{(m,k)}\|_p$ *is the* L_p *norm of the operator* $P_n^{(m,k)}\colon L_p(I) \to L_p(I)$.

10. The special basis in $C^0(I)$ and $L_p(I)$. In Section 5 we have introduced the orthonormal sets $(f_j^{(m)}, j = -m, -m+1, \ldots)$, $m \geqslant 0$. The aim of this section is to show that the system $(f_j^{(m,k)}, j = k-m, k-m+1, \ldots)$, where

$$f_j^{(m,k)} = D^k f_j^{(m)}, \quad 0 \leqslant k \leqslant m,\ j = k-m, k-m+1, \ldots \tag{38}$$

is a basis in $C^0(I)$ and $L_p(I)$ for $1 \leqslant p < \infty$.

Let us introduce the new system of functions $(g_j^{(m,k)}, j = k-m, k-m+1, \ldots)$, where $g_j^{(m,k)} = H^k f_j^{(m)}$, H is given as in (29). Integrating by parts we check

$$(g_j^{(m,k)}, D^k f) = (f_j^{(m)}, f) \quad \text{for} \quad j \geqslant k-m,\ f \in C^k(I). \tag{39}$$

In particular (39) gives

$$(g_j^{(m,k)}, f_i^{(m,k)}) = \delta_{i,j} \quad \text{for} \quad i, j = k-m, k-m+1, \ldots \tag{40}$$

Thus the system $(g_j^{(m,k)}, f_i^{(m,k)}, i, j = k-m, k-m+1, \ldots)$ is biorthogonal.

THEOREM 3. *Let the integers m, k, $0 \leqslant k \leqslant m$, and the real p, $1 \leqslant p < \infty$, be fixed. Then $(f_j^{(m,k)}, j = k-m, k-m+1, \ldots)$ is a basis simultaneously in $C^0(I)$ and $L_p(I)$ and in each of these spaces*

$$f = \sum_{j=k-m}^{\infty} (g_j^{(m,k)}, f) f_j^{(m,k)}. \tag{41}$$

Proof. First of all notice the operator $P_n^{(m,k)}\colon C^0(I) \to C^0(I)$ defined in (36) is a projection onto $C_n^{m-k}(I)$. This is a consequence of (40) and the formula (cf. (12) and (36)–(39))

$$P_n^{(m,k)} f = \sum_{j=k-m}^{n} (g_j^{(m,k)}, f) f_j^{(m,k)}.$$

Thus

$$P_n^{(m,k)} f = f \quad \text{for} \quad f \in C_n^{m-k}(I),$$

and therefore in a dense subset of $C^0(I)$ we have

$$\lim_{n\to\infty} P_n^{(m,k)} f = f,$$

whence by Lemma 5 we get (41).

Theorem 3 was established for $m = 0$ in [3] and for $m = 1$ in [10].

11. Tensor product of the special basis. Let $N_{\boldsymbol{k}} = N_{k_1} \times \ldots \times N_{k_d}$ for $\boldsymbol{k} = (k_1, \ldots, k_d)$, $k_i \geqslant 0$, $d \geqslant 1$, where $N_k = \{k+1, k+2, \ldots\}$ for $k \geqslant 0$ and $N = N_0$. For each multi-index $\boldsymbol{k} = (k_1, \ldots, k_d)$, a one-to-one mapping $\nu_{\boldsymbol{k}}\colon N_{\boldsymbol{k}} \to N$ is defined as follows:

For $d = 1$ $\nu_0(i) = i$, $i \in N_0$, and for $d = 2$

$$\nu_0(i_1, i_2) = \begin{cases} (i_1-1)^2 + i_2 & \text{for} \quad 1 \leqslant i_2 < i_1, \\ i_2(i_2-1) + i_1 & \text{for} \quad 1 \leqslant i_1 \leqslant i_2. \end{cases} \tag{42}$$

For $d > 2$

$$\nu_0(i_1, \ldots, i_d) = \nu_0(\nu_0(i_1, \ldots, i_{d-1}), \nu_0(i_d)),$$

where the index $\mathbf{0}$ in $\nu_0(i_1, \ldots, i_f)$, $1 \leqslant f \leqslant d$, is the zero element of the corresponding space R^f.

Now, let

$$\nu_{\boldsymbol{k}}^*(i_1, \ldots, i_d) = \#\{(j_1, \ldots, j_d) \in N_{\boldsymbol{k}}\colon \nu_0(j_1, \ldots, j_d) \leqslant \nu_0(i_1, \ldots, i_d)\}$$

for any $(i_1, \ldots, i_d) \in N_0$, where $\# E$ denotes the number of elements of the set E.

It should be clear that the restriction $\nu_{\boldsymbol{k}}$ of $\nu_{\boldsymbol{k}}^*$ to $N_{\boldsymbol{k}}$, i.e. $\nu_{\boldsymbol{k}}\colon N_{\boldsymbol{k}} \to N$, $\nu_{\boldsymbol{k}}(i_1, \ldots, i_d) = \nu_{\boldsymbol{k}}^*(i_1, \ldots, i_d)$, is one-to-one and onto.

For given $d \geqslant 1$, $\boldsymbol{k} = (k_1, \ldots, k_d)$, $1 \leqslant k_i \leqslant m$ and $n \in N_0$ we define

$$f_{n;d}^{(m,\boldsymbol{k})} = f_{i_1,1}^{(m,k_1)} \otimes \ldots \otimes f_{i_d;1}^{(m,k_d)}$$

and

$$g_{n;d}^{(m,\boldsymbol{k})} = g_{i_1,1}^{(m,k_1)} \otimes \ldots \otimes g_{i_d;1}^{(m,k_d)},$$

where $n = \nu_{\boldsymbol{k}}(i_1, \ldots, i_d)$, $f_{i;1}^{(m,k)} = f_{i-(m+1)}^{(m,k)}$, $g_{i;1}^{(m,k)} = g_{i-(m+1)}^{(m,k)}$ for $k = 0, \ldots, m$, and

$$(f_1 \otimes \ldots \otimes f_d)(t_1, \ldots, t_d) = f_1(t_1) \cdot \ldots \cdot f_d(t_d).$$

It is important to estimate the kernels

$$K_{n;d}^{(m,\boldsymbol{k})}(t, s) = \sum_{i=1}^{n} f_{i;d}^{(m,\boldsymbol{k})}(t) g_{i;d}^{(m,\boldsymbol{k})}(s), \quad n \geqslant 1,$$

where $t = (t_1, \ldots, t_d)$, $s = (s_1, \ldots, s_d)$. It is convenient to put $K_{0;d}^{(m,\boldsymbol{k})} = 0$ for any d, m and $\boldsymbol{k}$.

Notice that $r(n) = \nu_{\boldsymbol{k}}^*[\nu_0^{-1}(n)]$ approaches infinity as $n \to \infty$ and

$$D^{(\boldsymbol{k},0)} H^{(0,\boldsymbol{k})} K_{n;d}^{(m,0)} = K_{r(n);d}^{(m,\boldsymbol{k})}. \tag{43}$$

Conversely, for $n \in N$ we have

$$K_{n;d}^{(m,\boldsymbol{k})} = D^{(\boldsymbol{k},0)} H^{(0,\boldsymbol{k})} K_{s(n);d}^{(m,0)}, \tag{44}$$

where

$$s(n) = \nu_0(\nu_{\boldsymbol{k}}^{-1}(n)).$$

LEMMA 6. *Let $d \geqslant 1$, $m \geqslant 0$ and $\boldsymbol{k} = (k_1, \ldots, k_d)$, $0 \leqslant k_i \leqslant m$, be given. Then there exists a constant $P_{m;d}$ such that*

$$\int_{I^d} |K_{n;d}^{(m,\boldsymbol{k})}(t,s)|\,dt \leqslant P_{m;d} \quad \text{for} \quad n \geqslant 1.$$

Proof. We use the induction argument with respect to the dimension d. For $d = 1$ Lemma 6 follows from Theorem 2. Now, let us assume that Lemma 6 holds for $d-1$, $d \geqslant 2$. For given $s \in N$ let s_1 and s_2 be defined by the equality $s = \nu_0(s_1, s_2)$. According to (42) there are two cases to be considered. The first case corresponds to $s = (s_1-1)^2 + s_2$ with $1 \leqslant s_2 < s_1$, and then we have the following decomposition

$$K_{s;d}^{(m,0)} = K_{s_1-1;d-1}^{(m,0)} \otimes K_{s_1-1;1}^{(m,0)} + (K_{s_1;d-1}^{(m,0)} - K_{s_1-1;d-1}^{(m,0)}) \otimes K_{s_2;1}^{(m,0)}. \tag{45}$$

In the second case $s = s_2(s_2-1) + s_1$ with $1 \leqslant s_1 \leqslant s_2$, and then we have

$$K_{s,d}^{(m,0)} = K_{s_2;d-1}^{(m,0)} \otimes K_{s_2-1;1}^{(m,0)} + K_{s_1;d-1}^{(m,0)} \otimes (K_{s_2;1}^{(m,0)} - K_{s_2-1;1}^{(m,0)}). \tag{46}$$

In each case we apply the operator $D^{(k,0)} H^{(0,\boldsymbol{k})}$ to both sides of (45) and (46), respectively. To complete the proof it is sufficient then to use relations (43), (44) and the inductive assumption.

Now, like in the preceding section, it is not hard to prove with the help of Lemmas 1 and 6

THEOREM 4. *Let $m \geqslant 0$, $d \geqslant 1$, $1 \leqslant p < \infty$, $\boldsymbol{k} = (k_1, \ldots, k_d)$ with $0 \leqslant k_i \leqslant m$ for $i = 1, 2, \ldots, d$ be given. Then $(f_{n;d}^{(m,\boldsymbol{k})}, n = 1, 2, \ldots)$ is a basis simultaneously in $C^0(I^d)$ and $L_p(I^d)$ and*

$$f = \sum_{n=1}^{\infty} (f, g_{n;d}^{(m,\boldsymbol{k})}) f_{n;d}^{(m,\boldsymbol{k})} \tag{47}$$

holds for f in $C^0(I^d)$ and $L_p(I^d)$ respectively.

12. Construction of the basis in $C^m(I^d)$ and $W_p^m(I^d)$. It is important to notice that the Sobolev space $W_p^m(I^d)$ described in Section 1 can also be defined as the completion of $C^m(I^d)$ with respect to the norm $\| \;\|_p^{(m)}$, $1 \leqslant p < \infty$ (cf. the introduction in [2]). Thus, $C^m(I^d)$ is dense in $W_p^m(I^d)$.

LEMMA 7. *Let $d \geqslant 1$, $m \geqslant 0$ and $1 \leqslant p < \infty$. Then*

$$(g_{j;d}^{(m,\boldsymbol{k})}, D^{\boldsymbol{k}} f) = (f_{j;d}^{(m,0)}, f) \tag{48}$$

holds whenever $\boldsymbol{k} = (k_1, \ldots, k_d)$, $0 \leqslant |\boldsymbol{k}| \leqslant m$ and f is in $C^m(I^d)$ or $W_p^m(I^d)$.

Proof. Since both sides of (48) are continuous linear functionals on $W_p^m(I^d)$ in which $C^m(I^d)$ is dense it remains to check (48) for $f \in C^m(I^d)$. This follows from the definitions of $g_{j;d}^{(m,\boldsymbol{k})}$, $f_{j,d}^{(m,\boldsymbol{k})}$ and formula (39).

Now we are ready to prove the main result.

THEOREM 5. *Let $m \geqslant 0$, $d \geqslant 1$ and p, $1 \leqslant p < \infty$, be given. Then $(f_{j;d}^{(m,0)}, j = 1, 2, \ldots)$ is an orthogonal basis, with respect to the scalar product (1), simultaneously in $C^m(I^d)$ and $W_p^m(I^d)$ i.e. for each f in $C^m(I^d)$ and $W_p^m(I^d)$*

$$f = \sum_{j=1}^{\infty} (f_{j;d}^{(m,0)}, f) f_{j;d}^{(m,0)},$$

where the series is convergent in the norms $\| \;\|^{(m)}$ and $\| \;\|_p^{(m)}$, respectively.

Proof. Let

$$P_{n;d}^{(m,\boldsymbol{k})} f = \sum_{j=1}^{n} (g_{j;d}^{(m,\boldsymbol{k})}, f) f_{j;d}^{(m,\boldsymbol{k})}.$$

It is sufficient to show that

$$\|D^{\boldsymbol{k}}(f - P_{n;d}^{(m,0)} f)\|_p \to 0 \quad \text{for} \quad f \in W_p^m(I^d),$$

and

$$\|D^{\boldsymbol{k}}(f - P_{n;d}^{(m,0)} f)\|^{(0)} \to 0 \quad \text{for} \quad f \in C^m(I^d)$$

as $n \to \infty$, $|\boldsymbol{k}| \leqslant m$. To see this notice that Lemma 7 gives

$$\begin{aligned} D^{\boldsymbol{k}} P_{n;d}^{(m,0)} f(t) &= D^{\boldsymbol{k}} \int_{I^d} K_{n;d}^{(m,0)}(t,s) f(s)\,ds \\ &= \sum_{j=1}^{n} (f_{j;d}^{(m,0)}, f) D^{\boldsymbol{k}} f_{j;d}^{(m,0)}(t) \\ &= \sum_{j=1}^{r(n)} (g_{j;d}^{(m,\boldsymbol{k})}, D^{\boldsymbol{k}} f) f_{j;d}^{(m,\boldsymbol{k})}(t) \\ &= P_{r(n)}^{(m,\boldsymbol{k})} D^{\boldsymbol{k}} f(t), \end{aligned}$$

where $r(n)$ is given as in (43). Since $n \to \infty$ implies that $r(n) \to \infty$ we obtain from Theorem 4 the required result.

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